A NOTE ON A CONJECTURE OF GONEK

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Abstract: We derive a lower bound for a second moment of the reciprocal of the derivative of the Riemann zeta-function over the zeros of $\zeta(s)$ that is half the size of the conjectured value. Our result is conditional upon the assumption of the Riemann Hypothesis and the conjecture that the zeros of the zeta-function are simple.

Keywords: Riemann zeta-function, mean-value theorems, Gonek’s conjecture.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. Using a heuristic method similar to Montgomery’s study [13] of the pair-correlation of the imaginary parts of the non-trivial zeros of $\zeta(s)$, Gonek has made the following conjecture [7, 8].

**Conjecture.** Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple. Then, as $T \to \infty$,

$$
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3}{\pi^3} T
$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

The assumption of the simplicity of the zeros of the zeta-function in the above conjecture is so that the sum over zeros on the right-hand side of (1.1) is well defined. While the details of Gonek’s method have never been published, he announced his conjecture in [5]. More recently, a different heuristic method of Hughes, Keating, and O’Connell [10] based upon modeling the Riemann zeta-function and its derivative using the characteristic polynomials of random matrices has led to the same conjecture. Through the work of Ingham [11], Titchmarsh (Chapter 14 of [21]), Odlyzko and te Riele [17], Gonek (unpublished), and Ng [15],

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it is known that the behavior of this and related sums are intimately connected to
the distribution of the sum
\[ M(x) = \sum_{n \leq x} \mu(n) \]
where \( \mu(\cdot) \), the Möbius function, is defined by \( \mu(1) = 1 \), \( \mu(n) = (-1)^k \) if \( n \) is
divisible by \( k \) distinct primes, and \( \mu(n) = 0 \) if \( n > 1 \) is not square-free. See also
[9] and [20] for connections between similar sums and other arithmetic problems.

In support of his conjecture, Gonek [5] has shown, assuming the Riemann
Hypothesis and the simplicity of the zeros of \( \zeta(s) \), that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\gamma)|^2} \geq CT \]  
for some constant \( C > 0 \) and \( T \) sufficiently large. In this note, we show that the
inequality in (1.2) holds for any constant \( C < \frac{3}{2\pi} \).

**Theorem.** Assume the Riemann Hypothesis and that the zeros of \( \zeta(s) \) are simple.
Then, for any fixed \( \varepsilon > 0 \),
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\gamma)|^2} \geq \left( \frac{3}{2\pi} - \varepsilon \right) T \]  
for \( T \) sufficiently large.

While our result differs from the conjectural lower bound by a factor of 2,
any improvements in the strength of this lower bound have, thus far, eluded us. It
would be interesting to investigate whether for \( k > 0 \) there is a constant \( C_k > 0 \) such that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\gamma)|^{2k}} \geq C_k T (\log T)^{(k-1)^2} \]  
for \( T \) sufficiently large. However, a lower bound of this form is probably not of
the correct order of magnitude for all \( k \). This is because it is expected that for
each \( \varepsilon > 0 \) there are infinitely many zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \) satisfying
\[ |\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon} \]. If such a sequence of zeros were to exist, it would then follow that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\gamma)|^{2k}} = \Omega \left( T^{2k/3-\varepsilon} \right) \]  
and the lower bound in (1.4) would be significantly weaker than this \( \Omega \)-result when
\( k > \frac{3}{2} \).

2. **Proof of Theorem**

The method we use to prove our theorem is based on a recent idea of Rudnick and
Soundararajan [18]. Let
\[ \xi = T^g \]  
(2.1)
where $0 < \vartheta < 1$ is fixed and define the Dirichlet polynomial
\[ M_\xi(s) = \sum_{n \leq \xi} \mu(n)n^{-s} \]

where $\mu$ is the Möbius function. Assuming the Riemann Hypothesis, for any non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$, we see that $M_\xi(\rho) = M_\xi(1 - \rho)$. From this observation and Cauchy’s inequality it follows that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \frac{|M_1|^2}{M_2} \quad \text{(2.2)} \]

where
\[ M_1 = \sum_{0 < \gamma \leq T} \frac{1}{\zeta'(\rho)}M_\xi(1 - \rho) \quad \text{and} \quad M_2 = \sum_{0 < \gamma \leq T} |M_\xi(\rho)|^2. \]

Our Theorem is a consequence of the following proposition.

**Proposition.** Assume the Riemann Hypothesis and let $0 < \vartheta < 1$ be fixed. Then
\[ M_2 = \frac{3}{\pi^3} (\vartheta + \vartheta^2) T \log^2 T + O(T \log T). \quad \text{(2.3)} \]

If we further assume that the zeros of $\zeta(s)$ are all simple, then there exists a sequence $T := \{\tau_n\}_{n=3}^\infty$ such that $n < \tau_n \leq n + 1$ and for $T \in T$ we have
\[ M_1 = \frac{3\vartheta}{\pi^3} T \log T + O(T). \quad \text{(2.4)} \]

We now deduce our theorem from the above proposition.

**Proof of Theorem.** Let $T \geq 4$ and choose $\tau_n$ to satisfy $T - 1 \leq \tau_n < T$. Combining (2.2), (2.4), and (2.3) we see that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \sum_{0 < \gamma \leq \tau_n} \frac{1}{|\zeta'(\rho)|^2} \geq \frac{\vartheta^2}{(\vartheta + \vartheta^2)} \frac{3}{\pi^3} \tau_n + o(\tau_n) \geq \frac{1}{1 + \vartheta - 1} \frac{3}{\pi^3} T + o(T) \quad \text{(2.5)} \]

under the assumption of the Riemann Hypothesis and the simplicity of the zeros of $\zeta(s)$. From (2.5), our theorem follows by letting $\vartheta \to 1^-$. \qed

We could have just as easily estimated the sums $M_1$ and $M_2$ using a Dirichlet polynomial $\sum_{n \leq \xi} a_n n^{-s}$ for a large class of coefficients $a_n$ in place of $M_\xi(s)$. In the special case where
\[ a_n = \mu(n)P\left(\frac{\log \xi}{n}\right) \]
for polynomials $P$, we can show that the choice $P = 1$ is optimal in the sense that it leads to largest lower bound in (1.3).

We prove the above proposition in the next two sections; the sum $M_1$ is estimated in section 3 and the sum $M_2$ is estimated in section 4. The evaluation of sums like $M_1$ dates back to Ingham’s [11] important work on $M(x)$ in which he considered sums of the form

$$
\sum_{0 < \gamma < T} (T - \gamma)^k \zeta'(\rho)^{-1}
$$

for $k \in \mathbb{R}$. The sum $M_2$ is of the form

$$
\sum_{0 < \gamma < T} |A(\rho)|^2 \quad \text{where} \quad A(s) = \sum_{n \leq \xi} a_n n^{-s}
$$

is a Dirichlet polynomial with $\xi \leq T$. Such sums have played an important role in various applications. For instance, results concerning the distribution of consecutive zeros of $\zeta(s)$ and discrete mean values of the zeta-function and its derivatives are proven in [1, 2, 3, 6, 12, 13, 16, 19]. In each of these articles, the evaluation of the discrete mean (2.6) either makes use of the Guinand-Weil explicit formula or of Gonek’s uniform version [6] of Landau’s formula

$$
\zeta(\beta + i\gamma) = -\frac{T}{2\pi} A(x) + E(x, T) \quad (2.7)
$$

for $x, T > 1$ where $E(x, T)$ is an explicit error function uniform in $x$ and $T$. A novel aspect of our approach is that it does not require the use of the Guinand-Weil explicit formula or of the Landau-Gonek explicit formula (2.7). Instead we evaluate $M_2$ using the residue theorem and a version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [14]. Our approach is simpler and it is likely that it can be extended to evaluate the discrete mean (2.6) for a large class of coefficients $a_n$ when $\xi \leq T$.

3. The estimation of $M_1$

To estimate $M_1$, we require the following version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials.

**Lemma.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. For any real number $T > 0$, we have

$$
\int_0^T \left( \sum_{n=1}^{\infty} a_n n^{-it} \right) \left( \sum_{n=1}^{\infty} b_n n^{it} \right) dt = T \sum_{n=1}^{\infty} a_n b_n + O \left( \left( \sum_{n=1}^{\infty} n |a_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} n |b_n|^2 \right)^{\frac{1}{2}} \right) \quad (3.1)
$$
Proof. This is Lemma 1 of Tsang [22]. The special case where \( b_n = a_n \) is originally due to Montgomery and Vaughan [14]. It turns out, as shown by Tsang, that this special case is equivalent to the more general case stated in the lemma.

Let \( T \geq 4 \) and set \( c = 1 + (\log T)^{-1} \). It is well known (see Theorem 14.16 of Titchmarsh [21]) that assuming the Riemann Hypothesis there exists a sequence \( T = \{ \tau_n \}_{n=3}^{\infty}, \ n < \tau_n \leq n + 1, \) and a fixed constant \( A > 0 \) such that

\[
|\zeta(\sigma + i\tau_n)|^{-1} \ll \exp\left(\frac{A \log \tau_n}{\log \log \tau_n}\right)
\]  

uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \). We now prove the estimate (2.4) assuming that \( T \in T \).

Recall that \( |\gamma| > 1 \) for every non-trivial zero \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \). Thus, assuming that all the zeros of \( \zeta(s) \) are simple, the residue theorem implies that

\[
M_1 = \frac{1}{2\pi i} \left( \int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i} \right) \frac{1}{\zeta(s)} M_\xi(1-s) \, ds
\]

say. Here we are using the fact that the residue of the function \( 1/\zeta(s) \) at \( s = \rho \) equals \( 1/\zeta'(\rho) \) if \( \rho \) is a simple zero of \( \zeta(s) \).

The main contribution to \( M_1 \) comes from the integral \( I_1 \); the remainder of the integrals contribute an error term. Observe that

\[
I_1 = \frac{1}{2\pi} \int_1^T \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{c+it}} \sum_{n \leq \xi} \frac{\mu(n)}{n^{1-c-it}} \, dt.
\]

By (3.1) with \( a_m = \mu(m)m^{-c} \) and \( b_n = \mu(n)n^{-1+c} \) it follows that

\[
I_1 = \frac{(T-1)}{2\pi} \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O \left( \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2c-1}} \right)^{\frac{1}{2}} \left( \sum_{n \leq \xi} \frac{\mu(n)^2}{n^{2c-1}} \right)^{\frac{1}{2}}.
\]

Since

\[
\sum_{n \leq \xi} \frac{\mu(n)^2}{n} = \frac{6}{\pi^2} \log \xi + O(1),
\]

we conclude that

\[
I_1 = \frac{3}{\pi^3} T \log \xi + O \left( \xi \sqrt{\log T} + T \right)
\]

for our choice of \( c \). Here we have used the fact that

\[
\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2c-1}} \ll \zeta(2c-1) \ll \log T.
\]

To estimate the contribution from the integral \( I_2 \), we recall the functional equation for the Riemann zeta-function which says that

\[
\zeta(s) = \chi(s)\zeta(1-s)
\]
where
\[ \chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right). \]

Stirling’s asymptotic formula for the Gamma-function can be used to show that
\[
|\chi(\sigma + it)| = \left( \frac{|t|}{2\pi} \right)^{1/2 - \sigma} \left( 1 + O(|t|^{-1}) \right)
\]
uniformly for \(-1 \leq \sigma \leq 2\) and \(|t| \geq 1\). Combining this estimate and (3.2), it follows that, for \(T \in T\),
\[
|\zeta(\sigma + iT)|^{-1} \ll T^{\min(\sigma-1/2,0)} \exp \left( \frac{A \log T}{\log \log T} \right)
\]
uniformly for \(-1 \leq \sigma \leq 2\). In addition, we have the trivial bound
\[
|M_\xi(\sigma + it)| \ll \xi^{1-\sigma}.
\]
Thus, estimating the integral \(I_2\) trivially, we find that
\[
I_2 \ll \exp \left( \frac{A \log T}{\log \log T} \right) \int_{1-c}^c T^{\min(\sigma-1/2,0)} \xi^\sigma d\sigma \ll \xi \exp \left( \frac{A \log T}{\log \log T} \right).
\]

To bound the contribution from the integral \(I_3\), we notice that the functional equation for \(\zeta(s)\) combined with the estimate in (3.5) implies that, for \(1 \leq |t| \leq T\),
\[
|\zeta(1-c+it)|^{-1} \ll |t|^{1/2-c} |\zeta(c-it)|^{-1} \ll |t|^{1/2-c} \zeta(c) \ll |t|^{-1/2} \log T.
\]

It therefore follows that
\[
I_3 \ll \log T \left( \sum_{\eta \leq \xi} |\mu(n)| \right) \int_1^T t^{-1/2} dt \ll \sqrt{T} (\log T) \log \xi.
\]

Finally, since \(1/\zeta(s)\) and \(M_\xi(1-s)\) are bounded on the interval \([1-c+i, c+i]\), we find that \(I_4 \ll 1\). Hence, our combined estimates for \(I_1, I_2, I_3,\) and \(I_4\) imply that
\[
M_1 = \frac{3}{\pi^3} T \log \xi + O \left( \xi \exp \left( \frac{A \log T}{\log \log T} \right) + T \right).
\]

From this and (2.1), the estimate in (2.4) follows.

4. The estimation of \(M_2\)

We now turn our attention to estimating the sum \(M_2\). As before, let \(T \geq 4\) and \(c = 1 + (\log T)^{-1}\). Assuming the Riemann Hypothesis, we notice that
\[
M_2 = \sum_{0 < \gamma \leq T} M_\xi(\rho) M_\xi(1-\rho).
\]
Therefore, by the residue theorem, we see that
\[
M_2 = \frac{1}{2\pi i} \left( \int_{c+iT}^{c+iT} + \int_{1-c+iT}^{1-c+iT} + \int_{1-c-iT}^{1-c-iT} \right) M_\xi(s)M_\xi(1-s)\frac{\zeta'}{\zeta}(s) \, ds
= J_1 + J_2 + J_3 + J_4,
\]
say. In order to evaluate the integrals over the horizontal part of the contour we shall impose some extra conditions on \(T\). Without loss of generality, we may assume that \(T\) satisfies
\[
|\gamma - T| \gg \frac{1}{\log T} \quad \text{for all ordinates } \gamma \text{ and}
\]
(4.1)
\[
\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2 \quad \text{uniformly for all } 1 - c \leq \sigma \leq c.
\]
In each interval of length one such a \(T\) exists. This well known argument may be found in [4], page 108. Applying (3.6) we find that
\[
\sum_{T < \gamma < T + 1} |M_\xi(\rho)M_\xi(1-\rho)| \ll \xi(\log T).
\]
Here we have used the standard estimate that there are \(O(\log T)\) zeros of \(\zeta(s)\) with ordinates in the interval \([T, T + 1]\). Therefore our choice of \(T\) determines \(M_2\) up to an error term \(O(\xi \log T)\). First we estimate the horizontal portions of the contour. By (3.6) and (4.1), we have
\[
J_2 = \frac{1}{2\pi} \int_c^{1-c} M_\xi(\sigma + iT)M_\xi(1-\sigma - iT)\frac{\zeta'}{\zeta}(\sigma + iT) \, d\sigma \ll \xi(\log T)^2.
\]
Similarly, it may be shown that \(J_4 \ll \xi\). Next we relate \(J_3\) to \(J_1\). We have
\[
J_3 = \frac{1}{2\pi} \int_T^1 M_\xi(1-c + iT)M_\xi(c - it)\frac{\zeta'}{\zeta}(1-c + it) \, dt
= -\frac{1}{2\pi} \int_1^T M_\xi(1-c - it)M_\xi(c + it)\frac{\zeta'}{\zeta}(1-c - it) \, dt
\]
By differentiating (3.4), the functional equation, we find that
\[
-\frac{\zeta'}{\zeta}(1-c-it) = -\frac{\chi'}{\chi}(1-c-it) + \frac{\zeta'}{\zeta}(c+it)
\]
and hence that
\[
J_3 = -\frac{1}{2\pi} \int_1^T M_\xi(1-c - it)M_\xi(c + it)\frac{\chi'}{\chi}(1-c - it) \, dt
+ \frac{1}{2\pi} \int_1^T M_\xi(1-c - it)M_\xi(c + it)\frac{\zeta'}{\zeta}(c+it) \, dt.
\]
By (3.4) and Stirling’s formula it can be shown that

$$\frac{\chi'}{\chi}(1-c-it) = \log \left( \frac{|t|}{2\pi} \right) (1 + O(|t|^{-1}))$$

uniformly for $1 \leq |t| \leq T$. By (3.6), the term $O(|t|^{-1})$ contributes to $J_3$ an amount which is $O(\xi \log T)$ and, hence, it follows that

$$J_3 = K + \mathcal{J}_1 + O(\xi(\log T))$$

where

$$K = \int_1^T \log \left( \frac{t}{2\pi} \right) M_\xi(c+it)M_\xi(1-c-it) \, dt.$$  

Collecting estimates, we deduce that

$$M^2_2 = K + 2\Re J_1 + O(\xi^2(\log T)). \tag{4.2}$$

To complete our estimation of $M^2_2$, it remains to evaluate $K$ and then $J_1$. Integrating by parts, it follows that

$$K = \frac{1}{2\pi} \int_1^T \log \left( \frac{T}{2\pi} \right) M_\xi(c+it)M_\xi(1-c-it) \, dt$$

$$- \frac{1}{2\pi} \int_1^T \left( \int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du \right) \frac{dt}{t}. $$

By (3.1), we have

$$\int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du = (t-1) \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O(\xi \sqrt{\log T})$$

$$= \frac{6}{\pi^2} t \log \xi + O(\xi \sqrt{\log T} + t)$$

for $t > 1$. Substituting this estimate into the above expression for $K$, we see that

$$K = \frac{3}{\pi^3} T \log \left( \frac{T}{2\pi} \right) \log \xi + O(T \log T) + O(T \log \xi)$$

$$= \frac{3}{\pi^3} T \log \left( \frac{T}{2\pi} \right) \log \xi + O(T \log T). \tag{4.3}$$

We finish by evaluating the integral $J_1$ which is similar to the evaluation of the integral $I_1$ in the previous section. By another application of (3.1), we find that

$$J_1 = -\frac{1}{2\pi} \int_1^T \sum_{n=1}^\infty \frac{\alpha_n}{n^c+it} \sum_{n \leq \xi} \frac{\mu(n)}{n^{1-c-it}} \, dt$$

$$= -\frac{(T-1)}{2\pi} \sum_{n \leq x} \alpha_n \mu(n) \frac{\mu(n)}{n}$$

$$+ O \left( \left( \sum_{n=1}^\infty \frac{\alpha_n^2}{n^{2c-1}} \right)^{1/2} \left( \sum_{n \leq \xi} \frac{\mu(n)^2}{n^{1-2c}} \right)^{1/2} \right).$$
where the coefficients $\alpha_n$ are defined by

$$\alpha_n = \sum_{k\ell = n, \ell \leq \xi} \Lambda(k)\mu(\ell).$$

Observe that trivially $|\alpha_n| \leq \sum_{u \mid n} \Lambda(u) \leq \log n$. It follows that the error term in the above expression for $J_1$ is $\ll \zeta''(2c - 1)\frac{1}{2}\xi \ll (\log T)^{\frac{3}{2}}$. Finally, we note that

$$\sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{k \leq \frac{x}{\ell}} \frac{\Lambda(k)\mu(k\ell)}{k} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{p \prime \leq \ell, p \text{ prime}, j \geq 0} \frac{\mu(p^j\ell)\log p}{p^j}$$

$$= \sum_{\ell \leq \xi} \frac{\mu(\ell)}{\ell} \sum_{p \leq \xi/\ell} \frac{\mu(p\ell)\log p}{p} + O(\log \xi)$$

$$= -\sum_{\ell \leq \xi} \frac{\mu(\ell)^2}{\ell} \sum_{p \leq \xi/\ell} \frac{\log p}{p} + O\left(\log \xi + \sum_{\ell \leq \xi} \frac{1}{\ell} \sum_{p \mid \ell} \frac{\log p}{p}\right)$$

since $\mu(p\ell) = -\mu(\ell)$ if $(p, \ell) = 1$ and $\mu(p\ell) = 0 = O(1)$ if $p \mid \ell$. The sum in the error term is

$$\sum_{\ell \leq \xi} \frac{1}{\ell} \sum_{p \leq \xi/\ell} \frac{\log p}{p} = \sum_{p \leq x} \frac{(\log p)}{p^2} \sum_{\ell \leq \xi/\ell} \frac{1}{p} \ll \log \xi.$$

Hence, by the elementary result $\sum_{p \leq \xi} \frac{\log p}{p} = \log \xi + O(1)$, (3.3), and partial summation, we deduce that

$$\sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = -\sum_{l \leq \xi} \frac{\mu(l)^2 \log(l \xi)}{l} + O(\log \xi) = -\frac{3}{\pi^2} (\log \xi)^2 + O(\log \xi).$$

Therefore, combining formulae, we have

$$J_1 = -\frac{3}{2\pi^3} T(\log \xi)^2 + O(T \log T). \quad (4.4)$$

Finally (4.2), (4.3), and (4.4) imply that

$$M_2 = \frac{3}{\pi^3} T \log T \log \xi + \frac{3}{\pi^3} T(\log \xi)^2 + O(T \log T)$$

and, thus, by (2.1) we deduce (2.3).

References


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