Improving the power of the Diebold–Mariano–West test for least squares predictions

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Abstract

We propose a more powerful version of the test of Diebold and Mariano (1995) and West (1996) for comparing least squares predictors based on non-nested models when the parameter being tested is the expected difference between the squared prediction errors. The proposed test improves the asymptotic power by using a more efficient estimator of the parameter being tested than that used in the literature. The estimator used by the standard version of the test depends on the individual predictions and realizations only through the observations on the prediction errors. However, the parameter being tested can also be expressed in terms of moments of the predictors and the predicted variable, some of which cannot be identified separately by the observations on the prediction errors alone. Parameterizing these moments in a GMM framework and drawing on the theory of West (1996), we devise more powerful versions of the test by exploiting a restriction that is maintained routinely under the null hypothesis by West (1996, Assumption 2b) and later studies. This restriction requires only finite second-order moments and covariance stationarity in order to ensure that the population linear projection exists. Simulation experiments show that the potential gains in power can be substantial.

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1. Introduction

The test of out-of-sample predictive accuracy that was proposed by Diebold and Mariano (1995) and West (1996) is widely regarded as an important test for comparing two predictors. Applications include the studies by Andreou, Ghysels, and Kourtellos (2013), Corradi, Swanson, and Olivetta (2001), Hanson and Lunde (2005), Hong and Lee (2003), Mark (1995) and Swanson and White (1997).

Formal asymptotic theory was first presented by West (1996), and further developed by Clark and McCracken (2001, 2014), McCracken (2007) and others. The present paper proposes a more powerful version of the test, obtained by using a more efficient estimator of the parameter being tested that than which is used in the literature. The parameter being tested is the expected difference between functions of the prediction errors. The standard version of the test estimates it using the sample mean of the difference; as a consequence, the individual observations on the predictions and realizations generally enter the test statistic only through the prediction errors. However, the parameter being tested can also be expressed in terms of moments of the predictors and the predicted variable, some of which cannot be identified separately by the

Keywords: Prediction, Hypothesis tests
observations on the prediction errors alone.\footnote{Diebold and Mariano (1995, p. 254) note that the test is not limited to testing functions of the prediction error, but can be applied to functions in which the realization and the prediction enter separately. Of course, the same argument applies if the moments of the predicted variable and the predictors cannot be identified by the observations on such functions.} This raises the possibility of using the individual observations on the predictors and the predicted variable to construct more efficient estimators of the tested parameter,\footnote{The potential inefficiency of replacing a sample of observations (here, the individual observations on the predictors and predicted variable) with functions of the observations (here, the prediction errors) can be quantified in terms of information matrices. Specifically, if $g(X)$ is a measurable function of a random variable $X$ with a density that is a function of a vector $\theta$, it can be shown that the difference between the information matrices with respect to $\theta$, $I_\theta - J_{g(X)}$, is non-negative definite (Rao, 1973, pp. 330–331).} leading to tests with greater asymptotic power.

We pursue this for the commonly-tested hypothesis of equal mean squared errors in the case of least squares predictors, and propose a new version of the test based on a more efficient estimator of the expected difference between the squared prediction errors. We parameterize the moments of the predictors in a GMM framework, and derive a more efficient estimator by incorporating a restriction that is part of the weak regularity conditions that are maintained routinely under the null hypothesis by West (1996, Assumption 2b) and later studies. In the context of linear least squares prediction, this restriction requires only finite second-order moments and covariance stationarity. While this requirement is just a standard restriction on the data-generating process, it might be considered more restrictive than the setup of Diebold and Mariano (1995), in which the underlying predictors are completely unspecified. In applications that involve evaluating predictions from surveys, for example, it is clearly desirable to leave the underlying predictors unspecified and impose restrictions on the prediction errors directly. On the other hand, the proposed test should be useful given the large number of studies that evaluate predictions under the conditions of the present paper. Examples of studies that compare the mean squared errors of least squares predictors under covariance stationarity include those by Clark and McCracken (2006), Mark (1995), and Stock and Watson (2002, 2007), among others.

The proposed test and the GMM estimator on which it is based are described in Section 2. Section 3 reports evidence from simulation experiments on the efficiency of the estimator and the size and power of the tests. Consistent with the asymptotic efficiency of the GMM estimator, the simulations confirm substantial power gains relative to the estimator and the size and power of the tests. Confidence from simulation experiments on the efficiency of the Diebold–Mariano–West (DMW) test.

2. GMM version of the DMW test

2.1. Null hypothesis, predictors and restrictions

Given the data available at time $t$, we consider two competing predictors $\hat{y}_{1t:t+t}$ and $\hat{y}_{2t:t+t}$ of a variable $y_{1t:t+t}$ at time $t + \tau$ with prediction errors $e_{1t:t+t} = y_{1t:t+t} - \hat{y}_{1t:t+t}$ and $e_{2t:t+t} = y_{1t:t+t} - \hat{y}_{2t:t+t}$. The DMW test can be used to test the null hypothesis of equal predictive accuracy for a wide range of loss functions and predictors. An important special case is equal mean squared errors:

$$H_0 : \theta \equiv E(e_{1t:t+t}^2 - e_{2t:t+t}^2) = 0.$$ \hfill (1)

We assume that Eq. (1) is to be tested using $P$ of the $\tau$-step-ahead predictions. The predictors are of the form $\hat{y}_{jt:t+t} = X_{jt:t+t} \hat{\beta}_j (j = 1, 2)$, where $\hat{\beta}_j = (\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jt}, \hat{\beta}_{jt+1})$ is the least squares estimator computed from the regression of $y_{it}$ on $X_{jt:t+t}$ using a minimum of $R$ in-sample observations, where $X_{jt}$ denotes the vector of predictor variables that are specific to predictor $j$ and $X_t$ is the common set of predictor variables. Following West (1996), we assume a recursive forecasting scheme which is also used commonly in practice.\footnote{Recent studies that have used DMW tests for comparing non-nested models include those by Andreou et al. (2013) and Naes, Skjelte, and Ødegaard (2011).} Therefore, $\hat{\beta}_j = (\sum_{t=1}^{T} X_{jt} X_{jt})^{-1} \sum_{t=1}^{T} X_{jt} y_{jt}$ for $t = R, \ldots, R + P - 1$, and $j = 1$ and 2. Given the $P$ out-of-sample predictions, the standard DMW test of Eq. (1) is based on the following estimator:

$$\hat{\theta} = \frac{1}{P} \sum_{t=1}^{P} (e_{1t:t+t} - e_{2t:t+t}) (e_{1t:t+t} + e_{2t:t+t}),$$ \hfill (2)

where the subscript “$e$” emphasizes that Eq. (2) is computed from the observations on the prediction errors.

\footnote{See for example Clark and McCracken (2001, 2014) and McCracken (2007).}

\footnote{This assumption accommodates our application of West (1996, Theorem 4.1) in Section 2.3. West (1996) assumes a recursive forecasting scheme but considers extensions to fixed and rolling schemes in an unpublished working paper; see West (1994).}
Since \( \theta \) is a function of the moments of \( y_{1,t}, \ldots, y_{s,t} \) and \( y_{1,t+1}, \ldots, y_{s,t+1} \), the individual observations on \( y_{1,t}, \ldots, y_{s,t} \) and \( y_{1,t+1}, \ldots, y_{s,t+1} \) can be used to devise an asymptotically more efficient estimator than Eq. (2). The basis for our approach is the following assumption:

**Assumption 1.** For \( j = 1, 2 \), the sequence \( \{X_{jt}, Y_{jt}\} \) is covariance-stationary and ergodic for second moments, and \( E(X_{jt}'X_{jt}) \) is nonsingular.

Assumption 1 imposes weak regularity conditions that are maintained routinely under the null hypothesis in the literature; see for example West (1996, pp. 1070–1071) and Clark and McCracken (2014, p. 418). It ensures that the moments \( E(X_{jt}'X_{jt}) \) and \( E(X_{jt}'Y_{jt}) \) are finite, that \( E(X_{jt}'X_{jt})^{-1} \) exists, and therefore, that the linear projection \( \beta_j = [E(X_{jt}'X_{jt})^{-1}]E(X_{jt}'Y_{jt}) \) exists for \( j = 1, 2 \). Under Assumption 1, the law of large numbers can also be applied to show that \( \hat{\theta}_j \) converges in probability (as \( t \to \infty \)) to \( \theta_j^* \). One well-known property of the linear projection coefficient is that \( E(X_{jt}'(Y_{jt} - X_{jt}\hat{\beta}_j)) = 0 \), which is a special case of Assumption 2b of West (1996). This in turn implies the following property which is used to restrict \( \theta \) below:

\[
E[X_{jt}X_{jt}'(Y_{jt} - X_{jt}\hat{\beta}_j)] = 0. \tag{3}
\]

The predictors and prediction errors obviously depend on the estimators \( \hat{\beta}_j \). To reflect this, we let \( \hat{\beta}_j = (\hat{\beta}_{1j}, \ldots, \hat{\beta}_{s,j,m-1}) \) and \( \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_s) \), and rewrite \( \theta_j \) and \( \hat{\theta}_j \) as \( \theta_j(\hat{\beta}) \) and \( \hat{\theta}_j(\hat{\beta}) \) respectively. As was emphasized by Clark and McCracken (2013), hypotheses like that in Eq. (1) can be interpreted in terms of the finite-sample or population-level predictive accuracy. The former concerns \( \hat{\theta}(\hat{\beta}) \) for finite \( R \), whereas the latter concerns \( \theta(\beta) \) where \( \beta \) is replaced with \( \hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_s) \). Following most of the previous work, we focus on the population-level predictive accuracy, and therefore Eq. (1) as “\( \theta(\beta) = 0 \)”.

Assumption 1 provides a basis for efficiency gains over \( \hat{\theta}_j(\hat{\beta}) \) through restricted GMM estimation of the tested

\[
E[X_{jt}X_{jt}'(Y_{jt} - X_{jt}\beta_j)] = 0 \quad \text{and} \quad E[X_{jt}'Y_{jt}] = 0.
\]

The sample counterparts of Eq. (3) clearly hold for the \( R \) in-sample observations, since the least squares predictors, \( \hat{\beta}_j \), and \( \hat{\theta}_j(\hat{\beta}) \), are estimated consistently by their sample counterparts (Hamilton, 1994, Chap. 4). Consequently, \( \hat{\theta}_j \) converges in probability to \( \theta_j(\beta_j) = \theta_j(\hat{\beta}_j) \). Provided that \( X_{jt}X_{jt}' \) is nonsingular, for a discussion of sufficient conditions for second-moment ergodicity, see Hamilton (1994, Ch. 7).
2.2. Restricted GMM estimation of \( \theta(\beta) \)

Next, we devise a more efficient restricted estimator of \( \theta(\beta) \) using a GMM framework in which the moments in Eq. (5) are estimated jointly, subject to Eq. (3). Note that Eqs. (5) and (3) can be written as:

\[
\theta(\beta) = 2E(X_{2,t+1} \beta X_{2,t+1}^T) - E((X_{1,t+1} \beta)^2) \\
- 2E(X_{1,t+1} \beta y_{1,t+1}) + E((X_{1,t+1} \beta)^2)
\]  

\[
E((X_{2,t+1} \beta y_{1,t+1})) = E((X_{2,t+1} \beta)^2) + \beta_1 \beta_2
\]  

Let \( \mu \) denote the vector of the moments in Eq. (6):

\[
\mu = [E(y_{1,t+1} X_{2,t+1} \beta_1), E(y_{1,t+1} X_{2,t+1} \beta_2), E((X_{1,t+1} \beta_1)^2)], E((X_{2,t+1} \beta_2)^2)]^T
\]

In what follows, each element of \( \mu \) is treated as a parameter to be estimated. The orthogonality condition for the GMM estimator of \( \mu \) is \( E[g_{t+1}(\mu, \beta)] = 0 \), where \( g_{t+1}(\mu, \beta) = m_{t+1}(\beta) - \mu \) and

\[
m_{t+1}(\beta) = [y_{1,t+1} X_{2,t+1} \beta_1, y_{1,t+1} X_{2,t+1} \beta_2, (X_{1,t+1} \beta_1)^2], (X_{2,t+1} \beta_2)^2]
\]

Thus, the feasible sample analog of \( E[g_{t+1}(\mu, \beta)] \) is

\[
\hat{g}(\mu, \hat{\beta}) = p^{-1} \sum_{t=1}^p g_{t+1}(\mu, \hat{\beta})
\]

The restricted GMM estimator of \( \mu, \hat{\mu}(\hat{\beta}) \), then solves the problem:

\[
\min_{\mu} \hat{g}^T(\mu, \hat{\beta}) \hat{W} \hat{g}(\mu, \hat{\beta}), \quad \text{subject to } Q \mu = 0
\]

where \( \hat{W} \) is a weighting matrix. The solution to Eq. (8) is:

\[
\hat{\mu}(\hat{\beta}) = \hat{A} \hat{\mu}^*(\hat{\beta})
\]

where \( \hat{A} = I - \hat{W}^{-1}Q'[Q \hat{W}^{-1}Q']^{-1}Q \) and \( \hat{\mu}^*(\hat{\beta}) = p^{-1} \sum_{t=1}^p m_{t+1}(\hat{\beta}) \) is the unrestricted GMM estimator of \( \mu \). The restricted GMM estimator of \( \theta(\beta) \) is \( \hat{\theta}_R(\hat{\beta}) = c \hat{\mu}(\hat{\beta}) \), where \( c = (-2, 2, 1, -1) \), while \( \hat{\theta}_c(\hat{\beta}) = c \hat{\mu}^*(\hat{\beta}) \) is the unrestricted GMM estimator.

Under the following assumptions, Proposition 1 below follows from an application of West (1996, Theorem 4.1), provides the limiting distributions of \( \sqrt{p} \hat{\theta}_R(\hat{\beta}) \) and \( \sqrt{p} \hat{\theta}_c(\hat{\beta}) \), and establishes the asymptotic efficiency of \( \hat{\theta}_c(\hat{\beta}) \) relative to \( \hat{\theta}_R(\hat{\beta}) \).

Assumption 2. \( \text{plim}_{p \to \infty} \hat{W} = W \), where \( W \) is positive definite.

Assumption 3. (a) Let \( \nabla m_t(\beta) = \partial m_t(\beta)/\partial \beta, s_{1j} = y_t - X_{1j} \beta_1, c_{1j} = X_{1j} \beta_1, \) and \( \|n\| \) be the Euclidean norm. Then, for \( j = 1, 2 \) and some \( d > 1 \), sup \( E[\|\nabla m_t(\beta)\|^d] \) is strong mixing, with mixing coefficients of \( \text{size} - 3d/(d - 1) \).

(c) For \( j = 1, 2 \), \( \text{vec}(\nabla m_t(\beta)) \) is a weighting matrix. The solution to Eqs. (6) and (7) is treated as a parameter to be estimated. The orthogonality condition for the GMM estimator of \( \mu \) is \( E[g_{t+1}(\mu, \beta)] = 0 \), where \( g_{t+1}(\mu, \beta) = m_{t+1}(\beta) - \mu \) and

\[
m_{t+1}(\beta) = [y_{1,t+1} X_{2,t+1} \beta_1, y_{1,t+1} X_{2,t+1} \beta_2, (X_{1,t+1} \beta_1)^2], (X_{2,t+1} \beta_2)^2]
\]

Thus, the feasible sample analog of \( E[g_{t+1}(\mu, \beta)] \) is

\[
\hat{g}(\mu, \hat{\beta}) = p^{-1} \sum_{t=1}^p g_{t+1}(\mu, \hat{\beta})
\]

The restricted GMM estimator of \( \mu, \hat{\mu}(\hat{\beta}) \), then solves the problem:

\[
\min_{\mu} \hat{g}^T(\mu, \hat{\beta}) \hat{W} \hat{g}(\mu, \hat{\beta}), \quad \text{subject to } Q \mu = 0
\]

where \( \hat{W} \) is a weighting matrix. The solution to Eq. (8) is:

\[
\hat{\mu}(\hat{\beta}) = \hat{A} \hat{\mu}^*(\hat{\beta})
\]

where \( \hat{A} = I - \hat{W}^{-1}Q'[Q \hat{W}^{-1}Q']^{-1}Q \) and \( \hat{\mu}^*(\hat{\beta}) = p^{-1} \sum_{t=1}^p m_{t+1}(\hat{\beta}) \) is the unrestricted GMM estimator of \( \mu \). The restricted GMM estimator of \( \theta(\beta) \) is \( \hat{\theta}_R(\hat{\beta}) = c \hat{\mu}(\hat{\beta}) \), where \( c = (-2, 2, 1, -1) \), while \( \hat{\theta}_c(\hat{\beta}) = c \hat{\mu}^*(\hat{\beta}) \) is the unrestricted GMM estimator.

Under the following assumptions, Proposition 1 below follows from an application of West (1996, Theorem 4.1), provides the limiting distributions of \( \sqrt{p} \hat{\theta}_R(\hat{\beta}) \) and \( \sqrt{p} \hat{\theta}_c(\hat{\beta}) \), and establishes the asymptotic efficiency of \( \hat{\theta}_c(\hat{\beta}) \) relative to \( \hat{\theta}_R(\hat{\beta}) \).

Assumption 3. (a) Let \( \nabla m_t(\beta) = \partial m_t(\beta)/\partial \beta, s_{1j} = y_t - X_{1j} \beta_1, c_{1j} = X_{1j} \beta_1, \) and \( \|n\| \) be the Euclidean norm. Then, for \( j = 1, 2 \) and some \( d > 1 \), sup \( E[\|\nabla m_t(\beta)\|^d] \) is strong mixing, with mixing coefficients of \( \text{size} - 3d/(d - 1) \).
optimal GMM theory to address the problem of a singular asymptotic covariance matrix.\footnote{14} They proposed a two-step GMM estimator that is asymptotically equivalent to the infeasible optimal (asymptotically efficient) GMM based on the generalized inverse of the singular covariance matrix. For the present problem, this would entail replacing Assumption 2 with the assumption that \( p \lim_{n \to \infty} \hat{\Theta}^+ = \Theta^+ \) for some estimator \( \hat{\Theta}^+ \) of the generalized inverse \( \Theta^+ \). However, it is not clear how such an estimator could be constructed for the present problem. Unlike ordinary inverses, generalized inverses are discontinuous. Consequently, as noted by Andrews (1987), consistency of \( \hat{\Theta} \) for \( \Theta \) does not generally ensure consistency of \( \hat{\Theta}^+ \) for \( \Theta^+ \). Andrews (1987, Theorem 2) provides an additional necessary and sufficient condition for the latter to hold. The condition is \(
abla \text{prob} \{ \text{rank}(\hat{\Theta}) = \text{rank}(\Theta) \} \to 1 \) as \( p \to \infty \). However, this condition is unlikely to be satisfied in the present case if \( \Theta \) is a standard covariance estimator, because \( \hat{\Theta} \) will generally be nonsingular if \( X_{t+1} \beta_{1t} \neq X_{t+1} \beta_{2t} \), which will hold with probability one for all finite values of \( p \) if the components of \( X_{t+1} \) or \( X_{t+1} \) are distributed continuously. We leave the problem of constructing consistent estimators for \( \Theta^+ \) for future research.

Yet another possible approach would be a two-step procedure in which the first step consists of testing the hypothesis that the models are overlapping against the alternative that they are non-nested. Such tests have been proposed by Clark and McCracken (2014), Marcellino and Rossi (2008) and Vuong (1989). If the test fails to reject the hypothesis of overlapping models, the procedure stops. If the hypothesis of overlapping models is rejected, the second step tests the hypothesis \( \theta(\beta) = 0 \) with Eq. (11) using an estimate of the optimal weight matrix for \( \hat{\theta}(\bar{\beta}) \).

\section*{4. Additional moment restrictions}

Under Assumption 1, \( E(X'_{t+1}(y_{t+1} - X_{t+1} \beta_1)) = 0 \), which implies that the product of \( y_{t+1} - X_{t+1} \beta_1 \) and any linear combination of the predictor variables has zero expectation. As a consequence, many different specifications of \( \mu \) and \( m_{t+1}(\beta) \) are possible. Thus far, our specification consists of the four cross-product moments in the restriction \( E[X_{t+1}(\beta)(y_{t+1} - X_{t+1} \beta_1)] = 0, j = 1, 2 \), giving:

\[
\mu = [E(y_{t+1}X_{t+1} \beta_1), E(y_{t+1}X_{t+1} \beta_2),
E([X_{t+1} \beta_1]^2), E([X_{t+1} \beta_2]^2)]' \]

\[
m_{t+1}(\beta) = [y_{t+1}X_{t+1} \beta_1, y_{t+1}X_{t+1} \beta_2, (X_{t+1} \beta_1)^2, (X_{t+1} \beta_2)^2]' \]

Six additional restrictions implied by \( E(X'_{t+1}(y_{t+1} - X_{t+1} \beta_1)) = 0 \) are \( E[X_{t+1}(\beta)(y_{t+1} - X_{t+1} \beta_1)] = 0 \) and

\footnote{13} Hanson and Timmerman (2015) demonstrate that, for nested models, Eq. (10) can be expressed as the difference between two conventional Wald statistics testing the hypothesis that the coefficient in the larger model is zero, one statistic based on the full sample and the other on a subsample. They show that this dilutes the local power over the conventional full-sample Wald test, and argue that this raises “serious questions” about the testing of the population-level accuracy for nested models using out-of-sample tests.
on the optimal weight matrix for two models: unrestricted GMM is equivalent to equation-by-equation (15)
momently. In all experiments, tests were conducted at the 10%

is straightforward: it is simply a matter of replacing \( \hat{\mu}(\beta) \)
using Eq. (14) does not change \( \hat{\mu}(\beta) \), since unrestricted GMM is equivalent to equation-by-equation OLS in the present setting. However, it does change the asymptotic covariance of \( \sqrt{P\hat{\mu}(\beta)} \), and consequently, the optimal weighting matrix.

3. Simulation experiments

3.1. Design

Simulation experiments were conducted to examine the finite–sample properties of the various estimators and tests. Samples were drawn from the following model:

\[
y_{t+1} = 0.3y_t + \delta_{x1t+1} + \delta_{x2t+1} + u_{t+1},
\]

where \( u_t \) and \( x_t \) are independent and serially uncorrelated, \( u_t \sim N(0, 10) \) and \( x_t \sim N(0, 0.5) \). The tests were evaluated for one-step-ahead (\( \tau = 1 \)) predictions from the two models:

\[
y_{t+1} = \beta_0 + \beta_1X_{1t} + \beta_2X_{2t} + u_{t+1}, \quad (17a)
\]

\[
y_{t+1} = \beta_3 + \beta_{22}X_{1t} + \beta_{22}X_{2t} + u_{t+1}, \quad (17b)
\]

which were estimated by OLS, and a recursive scheme to generate predictions.

In all experiments, tests were conducted at the 10% significance level and for 5000 replications. Experiments were run for \( P = 20, 50, 100, 200, \) with \( R = 2P \) and \( j = 2, 3, 4, 5 \). Samples were generated from Eq. (16) using a non-nested specification in which we set \( \delta_1 = -2 \) in all experiments. The expected value of the squared prediction error in this case equals \( 10 + 0.5\delta_2^2 \) for Eq. (17a) 12 for Eq. (17b). We set \( \delta_2 = -2 \) in the size experiments, and \( \delta_2 = -1 \) in the power experiments. In what follows, the DMW test based on a given estimator is denoted by DMW(“estimator”). The restricted GMM estimator based on the optimal weight matrix for \( j \) moments is denoted by \( \hat{\theta}_{4j}(\hat{\beta}) \), \( j = 4, 13 \), and computed from the two-step GMM estimator \( \hat{\mu}(\hat{\beta}) \) given by Eq. (9). The first-step estimator is OLS. The weight matrix is the inverse of a consistent estimate of the asymptotic covariance. The latter covariance estimator is serial–correlation robust, and uses a Bartlett kernel and the automatic lag–selection algorithm proposed by Newey and West (1994).

3.2. Relative efficiencies of the GMM estimators

We have argued that tests based on \( \hat{\theta}_{41}(\hat{\beta}) \) and \( \hat{\theta}_{413}(\hat{\beta}) \) may have greater power than those based on \( \hat{\theta}_{4}(\hat{\beta}) \) because of the asymptotic efficiency of restricted estimation. Before evaluating the tests, we first examine the efficiency issue by comparing the sample means and standard deviations of \( \hat{\theta}_{4}(\hat{\beta}) \), \( \hat{\theta}_{44}(\hat{\beta}) \) and \( \hat{\theta}_{413}(\hat{\beta}) \) for the samples used to evaluate the tests. Under the null hypothesis of equal mean squared prediction errors, the standard deviations and means are reported in Tables 1 and 2 respectively. In all cases, the sample means are much smaller than the standard deviations, suggesting that biasness is not a significant source of estimation error. Consistent with the asymptotic efficiency of restricted estimation, the standard deviations of \( \hat{\theta}_{413}(\hat{\beta}) \) and \( \hat{\theta}_{44}(\hat{\beta}) \) are smaller than those of \( \hat{\theta}_{4}(\hat{\beta}) \) for all \( P \) and \( R \). Also as expected, the standard deviations of \( \hat{\theta}_{44}(\hat{\beta}) \) are (slightly) larger than those of \( \hat{\theta}_{413}(\hat{\beta}) \). Averaging the percentages over \( P \), the standard deviation of \( \hat{\theta}_{413}(\hat{\beta}) \) is about 61% of the standard deviation of \( \hat{\theta}_{4}(\hat{\beta}) \) for \( P = 20, 59\% \) of that for \( P = 50 \) and \( P = 100 \), and 57% of that for \( P = 200 \). The efficiency gains generally increase as \( P/R \) decreases. The percentages range from 64% to 67% for \( P/R = 1/2 \), from 57% to 61% for \( P/R = 1/3 \), from 52% to 59% for \( P/R = 1/4 \), and from 52% to 55% for \( P/R = 1/5 \). The efficiency gains for \( \hat{\theta}_{44}(\hat{\beta}) \) over \( \hat{\theta}_{4}(\hat{\beta}) \) are only slightly smaller.

3.3. Size and power results

Next, we evaluate the size and power of the tests. For all tests, the estimator of the asymptotic covariance

\[
\begin{array}{cccc}
P & R   & \hat{\theta}_{41}(\hat{\beta}) & \hat{\theta}_{413}(\hat{\beta}) & \hat{\theta}_{44}(\hat{\beta}) \\
20 & 40   & 3.21 & 2.16 & 2.18 \\
20 & 60   & 3.18 & 1.93 & 1.94 \\
20 & 80   & 3.25 & 1.91 & 1.92 \\
20 & 100  & 3.04 & 1.78 & 1.79 \\
50 & 100  & 1.97 & 1.27 & 1.33 \\
50 & 150  & 1.86 & 1.15 & 1.22 \\
50 & 200  & 1.90 & 1.11 & 1.18 \\
50 & 250  & 1.89 & 1.03 & 1.09 \\
100 & 200 & 1.38 & 0.881 & 0.947 \\
100 & 300 & 1.34 & 0.809 & 0.888 \\
100 & 400 & 1.34 & 0.755 & 0.816 \\
100 & 500 & 1.33 & 0.726 & 0.793 \\
200 & 400 & 0.966 & 0.627 & 0.684 \\
200 & 600 & 0.954 & 0.547 & 0.618 \\
200 & 800 & 0.945 & 0.510 & 0.590 \\
200 & 1000 & 0.942 & 0.483 & 0.529 \\
\end{array}
\]
Table 2
Means under the null hypothesis.

<table>
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<th>$\theta_{13}(\hat{P})$</th>
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matrix $\mathbf{\Omega}$ follows the discussion below Proposition 1. The estimator uses a serial-correlation robust Newey-West estimator that is specified with a Bartlett kernel and the automatic lag-selection algorithm proposed by Newey and West (1994). It is important to recognize that the choice of a truncation lag can affect the size and power of the tests in finite samples. The advantage of the lag-selection algorithm is that the truncation lag is at least optimal in an asymptotic mean square error sense.⁴¹ As Proposition 1 shows, $\mathbf{\Omega}$ also depends on the limit of $P/R$, $\pi$. When $R$ is large relative to $P$, one might consider setting $\pi = 0$, which greatly simplifies the estimated asymptotic covariance. However, initial experiments indicated that $DMW(\hat{\theta}_{41}(\hat{P}))$ and $DMW(\hat{\theta}_{13}(\hat{P}))$ were greatly oversized for all $P$ and $R$ when $\pi = 0$ was imposed. For example, for the 10% nominal size under the null hypothesis, the rejection rates for $DMW(\hat{\theta}_{41}(\hat{P}))$ with $\pi = 0$ imposed ranged from an average of 30% for $P = 200$ to an average of 57% for $P = 20$. Although $\hat{\beta}$ is asymptotically irrelevant in the distributions of $\theta_{41}(\hat{P})$ and $\theta_{13}(\hat{P})$ if the true value of $\pi$ is zero, the distributions generally depend on the distribution of $\theta_{41}(\hat{P})$ in finite samples. Imposing $\pi = 0$ in finite samples neglects this dependence, meaning that the estimated asymptotic variance may be a poor approximation to the finite-sample variance. Consistent with this, the sizes of $DMW(\hat{\theta}_{41}(\hat{P}))$ and $DMW(\hat{\theta}_{13}(\hat{P}))$ improved considerably when $\pi = 0$ was not imposed. As discussed below, the rejection rates of $DMW(\hat{\theta}_{41}(\hat{P}))$ and $DMW(\hat{\theta}_{13}(\hat{P}))$ are typically within 2%–3% of the nominal size when $\pi = 0$ is not imposed. In contrast, the value of $\pi$ is not an issue for $DMW(\hat{\theta}_{4}(\hat{P}))$. As Eqs. (12) and (13) reveal, unlike $\theta_{41}(\hat{P})$ and $\theta_{13}(\hat{P})$, $\hat{\beta}$ is asymptotically irrelevant in the distribution of $\theta_{4}(\hat{P})$ for all values of $\pi$.

Table 3 reports the rejection rates under the null hypothesis for a nominal size of 10%. In line with previous studies, the standard form of the DM test, $DMW(\hat{\theta}_{4}(\hat{P}))$, performs well in terms of size. It is slightly oversized in most cases, but the rejection rates are within 2% of the nominal size for all cases in which $P \geq 50$ and within about 3% for $P = 20$. Clark and McCracken (2014, pp. 425–426) also found $DMW(\hat{\theta}_{4}(\hat{P}))$ to be slightly oversized for similar sample sizes. They conjecture that both $P$ and $R$ might have to be larger “for the asymptotics to kick in”.

In all but two cases, the rejection rates of $DMW(\hat{\theta}_{13}(\hat{P}))$ are within about 2% of the nominal size for $P \geq 50$ with $P/R \leq 1/3$. The rejection rates of $DMW(\hat{\theta}_{41}(\hat{P}))$ are somewhat better, as they are within 2% of the nominal size for $P \geq 20$ with $P/R \leq 1/3$ for 14 cases out of 15. Therefore, $DMW(\hat{\theta}_{41}(\hat{P}))$ performs about as well as $DMW(\hat{\theta}_{4}(\hat{P}))$ in these cases. However, one difference is that, whereas $DMW(\hat{\theta}_{13}(\hat{P}))$ tends to be slightly oversized, $DMW(\hat{\theta}_{13}(\hat{P}))$ and $DMW(\hat{\theta}_{13}(\hat{P}))$ tend to be undersized, and therefore are more conservative. It should also be noted that the rejection rate of $DMW(\hat{\theta}_{4}(\hat{P}))$ is generally less sensitive to $P/R$ than that of either $DMW(\hat{\theta}_{41}(\hat{P}))$ or $DMW(\hat{\theta}_{13}(\hat{P}))$. Again, this might be due to the fact that the asymptotic variance of $\hat{\theta}_{4}(\hat{P})$ does not depend on $\pi$.

Table 4 reports the rejection rates under the alternative hypothesis. In terms of power, the rejection rate of $DMW(\hat{\theta}_{4}(\hat{P}))$ is also less sensitive to the value of $P/R$ than the other tests are. For the case of $P \geq 50$ with $P/R \leq 1/3$, in which the tests have similar sizes, $DMW(\hat{\theta}_{41}(\hat{P}))$ and $DMW(\hat{\theta}_{13}(\hat{P}))$ are much more powerful than $DMW(\hat{\theta}_{4}(\hat{P}))$. For example, with $P/R = 1/5$, the rejection rate of $DMW(\hat{\theta}_{13}(\hat{P}))$ is about 2.3 times greater than that of $DMW(\hat{\theta}_{4}(\hat{P}))$ for $P = 50$, 2.1 times greater for $P = 100$ and 1.5 times greater for $P = 200$. As expected, $DMW(\hat{\theta}_{41}(\hat{P}))$ is less powerful than $DMW(\hat{\theta}_{13}(\hat{P}))$ but more powerful than $DMW(\hat{\theta}_{4}(\hat{P}))$.

4. Illustrative empirical application

Next, we illustrate the tests using a forecasting application to monthly US industrial production. The data were downloaded from the FRED website of the Federal Reserve Bank of St. Louis. One forecasting model is a regression of the growth rate of the industrial production index on a constant, one lag of the growth rate and two lags of the spread between Moody’s Aaa and Baa corporate bond yields. The other model replaces the credit spread with the log of housing starts for privately owned housing. The data are monthly from 1959:03 to 2015:08. We applied the tests to one-month-ahead forecasts for the period 2011:06–2015:08. The models were estimated recursively. Using 4999 bootstrap draws, we then applied the bootstrap test of Clark and McCracken (2014, p. 421) to determine whether the models are overlapping or non-nested. The null hypothesis that the models are overlapping can be rejected at the 5% level. For the null hypothesis of equal predictive accuracy, we obtain $DMW(\hat{\theta}_{4}(\hat{P})) = 1.47$, which is insignificant at the 10% level, but $DMW(\hat{\theta}_{13}(\hat{P})) = 3.025$ and $DMW(\hat{\theta}_{41}(\hat{P})) = 2.162$, which are significant at the 1% and 5% levels, respectively.

⁴¹ For a summary of the parameters used in the lag-selection algorithm, see Table 1 of Newey and West (1994).
Table 3
Rejection rates for a nominal size of 10% under the null hypothesis.

<table>
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<tr>
<th>P</th>
<th>R</th>
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<th>DMW(\hat{\theta}_{115}(\hat{\beta}))</th>
<th>DMW(\hat{\theta}_{146}(\hat{\beta}))</th>
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Table 4
Rejection rates for a nominal size of 10% under the alternative hypothesis.

<table>
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<tr>
<th>P</th>
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<th>DMW(\hat{\theta}_1(\hat{\beta}))</th>
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5. Conclusion

The test developed by Diebold and Mariano (1995) and West (1996) is regarded widely as being an important test that addresses the need to formally assess whether differences in the accuracies of two predictors are purely sampling error. Using a GMM framework, we proposed a more powerful version that can be used to compare the accuracies of regression predictors based on non-nested models. One advantage of the version proposed by Diebold and Mariano (1995) is that it circumvents the problem of imposing assumptions on the predictors by imposing assumptions on the prediction errors directly. However, as we demonstrated, this comes at a cost in terms of power. Specifically, we showed that more powerful versions of the tests can be obtained by exploiting the properties of linear projections that only require assumptions as to the existence of certain moments and covariance-stationarity. The simulation experiments illustrate that the potential gains in power can be considerable. Possible directions for future research include extensions to the hypothesis of finite-sample (as opposed to population-level) predictive accuracy, and the estimation of the optimal weight matrix when the covariance matrix is singular for tests of overlapping and nested models.

Acknowledgments

The authors would like to thank the editor Michael McCracken, the associate editor, and an anonymous referee for helpful comments.

Appendix

Part (i) of Proposition 1 can be proved by showing that Assumptions 1–4 of West (1996) hold for \( m_t(\beta) \). The result then follows directly from Theorem 4.1 of West (1996), which also assumes a recursive forecasting scheme. As was noted above, our Assumptions 3 and 4 are Assumptions 3 and 4 of West (1996) with different notation. Clearly, \( m_t(\beta) \) is measurable and twice continuously differentiable, with second-order derivatives that do not depend on \( \beta \). Under our Assumption 1, the second-order derivatives have finite expectations. Thus, Assumption 1 of West (1996) holds. Next, note that under our Assumption 1 \( \tilde{\beta}_i - \beta = B(t)H(t) \), where \( B(t) = \text{diag}(B_1(t), B_2(t)) \), \( H(t) = \text{diag}(H_1(t), H_2(t)) \), and
\[ H_1(t')^\prime H_2(t') = t^1 \sum_{i=1}^n h_i(\bar{\theta}) \], \quad B = p \lim_{t \to \infty} B(t), \quad B \text{ has full column rank, and } Eh_i(\bar{\theta}) = 0. \] Therefore, Assumption 2 of West (1996) holds. Consequently, part (i) follows from West (1996, Theorem 4.1).

Part (ii) follows from part (i) by noting that \( \hat{\theta} \) converges in probability to \( \theta \) under Assumption 2 and \( \sqrt{p}(\hat{\theta}(\hat{\theta}) - \theta(\theta)) = cA\sqrt{p}A'c \), since \( Q_{\mu} = 0 \) under Assumption 1.

Part (iii) follows from part (i) by noting that \( \sqrt{p}(\hat{\theta}(\hat{\theta}) - \theta(\theta)) = c\sqrt{p}A'(\hat{\theta} - \mu) \). Part (iv) then follows from substituting \( W = \Omega^{-1} \) into \( A \) and multiplying out \( cA\Omega A'c \).

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