Correlations in Bivariate Pareto Distributions

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ABSTRACT
Bivariate Pareto distributions have proven very useful in modeling lifetime data, hydrology, competing risk data, and many other datasets. In this paper, we explore Kendall tau and Gini correlations in the bivariate Pareto distributions, comparing with the popular Pearson correlation and robust quadrant correlation. It is interesting to establish the fact that zero of those correlations mutually imply independence in this family. We derive the variance of the asymptotic normality for each sample correlation via the influence function approach. When the second moment is finite, we demonstrate that the symmetric Gini correlation is asymptotically efficient as well as relatively efficient among finite samples. However, Kendall tau is more appealing in terms of compromising efficiency and robustness.

KEYWORDS
Dependence; correlations; asymptotical efficiency; finite sample efficiency; influence function; heavy-tailed; robustness

1. Introduction

The Pareto distribution is a skewed and heavy-tailed distribution. It's named for Italian economist Vilfredo Pareto, who noticed the economic phenomenon that 80% of his country's wealth was distributed among 20% of its people. The Pareto distribution is used often in economics to model the distribution of incomes, city populations, and many other non-negative socio-economic issues. The Pareto distribution family has since been expanded to include many different types of Pareto distributions. The Pareto distribution of the second kind (also known as the Lomax distribution) is used frequently to model lifetime data in many applied sciences. As Lindley and Singpurwalla (1986, [11]) pointed out, using the univariate Pareto model to measure two lifetimes of components of a system is often inaccurate due to the influence of the common environment and its impact on the two components' correlations. For this reason, they introduce the Lindley-Singpurwalla bivariate Pareto distribution (LSBP). In this paper, our goal is to study correlations in a generalized bivariate Pareto distribution (SNBP), introduced by Sankaran and Nair (1993, [16]) that contains the Lindley-Singpurwalla bivariate Pareto as a special case. There are many applications of this bivariate Pareto distribution. For example, Nadarajah (2009, [12]) used a distribution equivalent to the LSBP as a model for drought. Sankaran and Kundu (2014, [15]) proposed using the SNBP distribution to model life tests of appliances, and also

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suggested this distribution is an effective model of competing risk data. Rootzén and Tajvidi (2006, [14]) proposed multivariate generalized Pareto distributions for modeling extreme values.

When using a bivariate model as a joint distribution of two components, it is often useful to measure the correlation between the two components. Correlation coefficients are an important tool used in many fields to measure the association between two variables. While there are many proposed measures of correlation, the most widely known is probably the Pearson correlation. For the Normal distribution, the Pearson correlation is known to be the most statistically efficient measure of correlation. The drawback of the Pearson correlation coefficient is its high sensitivity to outliers. We look at influence functions as a measure of the correlation's robustness [7]. The influence function for the Pearson correlation is unbounded, indicating the lack of robustness of this measure [5]. Also, the Pearson correlation requires the second moment assumption and hence it may not even exist for some Pareto distributions [6].

Without any moment assumption, the Kendall tau is a rank-based measure of association, useful for measuring monotonic relationships. It has a bounded influence function [4], making it more robust than the Pearson correlation coefficient. Another robust measure of correlation we consider is the quadrant correlation. This correlation involves centering around a coordinate-wise median and then it is defined as the difference of the probability in the first and third quadrant and the probability in the second or fourth quadrant. The influence function of the quadrant correlation is bounded, but it not smooth [22]. So it is more robust than the Pearson correlation, but not as robust as the Kendall tau.

The final correlations we consider are the Gini correlations. The Gini correlation is highly related to Gini mean difference and the Gini coefficient. One can find the Gini coefficient for a Pareto distribution by calculating the Lorenz curve for the Pareto distribution and doubling the area from the Lorenz curve to the equidistribution line. Arnold (1983, [1]) used study of the Gini index to show that the Pareto distribution of the second type (or Lomax), with shape parameter θ, is an accurate model of incomes with a wide gap by showing that if $\theta > 1$, the Gini index is greater than 0.5. The standard Gini covariances and correlations were introduced by [3] as natural bivariate extensions of the univariate Gini mean difference. The Gini correlations are considered a statistical compromise between the Pearson correlation and the Spearman correlation, as they are based on covariance of one variable and the rank of the other. Hence, there are two Gini correlations for each pair of random variables. The Gini correlations are very useful in measuring association between variables from heavy-tailed distributions [25]; since they share properties of both the Pearson correlation and the Spearman correlation, they balance robustness and efficiency. Unless the bivariate distribution is exchangeable up to a linear transformation, the Gini correlations are generally not symmetric in two variables X_1 and X_2 ([20], [21]), i.e., $\gamma(X_1, X_2) \neq \gamma(X_2, X_1)$. To combat the problem of asymmetry, Sang, Dang and Sang (2016, [17]) proposed a symmetric Gini correlation based on a joint rank. The other two symmetric Gini correlations were proposed by Yitzhaki and Olkin (1991, [24]): the arithmetic mean and the geometric mean, respectively, $(\gamma(X_1, X_2) + \gamma(X_2, X_1))/2$ and $\sqrt{|\gamma(X_1, X_2)\gamma(X_2, X_1)|}$. Vanderford *et al.* (2020, [23]) studied these correlations in detail. It is interesting to find that for the elliptical distributions in which two traditional Gini correlations and the average symmetric one are all equal, the sample symmetric Gini correlation is more statistically efficient than the traditional counterparts [23]. In the heavy-tailed Pareto distributions, we also have two traditional Gini correlations equal. It is worthwhile to explore their statistical efficiencies. In other words, we would like to know whether the symmetric versions have some statistical advantages over the traditional Gini correlations. We focus on the symmetric Gini that the average of the two traditional Gini's. We denote it as r_g.

We provide explicit formula for the Gini, Pearson, Kendall tau, and quadrant correlations in the SNBP distribution (of which the formula for the Kendall tau and Gini correlations appear to have never been published before). We are also able to use influence functions previously established for each correlation to give information about the asymptotic distributions of the correlations and calculate asymptotic variances where possible. We compare the asymptotic efficiencies by considering Fisher consistent correlations under the LSBP distribution. We also look at efficiencies of each estimator in a finite sample. Finally, we look at real data shown to fit the SNBP distribution and calculate each correlation coefficient in order to demonstrate a situation in which the proposed formulas are useful to determine the amount of correlation between variables in an SNBP distribution.

2. The SNBP distribution

There are several varieties and types of Pareto distributions; this paper will focus on one bivariate Pareto distribution in particular, what we call the SNBP distribution introduced by Sankaran and Nair (1993, [16]). This distribution is also called the bivariate Lomax distribution. As in the univariate case, the bivariate Lomax distribution has the Gamma-exponential connection, meaning that it can be derived either from two independent exponential random variables with parameters jointly from Kibble's bivariate Gamma distribution or from Gumbel's bivariate distribution with parameter from a Gamma distribution [2]. It has a simple distribution function, but is flexible enough to include many interesting and useful cases.

More specifically, its joint survival function and cumulative distribution function for $(X_1, X_2)^T$ from SNBP$(\alpha_1, \alpha_2, \delta, \theta)$ respectively are

$$S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2 + \delta\alpha_1\alpha_2 x_1 x_2)^{-\theta}, \tag{1}$$

$$H(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = 1 - (1 + \alpha_1 x_1)^{-\theta} - (1 + \alpha_2 x_2)^{-\theta} + S(x_1, x_2), \tag{2}$$

for $x_1 > 0, x_2 > 0$, where $\theta > 0, \alpha_1 > 0, \alpha_2 > 0$ and $0 \le \delta \le \theta + 1$. Among the parameters, α_1 and α_2 are scale parameters, while θ is the shape parameter that determines the tail heaviness of the distribution. A smaller θ value indicates a heavier tail.

The SNBP distribution has Pareto II (or Lomax) marginals. That is, X_i $(i = 1, 2)$ has a Pareto II distribution with distribution function of the form

$$F_i(x_i) = P(X_i \le x_i) = 1 - S_i(x_i) = 1 - (1 + \alpha_i x_i)^{-\theta},$$

with

$$\mathbb{E}[X_i] = \frac{1}{\alpha_i(\theta - 1)}, \text{ if } \theta > 1,$$

$$Var(X_i) = \frac{\theta}{\alpha_i^2(\theta - 1)^2(\theta - 2)}, \text{ if } \theta > 2.$$

Also for $\theta > 1$, the Gini mean difference (GMD) is

$$\Delta(X_i) = \mathbb{E}|X_i - X_i'| = \frac{2\theta}{\alpha_i(2\theta-1)(\theta-1)}, \tag{3}$$

where X_i' is an independent copy of X_i. Hence for $\theta > 1$, the Gini index, defined as the ratio of the GMD and twice the mean, is $\theta/(2\theta - 1)$, which is greater than 0.5 and approaches 1 as $\theta \to 1$.

Before we study various correlation measures, let us look at the parameter δ closely. The parameter, δ, determines the dependence properties of the distribution. If $\delta = 1$, we have $H(x_1, x_2) = F_1(x_1)F_2(x_2)$ and hence X_1 and X_2 are independent. If $\delta > 1$, H is negative quadrant dependent [8]. That is,

$$P(X_1 \le x_1, X_2 \le x_2) < P(X_1 \le x_1)P(X_2 \le x_2), \quad \text{for all } x_1 > 0, x_2 > 0, \tag{4}$$

or equivalently,

$$P(X_1 > x_1, X_2 > x_2) < P(X_1 > x_1)P(X_2 > x_2), \quad \text{for all } x_1 > 0, x_2 > 0. \tag{5}$$

This is a negative dependence concept because X_1 and X_2 are less likely to be small together or to be large together compared with independent X_1' and X_2', where X_i' has the same distribution of X_i, for $i = 1, 2$. Similarly, positive quadrant dependence can be defined if the inequalities in (4) and (5) are reversed. Clearly, SNBP is positive quadrant dependent when $0 \le \delta < 1$. For the case of $\delta = 0$, we see that the joint pdf is

$$f_{X_1,X_2}(x_1, x_2) = \frac{\theta(\theta+1)\alpha_1\alpha_2}{[1 + \alpha_1 x_1 + \alpha_2 x_2]^{\theta+2}},$$

which corresponds to the bivariate Pareto distribution presented by Lindley and Singpurwalla (1986, [11]), denoted LSBP throughout the remainder of this paper.

3. Correlations in SNBP

We study the Gini, Pearson, Kendall tau, and quadrant correlations in the SNBP distribution. Since all correlations are invariant under scale changes, they do not depend on the scale parameters α_1 and α_2. Without loss of generality, we assume $\alpha_1 = \alpha_2 = 1$. Below, we provide each correlation in the SNBP distribution.

3.1. Pearson correlation

For the SNBP distribution with $\theta > 2$, an explicit form for the Pearson correlation is given by Lai, et. al (2001, [10]) as

$$r_p = \frac{(1-\delta)(\theta-2)}{\theta^2} F(1, 2; \theta+1; 1-\delta), \tag{6}$$

where $F(a, b : c, z)$ is the Hypergeometric function.

Under the LSBP distribution with $\theta > 2$, the Pearson correlation is simplified to be

$$r_p = \frac{1}{\theta}. \tag{7}$$

Properties of the Pearson correlation in SNBP are listed below as remarks. Some of them are provided by [2].

Remark 1. The Pearson correlation is postive for $0 \leq \delta < 1$, negative for $1 < \delta \leq \theta+1$ and 0 for $\delta = 1$.

Remark 2. The Pearson correlation approaches 0 when θ goes to ∞ for each δ.

Remark 3. For a given $\theta > 2$, the Pearson correlation is decreasing in δ.

Remark 4. For any given $\theta > 2$, $r_p = 0$ mutually implies $\delta = 1$, i.e, X_1 and X_2 are independent. This is because $F(1, 2; \theta + 1; 1 - \delta) > 0$.

3.2. Gini correlations

Gini mean difference Δ, as an alternative to standard deviation, is the expected distance of two independent random variables from the distribution. It can be represented as 4 times the covariance between the variable and its cumulative distribution. A natural bivariate extension, Gini covariance is defined as 4 times the covariance between one variable and the cumulative distribution of the other. The Gini correlations are just normalized Gini covariances and defined as

$$\begin{aligned}
\gamma_1 = \gamma(X_1, X_2) &= \frac{cov(X_1, F_2(X_2))}{cov(X_1, F_1(X_1))} = \frac{cov(X_1, F_2(X_2))}{\Delta(X_1)/4} \\
\gamma_2 = \gamma(X_2, X_1) &= \frac{cov(X_2, F_1(X_1))}{cov(X_2, F_2(X_2))} = \frac{cov(X_2, F_1(X_1))}{\Delta(X_2)/4}.
\end{aligned}$$

The representation of the Gini correlation $\gamma(X_1, X_2)$ indicates that it has mixed properties of those of the Pearson and Spearman correlations. It is similar to Pearson in X_1 (the variable taken in its values) and similar to Spearman in X_2 (the variable taken in its ranks). Further the Pearson r_p, Spearman r_s, and Gini γ_1 can be written as

$$\begin{aligned}
r_p(X_1, X_2) &= \frac{\int\int H(x_1, x_2) - F_1(x_1)F_2(x_2)dx_1dx_2}{\sigma_1\sigma_2}, \\
r_s(X_1, X_2) &= 12 \int\int H(x_1, x_2) - F_1(x_1)F_2(x_2)dF_1(x_1)dF_2(x_2), \\
\gamma(X_1, X_2) &= \frac{\int\int H(x_1, x_2) - F_1(x_1)F_2(x_2)dx_1dF_2(x_2)}{cov(X_1, F_1(X_1))}.
\end{aligned} \tag{8}$$

It is clear that $\gamma(X_1, X_2)$ is invariant under all increasing transformation in X_2 and is invariant under changes of scale and location in X_1. Also $\gamma_1 = \gamma_2$ if (X_1, X_2) is exchangeable up to a linear transformation, meaning that $(aX_1 + b, cX_2 + d)$ has the same distribution as (X_2, X_1) for some constants a, b, c, d with a and $c > 0$. Details can be found in Yitzhaki and Schechtman (2013, [25]).

Under the SNBP distribution, it is easy to check that $\gamma_1 = \gamma_2$ since the joint survival function in (1) is symmetric in $\alpha_1 x_1$ and $\alpha_2 x_2$. Hence, the symmetric Gini correlation $r_g = \gamma_1 = \gamma_2$. The following theorem provides the formula for the Gini correlations in the SBNP distribution.

Theorem 3.1. *For the SNBP distributions with $\theta > 1$, the Gini correlations are equal and equal to*

$$\gamma_1 = \gamma_2 = \frac{(1-\delta)(2\theta-1)}{\theta(2\theta+1)} F(1,2;2\theta+2;1-\delta),$$

where $F(a,b;c;z)$ is the hypergeometric function.

When $\delta = 0$, by the result of $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, we immediately have the following corollary.

Corollary 3.2. *For the LSBP distributions with $\theta > 1$, the Gini correlations are*

$$\gamma_1 = \gamma_2 = \frac{1}{\theta}.$$

From the above corollary and (7), the Gini correlations and Pearson correlation are all equal to $1/\theta$, however r_p is bounded above by $1/2$ while the γ's are bounded above by 1.
Proof of Theorem 3.1. Using the result of (8), we have

$$\begin{aligned}
cov(X_1, F_2(X_2)) &= \int\int H(x_1,x_2) - F_1(x_1)F_2(x_2)dx_1 dF_2(x_2) \\
&= \int_0^\infty \int_0^\infty \frac{1}{(1+x_1+x_2+\delta x_1 x_2)^\theta} - \frac{1}{(1+x_1)^\theta}\frac{1}{(1+x_2)^\theta}dx_1 \frac{\theta}{(1+x_2)^{\theta+1}}dx_2 \qquad (9) \\
&= \frac{1}{\theta-1}\int_0^\infty \left(\frac{1}{(1+\delta x_2)(1+x_2)^{\theta-1}} - \frac{1}{(1+x_2)^\theta}\right)\frac{\theta}{(1+x_2)^{\theta+1}}dx_2 \\
&= \frac{\theta(1-\delta)}{\theta-1}\int_0^\infty \frac{x_2}{(1+x_2)^{2\theta+1}(1+\delta x_2)}dx_2 \\
&= \frac{\theta(1-\delta)}{\theta-1}\int_0^1 \frac{y(1-y)^{2\theta-1}}{1-(1-\delta)y}dy \qquad (10) \\
&= \frac{\theta(1-\delta)}{\theta-1}\frac{\Gamma(2)\Gamma(2\theta)}{\Gamma(2\theta+2)}F(1,2;2\theta+2;1-\delta) \qquad (11) \\
&= \frac{1-\delta}{2(\theta-1)(2\theta+1)}F(1,2;2\theta+2;1-\delta)
\end{aligned}$$

Equation (10) is due to the substitution of $y = x_2/(1+x_2)$. Equation (11) is obtained by using the integral representation of the hypergeometric function. See page 388 of the NIST Handbook of Mathematical Functions [13]. Utilizing (3) with $\alpha_1 = 1$ and the fact that $cov(X_1, F_1(X_1)) = \Delta(X_1)/4$, we complete the proof. □

Several conclusions can be drawn about Gini correlations in SNBP distributions and are stated as remarks below.

Remark 5. The Gini correlations are positive for $0 \le \delta < 1$, negative for $1 < \delta < \theta+1$ and 0 for $\delta = 1$ by (9).

Remark 6. The Gini correlations approach 0 when θ goes to ∞ since the hypergeometric function $F(1,2;2\theta+2,1-\delta)$ is absolutely convergent.

Remark 7. For a given $\theta > 1$, the Gini correlations are decreasing in δ.

Proof of Remark 7. Let us derive the derivative of γ with respective to δ.

$$\begin{aligned}
\frac{\partial \gamma}{\partial \delta} &= \frac{2\theta-1}{\theta(2\theta+1)}\left(-F(1,2;2\theta+2;1-\delta) + (1-\delta)\frac{\partial F(1,2;2\theta+2,1-\delta)}{\partial \delta}\right)\\
&= \frac{2\theta-1}{\theta(2\theta+1)}\left(-2\theta(2\theta+1)\int_0^1 \frac{t(1-t)^{2\theta-1}}{1-(1-\delta)t}dt - (1-\delta)2\theta(2\theta+1)\int_0^1 \frac{t^2(1-t)^{2\theta-1}}{(1-(1-\delta)t)^2}dt\right)\\
&= -2\theta(2\theta-1)\int_0^1 \frac{t(1-t)^{2\theta-1}}{1-(1-\delta)t}\left(1+\frac{(1-\delta)t}{1-(1-\delta)t}\right)dt\\
&= -2\theta(2\theta-1)\int_0^1 \frac{t(1-t)^{2\theta-1}}{(1-(1-\delta)t)^2}dt < 0.
\end{aligned}$$

This completes the proof. □

Remark 8. For any given $\theta > 1$, $\gamma_1 = \gamma_2 = 0$ mutually implies $\delta = 1$, i.e, X_1 and X_2 are independent. This is because $F(1,2;2\theta+2;1-\delta) > 0$ for any $\delta \geq 0$.

3.3. Kendall tau correlation

Two independent pairs of variables (X_1, X_2) and (X_1', X_2') from the distribution H are concordant if $(X_1 - X_1')(X_2 - X_2') \geq 0$ and discordant otherwise. The Kendall tau correlation is defined as

$$\begin{aligned}
\tau &= P[(X_1 - X_1')(X_2 - X_2') \geq 0] - P[(X_1 - X_1')(X_2 - X_2') < 0]\\
&= 2P[(X_1 - X_1')(X_2 - X_2') \geq 0] - 1,
\end{aligned}$$

the difference of probabilities of concordant pairs and discordant pairs. Since $P(X_1' \leq X_1, X_2' \leq X_2) = \int H dH$, we have, for continuous distribution H,

$$\tau = 4\int H(x_1,x_2)dH(x_1,x_2) - 1 = 4\mathbb{E}H(X_1,X_2) - 1. \tag{12}$$

Under the SNBP distribution, we have the following theorem.

Theorem 3.3. *The Kendall tau correlation, under the SNBP distribution, is*

$$\tau = \frac{2\theta(1-\delta)}{(2\theta+1)^2}F(1,1;2\theta+2;1-\delta), \tag{13}$$

where $F(a,b:c,z)$ is the hypergeometric function.

With $\delta = 0$, we can immediately obtain the following corollary.

Corollary 3.4. *Under the LSBP distribution, the Kendall tau correlation is*

$$\tau = \frac{1}{1+2\theta}. \tag{14}$$

Proof of Theorem 3.3. From (12), we have

$$
\begin{aligned}
\tau &= 4\int H(x_1,x_2)dH(x_1,x_2) - 1 \\
&= 4\int_0^\infty\int_0^\infty \frac{1}{(1+x_1+x_2+\delta x_1x_2)^\theta}\frac{\theta[\theta(1+\delta x_1)(1+\delta x_2)+1-\delta]}{(1+x_1+x_2+\delta x_1x_2)^{\theta+2}}dx_1dx_2 - 1 \\
&= -1+4\int_0^\infty\int_0^\infty \left(\frac{\theta[\theta(1+\delta x_2)+1-\delta]}{(1+x_1+x_2+\delta x_1x_2)^{2\theta+2}} + \frac{\theta^2\delta(1+\delta x_2)x_1}{(1+x_1+x_2+\delta x_1x_2)^{2\theta+2}}\right)dx_1dx_2 \\
&= -1+4\int_0^\infty \left(\frac{\theta[\theta(1+\delta x_2)+1-\delta]}{(2\theta+1)(1+\delta x_2)(1+x_2)^{2\theta+1}} + \frac{\theta\delta}{2(1+\delta x_2)(2\theta+1)(1+x_2)^{2\theta}}\right)dx_2 \\
&= -1+4\int_0^\infty \frac{\theta}{2(1+x_2)^{2\theta+1}} + \frac{\theta(1-\delta)}{2(2\theta+1)(1+\delta x_2)(1+x_2)^{2\theta+1}}dx_2 \\
&= -1+4\left(\frac{1}{4}+\frac{\theta(1-\delta)}{2(2\theta+1)}\int_0^\infty \frac{1}{(1+\delta x_2)(1+x_2)^{(2\theta+1)}}dx_2\right) \qquad (15) \\
&= \frac{2\theta(1-\delta)}{2\theta+1}\int_0^1 \frac{(1-y)^{2\theta}}{1-(1-\delta)y}dy \\
&= \frac{2\theta(1-\delta)}{(2\theta+1)^2}F(1,1;2\theta+2;1-\delta)
\end{aligned}
$$

Similar to the Gini correlation, Kendall tau also has the following properties.

Remark 9. The Kendall tau is positive for $0 \le \delta < 1$, negative for $1 < \delta < \theta+1$ and 0 for $\delta = 1$ by (15).

Remark 10. The Kendall tau correlation approaches 0 when θ goes to ∞ since the hypergeometric function $F(1,1;2\theta+2,1-\delta)$ is absolutely convergent.

Remark 11. The Kendall tau correlation is decreasing in δ for any $\theta > 0$ since $\partial\tau/\partial\delta < 0$.

Remark 12. For any SNBP distribution, $\tau = 0$ mutually implies $\delta = 1$, i.e, X_1 and X_2 are independent. This is because $F(1,1;2\theta+2;1-\delta) > 0$ for any $\delta \ge 0$.

3.4. Quadrant correlation

Rather than considering concordance between two independent pairs like in the Kendall tau, the quadrant correlation, r_Q, examines concordance of the random variables (X_1,X_2) with respect to the "center" of the distribution. More specifically,

$$
\begin{aligned}
r_Q &= 2P((X_1 - Med(X_1))(X_2 - Med(X_2) \ge 0) - 1 \\
&= 4H(Med(X_1), Med(X_2)) - 1 = 4[H(F_1^{-1}(1/2), F_2^{-1}(1/2)) - 1/4], \qquad (16)
\end{aligned}
$$

where $F_i^{-1}(q) = \inf\{x : F_i(x) \ge q\}$ for $q \in [0,1]$, the quantile function of X_i for $i = 1,2$. If X_1 and X_2 are independent, $H(F_1^{-1}(1/2), F_2^{-1}(1/2)) = F_1(F_1^{-1}(1/2))F_2(F_2^{-1}(1/2)) = 1/4$. Hence by (16), the quadrant correlation can be interpreted as 4 times the difference between the joint distribution function, H, and the product of marginal distributions, F_1F_2, evaluated at the marginal medians. In this sense, the quadrant correlation is also called medial coefficient. Using the copula function $C(u,v) = H(F_1^{-1}(u), F_2^{-1}(v))$, the quadrant correlation is

$$r_Q = 4C(1/2, 1/2) - 1.$$

For the SNBP distributions, Sankaran and Kundu (2014, [15]) provided the copula function to be

$$C(u,v) = u+v-1+[(1-u)^{-1/\theta}+(1-v)^{-1/\theta}-1+\delta((1-u)^{-1/\theta}-1)((1-v)^{-1/\theta}-1)]^{-\theta}.$$

Hence the quadrant correlation of the SNBP distribution is

$$r_Q = 4(2^{1+1/\theta} - 1 + \delta(2^{1/\theta} - 1)^2)^{-\theta} - 1 = \frac{4}{[\delta 2^{2/\theta} + 2(1-\delta)2^{1/\theta} - (1-\delta)]^\theta} - 1. \quad (17)$$

With $\delta = 0$, the quadrant correlation of LSBP distribution is $4(2^{1+1/\theta} - 1)^{-\theta} - 1$, which approaches 1 when θ goes to 0. It is easy to check the following remark.

Remark 13. The quadrant correlation, r_Q, in the SNBP distributions is strictly decreasing in δ for each $\theta \geq 0$.

This is because

$$\frac{\partial r_Q(\delta,\theta)}{\partial \delta} = -4\theta((2^{1+1/\theta} - 1 + \delta(2^{1/\theta} - 1)^2)^{-\theta-1}(2^{1/\theta} - 1)^2 < 0.$$

Combining this decreasing property with the result $r_Q = 0$ in the independence case when $\delta = 1$, we have the following result.

Remark 14. Under an SNBP distribution, $r_Q > 0$ if $0 \leq \delta < 1$ and $r_Q < 0$ if $1 < \delta \leq 1 + \theta$.

Remark 15. The quadrant correlation, r_Q, approaches 0 as θ goes to ∞.

Proof of Remark 15. Let us first establish a general result on

$$\lim_{n\to\infty} (ax^{2/n} + bx^{1/n} + c)^n = x^{(2a+b)/(a+b+c)}, \quad (18)$$

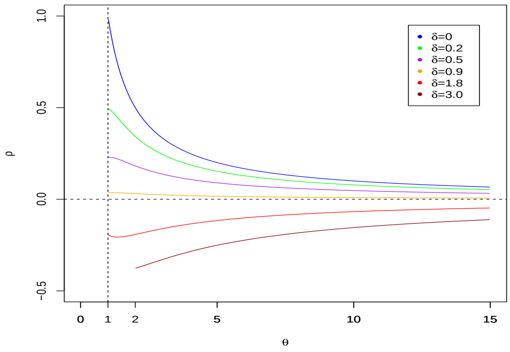
This can be proven by letting $\lim_{n\to\infty}(ax^{2/n} + bx^{1/n} + c)^n = y$ and

$$\begin{aligned}\ln y &= \lim_{n\to\infty} n \ln(ax^{2/n} + bx^{1/n} + c) = \lim_{n\to\infty} \frac{\ln(ax^{2/n} + bx^{1/n} + c)}{1/n} \\ &= \lim_{n\to\infty} \frac{ax^{2/n}(-2/n^2)\ln x + bx^{1/n}(-1/n^2)\ln x}{(-1/n^2)(ax^{2/n} + bx^{1/n} + c)} = \frac{2a+b}{a+b+c}\ln x.\end{aligned}$$

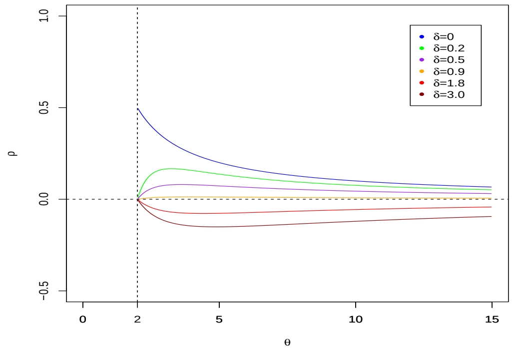
Plugging $a = \delta$, $b = 2(1-\delta)$, $c = -(1-\delta)$ and $x = 2$ in (18), we have

$$\lim_{\theta\to\infty} r_Q(\delta,\theta) = \frac{4}{\lim_{\theta\to\infty}[\delta 2^{2/\theta} + 2(1-\delta)2^{1/\theta} - (1-\delta)]^\theta} - 1 = \frac{4}{4} - 1 = 0. \quad \square$$

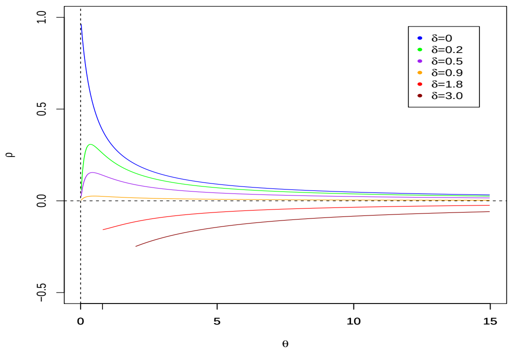
Remark 16. For any SNBP distribution, $r_Q = 0$ mutually implies $\delta = 1$, that is, X_1 and X_2 are independent. This result can be easily obtained from (17).



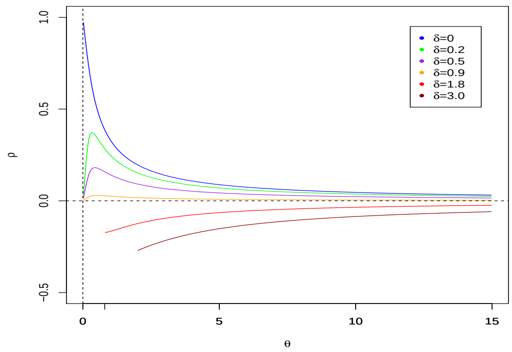
(a) Gini



(b) Pearson



(c) Kendall



(d) Quadrant

Figure 1. Plots of Gini, Pearson, Kendall, and Quadrant correlations as a function of θ under the SNBP distribution with different values of δ.

3.5. Correlation Plots

To visualize properties of each correlation measure, we plot Pearson, Gini, Kendall tau and quadrant correlations against the parameter θ under different values of $\delta = 0, 0.2, 0.5, 0.9, 1.8$ and 3.

Note that since the Gini correlation requires the existence of the first moment, θ must be greater than 1. For the Pearson correlation, the second moment is required, and thus we must restrict θ to be greater than 2. Also since $\delta \le 1 + \theta$ is required for the SNBP distribution, θ must be at least 2 for $\delta = 3.0$ and at least 0.8 for $\delta = 1.8$ as shown in the Kendall tau and quadrant correlation plots.

From the plots in Figure 1, we see that each correlation is positive for $\delta = 0, 0.2, 0.5, 0.9$ and negative for $\delta = 1.8, 3.0$, which demonstrates the results in Remarks 1, 5, 9, 14.

Each plot also clearly shows each correlation decreasing in δ for each θ value. In other words, the curve with a larger δ is below the curve with a smaller δ value. Hence, for each θ value, the largest correlations are achieved in the LSBP distribution, which are represented by the blue curve in the plots. For the LSBP distribution, although both the Gini correlation and Pearson correlation are equal to $1/\theta$, the Gini correlation can approach 1 while the Pearson correlation is bounded above by 0.5 due to the different moment requirements. From the plots, we see that the Kendall tau and quadrant correlation have an upper bound of 1 when θ goes to 0.

Although all correlations are strictly decreasing in θ for $\delta = 0$, this decreasing property does not hold uniformly for other values of δ. For $\delta = 0.2, 0.5, 0.9$, we see a short increase followed by a decrease as θ values increase. For example, we are able to find that the maximum quadrant correlation when $\delta = 0.2$ is $r_Q = 0.3717$ and occurs at $\theta = 0.3372$. The maximum quadrant correlation when $\delta = 0.5$ is $r_Q = 0.1807$ and occurs at $\theta = 0.4406$, and the maximum quadrant correlation when $\delta = 0.9$ is $r_Q = 0.0294$ and occurs at $\theta = 0.5325$. The Kendall tau correlation curves are very similar to those of the quadrant correlation.

From the plots, it is obvious to see that each correlation curve levels out close to 0 when θ becomes large, as stated in Remarks 2, 6, 10, 15. That means that the dependence of random variables in light-tailed SNBP distributions is very weak.

4. Asymptotic properties

Let $\mathcal{X} = \{\boldsymbol{x}_i\}_{i=1}^n = \{(x_{i1}, x_{i2})^T\}_{i=1}^n$ be a random sample of size n from the SBNP distribution. Then each correlation can be estimated by its sample counterpart, denoted accordingly as $\hat{r}_p$, $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{r}_g$, $\hat{\tau}$ and $\hat{r}_Q$. Note that although the two population Gini correlations and the symmetric (arithmetic mean version) all be the same, their sample values $\hat{\gamma}_1$, $\hat{\gamma}_2$, and $\hat{r}_g$ may be different. Using the influence function approach, we are able to establish asymptotic normality of each sample correlation. Influence functions also allow us to measure the robustness of an estimator by testing the sensitivity to small changes in the distribution.

4.1. Influence function approach

Let r be one of the correlation coefficients and $\hat{r}$ be its corresponding sample estimator. The influence function of r at $\boldsymbol{x} = (x_1, x_2)^T$ is defined to be the change rate under infinitesimal point mass contamination as follows.

$$\mathrm{IF}(\boldsymbol{x}; r, H) = \lim_{\epsilon \downarrow 0} \frac{r((1-\epsilon)H + \epsilon\delta_{\boldsymbol{x}}) - r(H)}{\epsilon},$$

where $\delta_{\boldsymbol{x}}$ denotes the point mass distribution at $\boldsymbol{x}$. Checking with the regularity conditions on r, we have the von Mises expansion

$$\hat{r} - r = \frac{1}{n}\sum_{i=1}^{n} \mathrm{IF}(\boldsymbol{x}_i; r, H) + o_p(n^{-1/2}). \tag{19}$$

This representation shows the connection between the IF and the robustness of r, observation by observation. Further, (19) yields the asymptotic normality of $\hat{r}$,

$$\sqrt{n}(\hat{r} - r) \xrightarrow{d} N(0, \mathbb{E}_H(\mathrm{IF}(\boldsymbol{X}; r, H))^2. \tag{20}$$

For more details on the influence function approach refer to Hampel *et al.* (1986, [7]), or Serfling (1980, [18]). We use this method to establish the asymptotic distribution of each correlation below.

4.2. Limiting distributions

Without loss of generality, we can assume that $\alpha_1 = \alpha_2 = 1$ since both r and $\hat{r}$ are invariant under changes of scale. All proofs in this section are reserved in Appendix.

Proposition 4.1. *For a sample from SNBP$(1, 1, \delta, \theta)$ with $\theta > 4$, we have*

$$\sqrt{n}(\hat{r}_p - r_p) \xrightarrow{d} N(0, \nu_p)$$

with

$$\nu_p = \frac{(\theta - 2)^2}{\theta^2} \left[(1 + \frac{r_p^2}{2}) \mathbb{E}(u_1^2 u_2^2) - 2 r_p \mathbb{E}(u_1^3 u_2) + \frac{r_p^2}{2} \mathbb{E}(u_1^4) \right], \tag{21}$$

where $u_i = (\theta - 1)X_i - 1, i = 1, 2$ and the expectations are with respect to the SNBP$(1, 1, \delta, \theta)$.

The condition of $\theta > 4$ guarantees the existence of the three expectations in (21). In general, an explicit derivation of the asymptotic variance won't be possible and numerical integration will have to be utilized to evaluate ν_p. However, in the cases of $\delta = 0$ and $\delta = 1$, we are able to find the explicit formula given below.

Corollary 4.2. *Under LSBP$(1, 1, \theta)$ with $\theta > 4$, sample Pearson correlation is asymptotically normally distributed as below.*

$$\sqrt{n}(\hat{r}_p - r_p) \xrightarrow{d} N(0, \nu_p),$$

where

$$\nu_p = \frac{(\theta^2 + 2)(\theta - 1)^2 (\theta + 1)(\theta + 2)}{\theta^4 (\theta - 4)(\theta - 3)}. \tag{22}$$

From (22), it is obvious that $\theta > 4$ is required for a positive asymptotic variance. Also $v_p \downarrow 1$ as θ increases.

For $\delta = 1$, the SNBP distribution has two independent components, X_1 and X_2, with $r_p = 0$. The asymptotic variance of the sample Pearson correlation, $\hat{r}_p$, is 1 as shown below.

Corollary 4.3. *Under SNBP$(1, 1, \delta = 1, \theta)$ with $\theta > 4$, we have*

$$\sqrt{n}\hat{r}_p \xrightarrow{d} N(0, 1).$$

Remark 4 and the above corollary 4.3 enable us to conduct independence tests in the SNBP distributions.

Proposition 4.4. *For a sample from SNBP$(1, 1, \delta, \theta)$ with $\theta > 2$, the limiting distribution of the Gini correlation is*

$$\sqrt{n}(\hat{\gamma}_1 - \gamma_1) \xrightarrow{d} N(0, \nu_\gamma)$$

with

$$\nu_\gamma = \frac{(2\theta - 1)^2}{\theta^2}(\mathbb{E}u_1^2 v_2^2 - 2\gamma_1 \mathbb{E}u_1^2 v_1 v_2 + \gamma_1^2 \mathbb{E}u_1^2 v_1^2), \tag{23}$$

where $u_i = (\theta - 1)X_i - 1, i = 1, 2$ and $v_j = 1 - 2/(1 + X_j)^\theta, j = 1, 2$, and the expectations are with respect to the SNBP$(1, 1, \delta, \theta)$.

For $\delta = 1$ (the independent case) the asymptotic variance ν_γ can be expressed explicitly as follows.

Corollary 4.5. *Under SNBP$(1, 1, \delta = 1, \theta)$ with $\theta > 2$, we have $\sqrt{n}\hat{\gamma}_1 \xrightarrow{d} N(0, \nu_\gamma)$, where*

$$\nu_\gamma = \frac{(2\theta - 1)^2}{3\theta(\theta - 2)}.$$

Note that the other sample Gini correlation, $\hat{\gamma}_2$, has the same limiting distribution as $\hat{\gamma}_1$. This results in the symmetric Gini correlation that takes the arithmetic mean of the two traditional Gini correlations having more statistical efficiency. More specifically, under the same conditions and using the same notations of Proposition 4.4, we have

Proposition 4.6. *For a sample from SNBP$(1, 1, \delta, \theta)$ with $\theta > 2$, the limiting distribution of the sample symmetric Gini correlation, $\hat{r}_g$, is*

$$\sqrt{n}(\hat{r}_g - \gamma_1) \xrightarrow{d} N(0, \nu_g),$$

where

$$\nu_g = \frac{(2\theta - 1)^2}{2\theta^2}\left(\mathbb{E}u_1^2 v_2^2 - 2\gamma_1 \mathbb{E}u_1^2 v_1 v_2 + \gamma_1^2 \mathbb{E}u_1^2 v_1^2 + (1 + \gamma_1^2)\mathbb{E}u_1 u_2 v_1 v_2 - 2\gamma_1 \mathbb{E}u_1 u_2 v_1^2\right). \tag{24}$$

For $\delta = 1$ (the independent case) the asymptotic variance ν_g can be expressed explicitly as follows.

Corollary 4.7. *Under SNBP$(1, 1, \delta = 1, \theta)$ with $\theta > 2$, we have $\sqrt{n}\hat{r}_g \xrightarrow{d} N(0, \nu_g)$, where*

$$\nu_g = \frac{7\theta^2 - 10\theta + 1}{6\theta(\theta - 2)}.$$

It is noted that $\nu_g \leq \nu_\gamma$. For example, for $\delta = 1$ and $\theta > 2$, $\nu_g - \nu_\gamma = -(\theta + 1)^2/(6\theta(\theta - 2)) < 0$. This means that $\hat{r}_g$ is more statistically efficient than $\hat{\gamma}_1$. Indeed, this result can be applied to a more general class of distributions.

Theorem 4.8. *Under a bivariate distribution exchangeable up to a linear transformation with a finite second moment, the symmetric Gini correlation $\hat{r}_g$ is more statistically efficient than any traditional Gini correlation of $\hat{\gamma}_1$ or $\hat{\gamma}_2$.*

Theorem 4.8 states that a simple average of the two Gini correlations has statistical advantage in distributions exchangeable up to a linear transformation. This class of distributions includes the popular elliptical distributions. The conclusion in regards to the normal distribution studied in [23] is an example of the above result.

For the Kendall tau correlation, we are unable to find an explicit formula for the asymptotic variance. One must rely on numerical integration techniques to evaluate variance based on the following result.

Proposition 4.9. *For a sample from SNBP$(1, 1, \delta, \theta)$, as $n \to \infty$,*

$$\sqrt{n}(\hat{\tau} - \tau) \xrightarrow{d} N(0, \nu_\tau),$$

where

$$\nu_\tau = 4\mathbb{E}[4/(1 + X_1 + X_2 + \delta X_1 X_2)^\theta - 2/(1 + X_1)^\theta - 2/(1 + X_2)^\theta + 1 - \tau]^2. \quad (25)$$

For $\delta = 1$ (the independent case) the asymptotic variance ν_τ can be expressed explicitly as follows.

Corollary 4.10. *Under SNBP$(1, 1, \delta = 1, \theta)$, we have*

$$\sqrt{n}\hat{\tau} \xrightarrow{d} N(0, 4/9).$$

An explicit formula for the asymptotic variance of the quadrant correlation coefficient is derived and its limiting distribution is as follows.

Proposition 4.11. *For a sample from SNBP$(1, 1, \delta, \theta)$, as $n \to \infty$,*

$$\sqrt{n}(\hat{r}_Q - r_Q) \xrightarrow{d} N(0, \nu_Q),$$

where

$$\nu_Q = 1 - r_Q^2. \quad (26)$$

In Propositions 4.9 and 4.11, there are no moment requirements for the limiting distributions of the Kendall tau and quadrant correlations. In this sense, Kendall tau and quadrant correlations are more robust than the Gini correlations that require a second finite moment and the Pearson correlation which requires a finite fourth moment.

From (26), the asymptotic variance of the sample quadrant correlation is decreasing with r_Q^2 and is less than or equal to 1. In the case of $\delta = 1$, $r_Q = 0$ and $\nu_Q = 1$, which is the same as ν_p. This means that the independence test based on the sample quadrant correlation is asymptotically equivalent to the test based on the sample Pearson correlation for $\theta > 4$.

For $\delta \neq 1$, $\nu_Q < 1$ and is smaller than ν_p in (22). This seems to suggest that under an LSBP distribution, the quadrant correlation is more efficient than the Pearson correlation. However, this is not true because the two correlations estimate different quantities. In order to compare asymptotic efficiency, we must consider Fisher consistent estimators so that they estimate the same parameter.

4.3. Asymptotic relative efficiency

We compare the asymptotic efficiencies of the correlations under the LSBP distribution. We consider Fisher consistent estimators based on different correlations so that they estimate the same parameter, $\rho = 1/\theta$, and their asymptotic variances are comparable. Let H_n be the empirical distribution of a random sample of size n from distribution H. An estimator $\hat{\rho} = \hat{\rho}(H_n)$ is Fisher consistent for ρ if $\hat{\rho}(\lim_{n\to\infty} H_n) = \rho$.

Under LSBP$(1, 1, \theta)$, the Pearson correlation is $1/\theta$, hence the corresponding sample Pearson correlation, $\hat{r}_p$, is Fisher consistent and denoted $\hat{\rho}_p$. The same is true for the various Gini correlations, denoted $\hat{\rho}_g$ and $\hat{\rho}_\gamma$, symmetric Gini and standard Gini, respectively. For the Kendall tau, by the relationship in (14), we derive the Fisher consistent estimator $\hat{\rho}_\tau$ to be $\hat{\rho}_\tau = 2\hat{\tau}/(1-\hat{\tau})$, and its asymptotic variance, derived through the Delta method, is

$$v_\tau = 4\nu_\tau/(1-\tau)^4.$$

For the quadrant correlation $r_Q = 4(2^{1+1/\theta}-1)^{-\theta}-1 = 4(2^{1+\rho}-1)^{-1/\rho}-1 := g(\rho)$, then the Fisher consistent estimator based on the quadrant correlation is $\hat{\rho}_Q = g^{-1}(\hat{r}_Q)$, and its asymptotic variance is $v_Q = k(\rho)^{-2}\nu_Q$, where

$$k(\rho) = \frac{\partial r_Q}{\partial \rho} = \frac{4}{\rho^2}(2^{1+\rho}-1)^{-1/\rho}\left(\ln(2^{1+\rho}-1) - \ln(2)[1+(2^{1+\rho}-1)^{-1}]\right).$$

We compute the asymptotic variance (ASV) of the quadrant correlation, $\hat{\rho}_Q$, and asymptotic relative efficiencies (ARE) of estimators $\hat{\rho}_p$, $\hat{\rho}_\gamma$, $\hat{\rho}_g$, and $\hat{\rho}_\tau$ relative to $\hat{\rho}_Q$, which are reported in Table 1. The asymptotic relative efficiency of one estimator, $\hat{\rho}_1$, with respect to another, $\hat{\rho}_2$, is defined by

$$ARE(\hat{\rho}_1, \hat{\rho}_2) = ASV(\hat{\rho}_2)/ASV(\hat{\rho}_1).$$

When explicit formulas for the asymptotic variances of $\hat{\rho}_\gamma$, $\hat{\rho}_g$, and $\hat{\rho}_\tau$ are not available, numerical integration techniques (the *integral* or *integral2* functions of the "pracma" package in R) are used to approximate them.

ρ	ARE($\hat{\rho}_p$)	ARE($\hat{\rho}_\gamma$)	ARE($\hat{\rho}_g$)	ARE($\hat{\rho}_\tau$)	ASV($\hat{\rho}_Q$)
0.100	3.1625	3.2454	3.8278	2.3574	5.1529
0.200	1.2516	3.1197	3.8157	2.2956	6.0552
0.333	N/A	2.5953	3.3355	2.2321	7.3860
0.500	N/A	N/A	N/A	2.1735	9.2563
1.000	N/A	N/A	N/A	2.0979	16.4344

Table 1. Asymptotic relative efficiencies (ARE) of estimators $\hat{\rho}_p$, $\hat{\rho}_\gamma$, $\hat{\rho}_g$, and $\hat{\rho}_\tau$ relative to $\hat{\rho}_Q$ for the LSBP distribution, with asymptotic variance (ASV($\hat{\rho}_Q$)) of $\hat{\rho}_Q$.

From Table 1, it is apparent that the ASV of the quadrant correlation increases as ρ increases. Kendall tau is twice as statistically efficient as the quadrant correlation.

The ARE of the Pearson correlation is 3.16 for $\rho = 0.1$, but it reduces dramatically to 1.25 when $\rho = 0.2$. The ARE of the standard Gini correlation maintains better with a value of 3.12 when $\rho = 0.2$ and a value of 2.60 for $\rho = 0.3$. The symmetric Gini correlation, $\hat{\rho}_g$, is the most efficient; it is 18%, 22% and 29% more efficient than $\hat{\rho}_\gamma$ for $\rho = 0.1, 0.2$, and 0.3, respectively. Although $\hat{\rho}_g$ is most efficient, it remains restricted by the moment requirement. If $\theta \leq 2$ or $\rho \geq 0.5$, the ASV of $\hat{\rho}_g$ does not exist, which explains the appearance of N/A in the table.

5. Finite sample performance

To explore finite sample performance of these correlations, we conduct two simulations to study finite sample efficiency under LSBP distributions and to compare empirical power of independence tests based on those correlations.

5.1. Finite sample efficiency

A small simulation is conducted to compare the efficiency of these correlations among finite samples. Samples of sizes $n = 30$ and $n = 300$ were drawn from LSBP distributions. The R Package "Bivariate.Pareto" was used to generate data from the SNBP distributions, setting $\alpha_1 = \alpha_2 = 1$ and $\delta = 0$. Again, we consider $\rho = 1/\theta$ as the parameter and its Fisher consistent estimators $\hat{\rho}_p$, $\hat{\rho}_\gamma$, $\hat{\rho}_g$, $\hat{\rho}_\tau$, and $\hat{\rho}_Q$.

ρ	n	$\hat{\rho}_p$	$\hat{\rho}_\gamma$	$\hat{\rho}_g$	$\hat{\rho}_\tau$	$\hat{\rho}_Q$
0.10	30	1.1423 (.0122)	1.2222 (.0133)	1.1403 (.0130)	1.6975 (.0315)	1.9733 (.0351)
	300	1.2633 (.0204)	1.2539 (.0177)	1.1591 (.0162)	1.4948 (.0199)	2.0080 (.0298)
0.20	30	1.2793 (.0143)	1.2914 (.0150)	1.1969 (.0121)	1.8637 (.0320)	2.0589 (.0302)
	300	1.6887 (.0345)	1.3720 (.0188)	1.2465 (.0179)	1.6423 (.0227)	2.4068 (.0314)
0.33	30	1.4534 (.0160)	1.3616 (.0169)	1.2597 (.0167)	2.0999 (.0441)	2.1153 (.0176)
	300	2.4106 (.0395)	1.5581 (.0255)	1.3974 (.0223)	1.8451 (.0233)	2.7723 (.0372)
0.50	30	1.6600 (.0176)	1.4437 (.0174)	1.3390 (.0176)	2.4083 (.0300)	2.1238 (.0167)
	300	3.3188 (.0402)	1.8175 (.0223)	1.6307 (.0232)	2.0913 (.0199)	3.0728 (.0399)
1.00	30	2.7374 (.0266)	2.0692 (.0289)	2.0163 (.0248)	3.3088 (.0587)	2.1988 (.0296)
	300	7.3578 (.0775)	3.5521 (.0337)	3.4862 (.0285)	2.8389 (.0327)	2.6653 (.0490)

Table 2. The mean and standard deviation (in parenthesis) of $\sqrt{n}$RMSE of $\hat{\rho}_p$, $\hat{\rho}_\gamma$, $\hat{\rho}_g$, $\hat{\rho}_\tau$ and $\hat{\rho}_Q$ under LSBP distributions.

An estimator $\hat{\rho}^{(m)}$ is calculated for the m^{th} sample $(m = 1, 2, ...M)$, and the root mean squared error (RMSE) is computed using the formula below.

$$\text{RMSE}(\hat{\rho}) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}(\hat{\rho}^{(m)} - \rho)^2}.$$

With $M = 3000$, the procedure is repeated 30 times to procure the mean and standard deviation of $\sqrt{n}$RMSE. In Table 2, we display the mean and standard deviation (in parenthesis) of $\sqrt{n}$RMSE of $\hat{\rho}_p$, $\hat{\rho}_\gamma$, $\hat{\rho}_g$, $\hat{\rho}_\tau$, and $\hat{\rho}_Q$.

We notice an increasing trend in $\sqrt{n}$RMSEs as ρ increases for each sample size. We see that $\hat{\rho}_g$ outperforms the regular Gini correlation $\hat{\rho}_\gamma$ in all cases, demonstrating the efficiency advantage of the symmetric Gini correlation. Also, it is the most efficient among all correlations except in the large sample case when $\rho = 1$. In this instance, the quadrant and Kendall tau correlations perform better, and the Pearson correlation performs the worst. It is interesting to see that the Gini correlations, $\hat{\rho}_g$ and $\hat{\rho}_\gamma$, perform extremely well in small sample size $n = 30$. As expected, the quadrant correlation performs the worst of the measures evaluated here.

5.2. Independence tests

We conduct independence tests based on different correlations. The following null and alternative hypotheses are considered.

$$H_0 : \delta = 1 \quad vs \quad H_a : \delta = 0. \tag{27}$$

The test based on $\hat{r}$ is to reject independence if $\hat{r} > q_{1-\alpha}\sqrt{\nu_r/n}$ where $\hat{r}$ represents one of the sample correlations $\hat{r}_p$, $\hat{\gamma}_1$, $\hat{r}_g$, $\hat{\tau}$ and $\hat{r}_Q$, while ν_r is the asymptotic variance of $\hat{r}$ under H_0 and $q_{1-\alpha}$ is the quantile of the standard normal distribution.

We generate samples of size $n = 30$ and $n = 300$ from SNBP$(1, 1, \delta = 1, \theta)$ and LSBP$(1, 1, \theta)$ for $\theta = 1, 2, 3, 4, 5$ and 10. Empirical size and power of tests are calculated based on $M = 10000$ repetitions. The results are listed in Table 3.

		Size					Power				
θ	n	$\hat{r}_p$	$\hat{\gamma}$	$\hat{r}_g$	$\hat{\tau}$	$\hat{r}_Q$	$\hat{r}_p$	$\hat{\gamma}$	$\hat{r}_g$	$\hat{\tau}$	$\hat{r}_Q$
10	30	.0696	.0485	.0517	.0628	.0744	.1626	.1050	.1260	.1169	.1090
	300	.0608	.0525	.0527	.0509	.0402	.5003	.4162	.4580	.3603	.1820
5	30	.0712	.0417	.0482	.0630	.0774	.2814	.1809	.2229	.1945	.1640
	300	.0598	.0483	.0506	.0540	.0421	.8679	.8185	.8829	.7607	.4230
4	30	N/A	.0318	.0384	.0618	.0745	N/A	.2107	.2747	.2441	.1851
	300	N/A	.0440	.0491	.0517	.0431	N/A	.9140	.9573	.8930	.5627
3	30	N/A	.0172	.0299	.0323	.0704	N/A	.2208	.3259	.3297	.2340
	300	N/A	.0375	.0404	.0525	.0436	N/A	.9676	.9922	.9771	.7478
2	30	N/A	N/A	N/A	.0606	.0728	N/A	N/A	N/A	.5027	.3366
	300	N/A	N/A	N/A	.0505	.0391	N/A	N/A	N/A	.9996	.9547
1	30	N/A	N/A	N/A	.0609	.0700	N/A	N/A	N/A	.8582	.6395
	300	N/A	N/A	N/A	.0510	.0375	N/A	N/A	N/A	1.000	1.000

Table 3. Empirical size and power of independence tests (with $\alpha = 0.05$) based on each correlation.

From Table 3, we see that all tests maintain the nominal size 0.05 well, especially when $n = 300$. The power of tests increase substantially when sample size n increases from 30 to 300. Also, there is a clear trend of increasing power as θ decreases. This is because for large values of θ, the difference between the independent distribution under H_0 and the LSBP distribution under H_a is small, making it harder for the tests to reject H_0; hence, the power of each test is relatively low compared to those with small values of θ. The Kendall tau test performs much better than the quadrant correlation test. Although the quadrant correlation test is asymptotically equivalent to the Pearson correlation test for $\theta > 4$, its finite sample performance is extremely poor, about 32% and 46% lower than the power of the Pearson correlation test for $\theta = 10$ and $\theta = 5$, respectively. Again, the test based on the symmetric Gini correlation, $\hat{r}_g$, is more powerful than the test based on traditional Gini correlation in any case. Also, the symmetric Gini test performs the best when $\theta = 3, 4, 5$. However, since $\hat{r}_g$ requires the existence of finite second moment, it is not suitable for inferences on bivariate Pareto distributions with $\theta \leq 2$. In that case, the Kendall tau is the best, most robust, choice.

6. Real data analysis

For the purpose of illustrating a situation in which these formulas prove useful for determining correlations in the SNBP distribution, we gather the Palmer Drought Severity Index (PDSI) for Mississippi and Tennessee for every month from January 1896 until December 2017. The data is freely accessible at: *https://www.ncdc.noaa.gov/cag/statewide/time-series/22/pdsi*. The PDSI is an often used measurement of aridity. A negative PDSI corresponds to the occurrence of a drought, hence we define drought duration as the lengths of negative runs in our PDSI values. Thus, for all PDSI values, we obtain 90 occurrences of drought in both states. The longest drought duration in MS occurred from June, 1953 to March, 1957, lasting 44 months, while in TN the longest drought duration was 55 months and continued from July, 1939 to January, 1944.

Figure 2 shows histograms of drought duration in each state and a scatterplot of the data. In the histogram plots, we add the kernel smoothing density curves (KSD) that are estimated using the default settings of the "density" function in R. Marginal Pareto density curves are added to the histograms and the density contours in the scatterplot are based on the fitted SNBP model described below.

The model is fitted to the drought duration data using the two stage Maximum Likelihood procedure as laid out in [15]. The estimated parameters are $\hat{\alpha}_1 = 0.0608$, $\hat{\alpha}_2 = 0.0625$, $\hat{\theta} = 2.6811$, and $\hat{\delta} = 1.1718$. We then implement the bivariate Kolmogorov-Smirnov goodness of fit test laid out in [9]. The test statistic we get from our data is 0.3074. Based on 2000 test statistics from the Monte Carlo simulation procedure, the p-value of the test is 0.6735. Hence we can conclude that this data comes from the SNBP distribution.

	Correlations					
	r_p	γ_1	γ_2	r_g	τ	r_Q
SNBP model	-0.0163	-0.0439	-0.0439	-0.0439	-0.0228	-0.0235
Drought data	0.0717	-0.0170	0.0154	-0.0008	-0.0959	-0.0750
	Standard Errors					
	r_p	γ_1	γ_2	r_g	τ	r_Q
SNBP model	NA	0.1943	0.1943	0.1555	0.0749	0.0999
Jackknife method	0.1294	0.1458	0.1449	0.1346	0.0813	0.1175

Table 4. Sample and population correlations between drought duration of MS and TN. Standard errors of sample correlations are estimated by the model and by the Jackknife method.

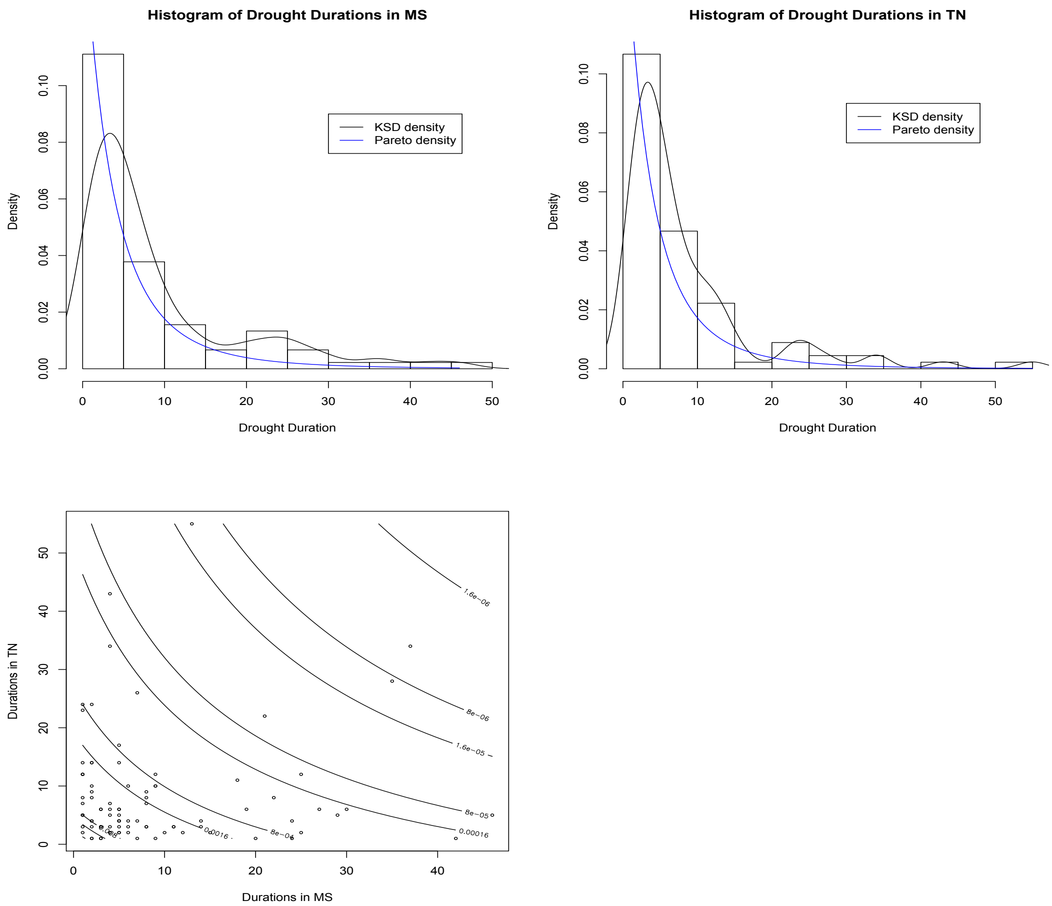


Figure 2. Drought Durations in MS and TN during 1896 and 2017. (a) Histogram of drought durations in MS. (b) Histogram of drought durations in TN. (c) Scatter plot with bivariate Pareto density contours.

With the fitted SNBP model, we are able to compute population correlations and compare with sample correlations computed from data. For each sample correlation, we use two methods to estimate its standard error. One is computed from the asymptotic variance established previously in Section 4.2. The other is computed using the Jackknife method. Let $\hat{r}_{(-i)}$ be the jackknife pseudo value of a correlation estimator $\hat{r}$ based on the sample with the i^{th} observation deleted. Then the jackknife variance is

$$\hat{v}_r = \frac{n-1}{n} \sum_{i=1}^{n} (\hat{r}_{(-i)} - \bar{\hat{r}}_{(\cdot)})^2 \tag{28}$$

where $\bar{\hat{r}}_{(\cdot)} = 1/n \sum_{i=1}^{n} \hat{r}_{(-i)}$. See [19] for more details.

The results are listed in Table 4. From the table, we can see that each sample correlation is close to its population correlation, indicating that the SNBP model is a good fit to the data. Standard errors based on the model and nonparametric jackknife are also close to each other. Note that the asymptotic variance of the Pearson correlation is not available since it requires $\theta > 4$ but in the fitted model, $\theta = 2.6811$. From the table, we see that all correlations are close to zero, suggesting independence in the drought durations of the two states. It is a little surprising to find that geographic closeness of MS and TN does not necessarily imply a high correlation in the lengths of droughts in the two states. The formal tests based on each correlation studied in Section 5.2, using either standard error, clearly confirm independence.

Models	AIC	BIC
SNBP(α_1, α_2, δ, θ)	1150.3	1160.3
SNBP(α_1, α_2, $\delta = 1$, θ)	1142.7	1150.2
ParetoII(α_1, θ_1)ParetoII(α_2, θ_2)	1143.5	1153.5

Table 5. Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for comparison of three models.

The independence leads us to also look at two independent models. The first is the SNBP model with $\delta = 1$ and the second is univariate Pareto II modeling of the drought duration of each state separately. We gather the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for comparing those three models and present the information in Table 5. From the values, we see that the independent models for drought durations in MS and TN are preferred to the SNBP with $\delta = 1.172$, the conclusion also supported by each correlation. We also see that the bivariate Pareto model SNBP(α_1, α_2, $\delta = 1$, θ) is slightly preferred over modeling each state separately. It is interesting to see that though the two variables are independent, a joint bivariate modeling might be beneficial. Sharing a common shape parameter results in a better modeling of drought durations of the two states.

7. Conclusion

In this paper, we take a close look at correlations in the Sankaran and Nair bivariate Pareto distribution, which contains the Lindley-Singpurwalla bivariate Pareto as a special case. These distributions have proven to be very useful in modelling lifetime data, hydrology, competing risk data, and many other datasets. When using a bivariate model as a joint distribution of two components, it is often useful to measure the correlation between the two components. We provide explicit formulas for the Gini, Pearson, Kendall tau, and quadrant correlations in the SNBP distribution (of which the formulas for the Gini and Kendall tau correlations appear to have never been published before). We show that all correlations considered capture dependence in the SNBP distributions. That is, zero values of those correlations mutually imply independence. The parameter δ in the SNBP also allows negative correlations in the SNBP distributions.

We are able to use influence functions previously established for each correlation to give information about the asymptotic distributions of the correlations and calculate asymptotic variances. Pearson correlation is shown to be least robust since it requires a finite fourth moment for asymptotic variance. Quadrant and Kendall tau correlations are most robust and have no moment assumptions. Robustness of the Gini correlation lies somewhere between that of Pearson and that of Kendall tau and Quadrant. We also consider Fisher consistent correlations in the LSBP distributions in order to compare asymptotic variances. We find that the quadrant correlation is least efficient. The symmetric Gini correlation, the simple average of the two traditional Gini correlations, is more efficient than either traditional one. This result is extended to a more general class of distributions including elliptical distributions with normal models as a special case. Finite sample performance comparison has been conducted. Again, quadrant correlation performs worse than Kendall tau correlation in all cases. The symmetric Gini correlation is superior to all others when the second moment of the distribution is finite. In terms of compromising efficiency and robustness, the Kendall tau correlation proves to be more appealing.

8. Appendix

Proof of Proposition 4.1. Devlin, Gnanadesikan and Kettering (1975, [5]) give the influence function for the Pearson correlation,

$$\mathrm{IF}((x_1,x_2)^T;r_p,H)=\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}-\frac{1}{2}r_p[\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}],$$

where μ_i and σ_i are the mean and standard deviation of X_i for $i=1,2$. Then, for $(X_1,X_2)\sim H=\mathrm{SNBP}(1,1,\delta,\theta)$,

$$\begin{aligned}&\mathrm{IF}((x_1,x_2)^T;r_p,H)\\&=\frac{\theta-2}{\theta}\left[((\theta-1)x_1-1)((\theta-1)x_2-1)-\frac{r_p}{2}[((\theta-1)x_1-1)^2+((\theta-1)x_1-1)^2]\right].\end{aligned}$$

Let $u_1=(\theta-1)X_1-1$ and $u_2=(\theta-1)X_2-1$. Then,

$$\begin{aligned}\nu_p=\mathbb{E}[\mathrm{IF}((X_1,X_2)^T;r_p,H)^2]&=\frac{(\theta-2)^2}{\theta^2}\mathbb{E}[u_1u_2-\frac{r_p}{2}(u_1^2+u_2^2)]^2\\&=\frac{(\theta-2)^2}{\theta^2}\left[(1+\frac{r_p^2}{2})\mathbb{E}(u_1^2u_2^2)-2r_p\mathbb{E}u_1^3u_2+\frac{r_p^2}{2}\mathbb{E}u_1^4\right].\end{aligned}$$

The last equation is due to the symmetric role of u_1 and u_2 in the expectations so that $\mathbb{E}u_1^4 = \mathbb{E}u_2^4$ and $\mathbb{E}u_1^3 u_2 = \mathbb{E}u_1 u_2^3$. $\square$

Proof of Corollary 4.2. For a LSBP distribution, we have $r_p = 1/\theta$ and

$$\mathbb{E}u_1^2 u_2^2 = \frac{\theta^3 - \theta^2 + 14\theta + 4}{(\theta-4)(\theta-3)(\theta-2)}, \qquad \mathbb{E}u_1^3 u_2 = \frac{3(3\theta^2 + \theta + 2)}{(\theta-4)(\theta-3)(\theta-2)},$$
$$\mathbb{E}u_1^4 = \frac{3\theta(3\theta^2 + \theta + 2)}{(\theta-4)(\theta-3)(\theta-2)}.$$

Then,

$$\nu_p = \frac{(\theta-2)^2}{\theta^4}[(\theta^2 + \frac{1}{2})\mathbb{E}[u_1^2 u_2^2] - 2\theta\mathbb{E}[u_1^3 u_2] + \frac{1}{2}\mathbb{E}[u_1^4]] = \frac{(\theta^2+2)(\theta-1)^2(\theta+1)(\theta+2)}{\theta^4(\theta-4)(\theta-3)}.$$

$\square$

Proof of Corollary 4.3. For $\delta = 1$, $r_p = 0$ and $\mathbb{E}u_1^2 u_2^2 = \mathbb{E}u_1^2 \mathbb{E}u_2^2 = [\mathbb{E}u_1^2]^2 = \theta^2/(\theta-2)^2$. Plugging them in (21), we have $\nu_p = 1$. $\square$

Proof of Poposition 4.4. Vanderford, Sang, and Dang (2020, [23]) proposed the influence function of the traditional Gini correlation for any nondegenerate bivariate distribution H with finite first moment,

$$\begin{aligned}&\mathrm{IF}((x_1,x_2)^T;\gamma_1,H)\\ &= \frac{(x_1 - \mathbb{E}X_1)(F_2(x_2) - \mathbb{E}F_2(X_2))}{cov(X_1, F_1(X_1))} - \gamma_1 \frac{(x_1 - \mathbb{E}X_1)(F_1(x_1) - \mathbb{E}F_1(X_1))}{cov(X_1, F_1(X_1))}.\end{aligned} \tag{29}$$

Thus, for $(X_1, X_2) \sim \mathrm{SNBP}(1,\ 1,\ \delta,\ \theta)$,

$$\begin{aligned}&\mathrm{IF}((x_1,x_2)^T;\gamma_1,H)\\ &= \frac{2\theta-1}{\theta}\left[((\theta-1)x_1 - 1)(1 - \frac{2}{(1+x_2)^\theta}) - \gamma_1((\theta-1)x_1 - 1)(1 - \frac{2}{(1+x_1)^\theta})\right].\end{aligned}$$

Then, with the notations $u_i = (\theta-1)X_i - 1$ and $v_j = 1 - 2(1+X_i)^{-\theta}$ for $i = 1, 2$, we have

$$\begin{aligned}\nu_{\gamma_1} &= \mathbb{E}[\mathrm{IF}((X_1,X_2)^T;\gamma_1,H)^2] = \frac{(2\theta-1)^2}{\theta^2}\mathbb{E}(u_1 v_2 - \gamma_1 u_1 v_1)^2\\ &= \frac{(2\theta-1)^2}{\theta^2}(\mathbb{E}[u_1^2 v_2^2] - 2\gamma_1\mathbb{E}[u_1^2 v_1 v_2] + \gamma_1^2\mathbb{E}[u_1^2 v_1^2]).\end{aligned}$$

$\square$

Proof of Corollary 4.5. For $\delta = 1$, we have $\gamma = 0$, $\mathbb{E}u_1^2 v_2^2 = \mathbb{E}u_1^2 \mathbb{E}v_2^2 = \theta/(3(\theta-2))$. Plugging in (23), we derive the explicit formula of ν_γ.

$\square$

Proof of Proposition 4.6. The influence function of symmetric Gini correlation for $H = \mathrm{SNBP}(1,1,\delta,\theta)$ is

$$\mathrm{IF}((x_1,x_2)^T;r_g,H) = (\mathrm{IF}((x_1,x_2)^T;\gamma_1,H) + \mathrm{IF}((x_1,x_2)^T;\gamma_2,H))/2.$$

Hence, with $\gamma_1 = \gamma_2$, symmetric roles of u_1, u_2, and symmetric roles of v_1, v_2, we have

$$\begin{aligned}\nu_g &= 1/4\mathbb{E}(\mathrm{IF}((X_1, X_2)^T; \gamma_1, H) + \mathrm{IF}((X_1, X_2)^T; \gamma_2, H))^2 \\ &= \frac{(2\theta-1)^2}{2\theta^2}\left(\mathbb{E}[u_1^2 v_2^2 - 2\gamma_1 u_1^2 v_1 v_2 + \gamma_1^2 u_1^2 v_1^2 + (1+\gamma_1^2) u_1 u_2 v_1 v_2 - 2\gamma_1 u_1 u_2 v_1^2]\right).\end{aligned}$$

□

Proof of Corollary 4.7. For $\delta = 1$, we have $\gamma = 0$, $\mathbb{E}u_1^2 v_2^2 = \mathbb{E}u_1^2 \mathbb{E}v_2^2 = \theta/(3(\theta-2))$ and $\mathbb{E}u_1 u_2 v_1 v_2 = [\mathbb{E}u_1 v_1]^2 = \theta^2/(2\theta-1)^2$. Plugging in (24), we derive the explicit formula of ν_g. □

Proof of Theorem 4.8. From (29), we denote $\mathrm{IF}((x_1, x_2)^T; \gamma_1, H) = \delta_1(x_1, x_2)$ and similarly, $\mathrm{IF}((x_1, x_2)^T; \gamma_2, H) = \delta_2(x_1, x_2)$. For a bivariate distribution H with finite second moments, exchangeable up to a linear combination, we have $\gamma_1 = \gamma_2$ and $\mathbb{E}\delta_1(X_1, X_2)^2 = \mathbb{E}\delta_2(X_1, X_2)^2$. Thus, the asymptotic variance of $\hat{r}_g$ is

$$\begin{aligned}\nu_g &= \frac{1}{4}\mathbb{E}[\delta_1(X_1, X_2)^2 + \delta_2(X_1, X_2)^2 + 2\delta_1(X_1, X_2)\delta_2(X_1, X_2)] \\ &\leq \frac{1}{4}\mathbb{E}[2\delta_1(X_1, X_2)^2 + 2\delta_2(X_1, X_2)^2] \\ &= \mathbb{E}[\delta_1(X_1, X_2)^2] = \nu_\gamma.\end{aligned}$$

□

Proof of Proposition 4.9. Croux and Dehon (2010, [4]) published the influence function of the Kendall tau correlation as follows,

$$\mathrm{IF}((x_1, x_2)^T; \tau, H) = 2[2P((x_1 - X_1)(x_2 - X_2) > 0) - 1 - \tau].$$

Then,

$$\begin{aligned}\nu_\tau &= \mathbb{E}[\mathrm{IF}((X_1, X_2)^T; \tau, H)^2] \\ &= \mathbb{E}[(2[2P((X_1 - X_1')(X_2 - X_2') > 0) - 1 - \tau])^2] \\ &= 4\mathbb{E}[2(1 + 2(1 + X_1 + X_2)^{-\theta} - (1 + X_1)^{-\theta} - (1 + X_2)^{-\theta}) - 1 - \tau]^2 \\ &= 4\mathbb{E}[4(1 + X_1 + X_2)^{-\theta} - 2(1 + X_1)^{-\theta} - 2(1 + X_2)^{-\theta} + 1 - \tau]^2.\end{aligned}$$

□

Proof of Corollary 4.10. For $\delta = 1$, $\tau = 0$ and

$$\begin{aligned}\nu_\tau &= 4\mathbb{E}[4(1 + X_1 + X_2 + X_1X_2)^{-\theta} - 2(1 + X_1)^{-\theta} - 2(1 + X_2)^{-\theta} + 1]^2 \\ &= 4\left(\mathbb{E}\left[\frac{4}{(1+X_1)^\theta(1+X_2)^\theta}\right]^2 + 2\mathbb{E}\left[\frac{2}{(1+X_1)^\theta}\right]^2 + 1 - 4\mathbb{E}\frac{8}{(1+X_1)^{2\theta}(1+X_2)^\theta}\right. \\ &\left. + 4\mathbb{E}\frac{4}{(1+X_1)^\theta(1+X_2)^\theta} - 4\mathbb{E}\frac{2}{(1+X_1)^\theta}\right) \\ &= 4(\frac{16}{9} + \frac{8}{3} + 1 - \frac{16}{3} + 4 - 4) \\ &= \frac{4}{9}.\end{aligned}$$

Proof of Proposition 4.11. Shevlyakov and Vilchevski (2002, [22]) give the influence function for the quadrant correlation,

$$\mathrm{IF}((x_1, x_2)^T; r_Q, H) = \mathrm{sign}[(x_1 - \mathrm{med}(X_1))(x_2 - \mathrm{med}(X_2))] - r_Q.$$

Thus,

$$\begin{aligned}
\nu_Q &= \mathbb{E}[\mathrm{sign}[(X_1 - \mathrm{med}(X_1))(X_2 - \mathrm{med}(X_2))] - r_Q]^2 \\
&= 1 + r_Q^2 - 2r_Q \mathbb{E}sign((X_1 - Med(X_1))(X_2 - Med(X_2))) \\
&= 1 + r_Q^2 - 2r_Q^2 \\
&= 1 - r_Q^2.
\end{aligned}$$

□

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