

Research Statement

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Overview. My research interests lie in Commutative Algebra, with some overlap into Algebraic Geometry. Commutative Algebra is a branch of pure mathematics which is essentially the study of commutative rings and ideals. My main area of study is the behavior of maps on divisor class groups and Chow groups determined by hypersurface sections. My results represent steps towards understanding the injectivity of these maps on divisor class groups and the commutativity of intersection in the Chow group.

Additional research projects lie in other topics of Commutative Algebra. These include the study of growth of ideals, namely basic questions which involve the theories of integral closure of ideals and valuations, and graphs of zero divisors. These are discussed in the last two sections.

1. BACKGROUND

The development of the divisor class group of a Noetherian normal domain A is due, in large part, to Samuel's work [23], [25] on unique factorization domains (UFD's) in the 1960's. Roughly speaking, the divisor class group of A , denoted by $\text{Cl}(A)$, is a measure of the extent to which A fails to be a UFD. In particular, the divisor class group of A is trivial if and only if A is a UFD. It has long been known that if A is a UFD, then so is a polynomial ring over A , a result due to Gauss. However, the same is not true for the power series ring, and the theory of UFD's includes extensive study of when a similar result holds for these rings. Danilov [5] defined a map from the divisor class group of the power series of A to the divisor class group of A , and in a series of articles [3], [4], [5], characterized its injectivity. These results in some ways parallel those of Grothendieck [11], who found conditions under which the homomorphism from the Picard group of the punctured spectrum of A to that of a hypersurface is injective, and provide motivation for some of my own work.

Although I am mostly interested in the purely algebraic notions of factoriality and the divisor class group, these concepts have geometric interpretations. More specifically, in geometry a domain A arises as the homogeneous coordinate ring of an irreducible variety V . To say that A is factorial means that for every subvariety $Y \subset V$ of codimension one, the ideal of functions that vanish on Y is principal. The divisor class group of V , an intrinsic invariant of V , consists of isomorphism classes of codimension one cycles. It has much importance in algebraic geometry since it is useful in classifying algebraic varieties.

2. DIVISOR CLASS GROUPS

Let f be a prime element such that the hypersurface determined by f is normal. Lipman [14] generalized Danilov's map by showing that there is a homomorphism, often referred to as a restriction map, from the divisor class group of A to the divisor class group of the hypersurface determined by f . This map need not be injective. However, C. Miller [15] used a generalized notion of the divisor class group to prove that the intersection of the kernels of $\text{Cl}(A[[T]]) \rightarrow \text{Cl}(A[[T]]/(T^n))$, where n ranges from one to infinity, is trivial. In other words, no non-trivial divisor class can be in each of the kernels.

This motivates the investigation into whether a similar result will hold more generally for a sequence of distinct elements: let $\{f_n\}_{n=1}^{\infty}$ be a sequence of prime elements such that each hypersurface $A/f_n A$ is normal and f_n approaches zero as n goes to infinity. Is the intersection

of the kernels of $\text{Cl}(A) \rightarrow \text{Cl}(A/f_nA)$ trivial? Moreover, does there exist a positive integer N such that $\text{Cl}(A) \rightarrow \text{Cl}(A/f_nA)$ is injective when n exceeds N ? And if so, are there effective methods to determine N ?

The work in my thesis shows that the answer to the first question is affirmative when the ambient ring is *excellent*, a very mild hypothesis which is satisfied by Noetherian rings that arise in algebraic and arithmetic geometry. (I also relax the condition of normality on the hypersurfaces to regularity in codimension one and define a map to the divisor class group of the integral closure of A/f_nA , denoted $(A/f_nA)'$.) More importantly, it seems to suggest that the pathology of the map lies near the “top” of the maximal ideal. The basic philosophy of my thesis is that the deeper f lies in powers of the maximal ideal, the better the injective behavior of the restriction map.

Theorem 2.1. ([26]) *Let A be an excellent, normal, local \mathbb{Q} -algebra, such that A is an isolated singularity of dimension at least four. In addition, suppose that A has a small Cohen-Macaulay module. Then there is a positive integer N , depending only on the ring A , such that the following holds: If f lies in the N -th power of the maximal ideal and A/fA is regular in codimension one, then $\text{Cl}(A) \rightarrow \text{Cl}((A/fA)')$ is injective.*

This result has a connection to one of the biggest open problems in Commutative Algebra, namely the Small Cohen-Macaulay Modules Conjecture. The conjecture asserts that if A is a complete Noetherian local ring, then A has a small Cohen-Macaulay module. One consequence of Theorem 2.1 is that if there is an element f satisfying the hypotheses such that the restriction map is not injective, then A could not possess a small Cohen-Macaulay module. Such an example would disprove the long-standing conjecture.

Corollary 2.2. *In the context of Theorem 2.1, if there exists a non-zero element f in the N -th power of the maximal ideal such that $(A/fA)'$ is a UFD, then A is a UFD.*

In the case of characteristic $p > 0$, Phillip Griffith and I have attacked similar questions. One difference in this case is the possibility of p -torsion elements in the kernel.

Theorem 2.3. ([10]) *Let k be a perfect field of positive characteristic p and let S denote an \mathbb{N} -graded ring of dimension at least four such that $S_0 = k$ and S is a normal domain. Assume that $\text{Spec}(S) - S_+$ is regular. Let x_1, \dots, x_d be a system of parameters of S contained in the Noetherian different of the completion of S . If f is a homogeneous prime element in the ideal generated by the x_i^2 such that the hypersurface determined by f is regular in codimension one, then the kernel of the restriction map is at worst a bounded p -group.*

For a geometric interpretation of this theorem, assume that S is generated over k in degree one and $\text{Proj}(S) = V$ is smooth over k . Let H be the hypersurface, necessarily smooth in codimension one, defined by the homogeneous element f . Then there is a commutative diagram in which the column homomorphisms amount to restriction [25], [12].

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Cl}(V) & \longrightarrow & \text{Cl}(S) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Cl}(H) & \longrightarrow & \text{Cl}((S/fS)') \longrightarrow 0 \end{array}$$

It follows that there is a natural identification between the kernels of the maps on the divisor class groups of the varieties and the divisor class groups of the rings. Thus, the results of the above theorem apply to $\text{Cl}(V) \rightarrow \text{Cl}(H)$ as well.

Remark 2.4. *As a corollary to Theorem 2.3, when it is assumed that the hypersurfaces are normal, rather than just regular in codimension one, the result is the characteristic p analogue of Theorem 2.1.*

In addition, Griffith and I used the work in my thesis to construct some important examples where the restriction map is not injective. Most examples for which the kernel of the restriction of divisor classes is non-trivial come about in one of two ways: either A is a complete intersection of dimension less than or equal to three, or A is a power series ring, as in Danilov’s theory. In particular, our goal was to construct examples of rings that were local isolated singularities of dimension at least four. To do this, we appealed to a combination of the Danilov results together with those in [26]. Such examples are important in light of Grothendieck’s results [11].

Further work on divisor class groups includes a joint paper with Anurag Singh at the University of Utah. Using the theory of rational coefficient Weil divisors due to Demazure [6] and related results of Watanabe [27], we provide a simple technique to compute divisor class groups of affine normal hypersurfaces of the form

$$k[z, x_1, \dots, x_d]/(z^n - g),$$

where g is a weighted homogeneous polynomial in x_1, \dots, x_d of degree relatively prime to n .

Proposition 2.5. *Let $R = k[z, x_1, \dots, x_d]/(z^n - g)$ be a normal hypersurface over a field k , with g as above. If $g = h_1 \cdots h_r$, where the h_i are irreducible polynomials in x_1, \dots, x_d , then*

$$\text{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{r-1},$$

and the images of $(z, h_1), \dots, (z, h_{r-1})$ form a minimal generating set for $\text{Cl}(R)$.

For future work on the topic of divisor class groups, Sean Sather-Wagstaff (North Dakota State University) and I will investigate ideas similar to those in my thesis, as well as some based on the work of Griffith and Weston [9]. We are studying sequences of prime ideals with finite projective dimension. Once again, one can give some restriction on the kernels of the maps involved.

3. CHOW GROUPS

Recently I have been investigating related ideas for Chow groups, in particular, intersection with divisors. The Chow group of a Noetherian ring A is defined to be the group of cycles of A modulo rational equivalence. When the ring is a normal domain of dimension d , the $(d - 1)^{\text{st}}$ component of the Chow group is simply the divisor class group.

The operation given by intersecting with a Cartier divisor is one of the basic ideas of Intersection Theory, and the fact that it defines a commutative operation in the Chow group is fundamental in making the theory work. If the intersection is proper, that is, if one intersects a divisor D with a variety W not contained in D , this concept is quite simple. However, if W is contained in D , then the situation is considerably more complicated. A classical approach to this question is to use a “moving lemma” to move D to another divisor which meets W properly, while a newer method, introduced by Fulton [7], is to use a theory of “pseudo-divisors”. These proofs, which involve a considerable amount of machinery from algebraic geometry, can be found in [7] and [22]; they use a pullback to the blow-up of an ideal, the general theory for the resulting divisors, and properties of proper morphisms of schemes.

Algebraically, we have the following: Let f be an element of A . Intersection with the divisor (f) gives a map from the Chow group of A to the Chow group of the hypersurface determined by f . It is denoted by $(f) \cap -$, and maps a generator $[A/\mathfrak{p}]$ to the class $[A/(\mathfrak{p}, f)]$ if f is not an element of \mathfrak{p} , and to zero otherwise. If g is another element of A , then

$$(f) \cap (g) \cap [A/\mathfrak{p}] = (g) \cap (f) \cap [A/\mathfrak{p}].$$

When (f) and (g) intersect properly there is a straightforward proof of this equality using Koszul homology. However, if the codimension of the ideal generated by f and g is one, then the situation is more difficult. For many years the only known proofs were the ones mentioned in [7] and [22].

Recently, Paul Roberts (University of Utah) and I collaborated on a purely algebraic proof of the commutativity of intersection with divisors. Our argument involves a reduction to the case of a normal domain. The statement of the main theorem, as well as the specific formula for the difference of the two cycles after the reduction, is given below.

Theorem 3.1. *Let f and g be elements of a Noetherian ring A , and let α be a cycle of dimension i . Then $(f) \cap (g) \cap \alpha$ and $(g) \cap (f) \cap \alpha$ are rationally equivalent in the group of cycles of A of dimension $i - 2$.*

Theorem 3.2. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the height one prime ideals of a Noetherian normal domain A containing f and g and let v_i be the valuation defined on $A_{\mathfrak{p}_i}$. For each i , there exist elements a_i and b_i of A not in \mathfrak{p}_i such that*

$$\frac{a_i}{b_i} = \frac{g^{v_i(f)}}{f^{v_i(g)}}.$$

Moreover, for any such pair, there is an equality of cycles

$$(f) \cap (g) \cap [A] - (g) \cap (f) \cap [A] = \sum_{i=1}^r \operatorname{div}(\mathfrak{p}_i, a_i/b_i).$$

This equality of cycles has an interpretation in K -theory. Basically, the formula shown above amounts to the assertion that the composition of the tame symbol and the div map in the Gersten complex is zero. More specifically, for any Noetherian domain R with field of fractions K , there is a complex [18]

$$K_2(K) \rightarrow K_1(K) \rightarrow K_0(K).$$

When R is normal the first map is the tame symbol [8] and the second map is div.

Although my focus thus far has been on injectivity and commutativity, the study of surjectivity of the intersection map has a rich history. The classical ‘‘Lefschetz-type’’ theorems state that $(f) \cap -$ is an isomorphism under certain conditions. Grothendieck, Danilov, and Srinivas [19], for example, have studied these types of problems from the standpoint of algebraic geometry. In [14], Lipman gives a purely algebraic proof of a theorem of Danilov-Samuel. One avenue of future research is investigating more purely algebraic proofs for ‘‘Lefschetz-type’’ results that have been established through methods of algebraic geometry, thereby providing a better understanding of the algebraic structure involved.

4. IDEAL FILTRATIONS

This project is joint work with Florian Enescu at Georgia State University and Cătălin Ciupercă at North Dakota State University. We begin with the classical situation where J and I are ideals in a Noetherian ring A such that J^p is contained in I^q , for some non-negative integers p and q . Obviously $J^{p'} \subseteq I^{q'}$ for q' less than q and p' greater than p , but what can be said about the supremum of q/p when J^p is contained in I^q ?

In the 1950's, Samuel [24] studied the sequence $\{v_I(J, m)/m\}$, where for each m , the numerator is the largest integer such that $I^{v_I(J, m)}$ contains J^m . Here I and J are two ideals in a commutative Noetherian ring A such that (i) the radical of I contains that of J , (ii) I and J are non-nilpotent, and (iii) the intersection of all the powers of I is zero. Samuel showed that the sequence has a limit $l_I(J)$, and conjectured that this value was a rational number. Nagata [17] and Rees [20] verified this conjecture using valuation theory.

Our main result is a generalization of the problem described above. Let J_1, \dots, J_k , and I be ideals in a locally analytically unramified ring A such that the conditions (i)-(iii) above hold for each pair I, J_i , and let $C \subseteq \mathbb{R}^{k+1}$ be the cone generated by the $(k+1)$ -tuples (m_1, \dots, m_k, n) in \mathbb{N}^{k+1} such that $J_1^{m_1} \dots J_k^{m_k} \subseteq I^n$. We prove that the topological closure of C is a rational polyhedral cone; i.e., a polyhedral cone bounded by hyperplanes whose equations have rational coefficients. [The case $k = 1$ follows from the results proved by Samuel, Nagata, and Rees.] An example in the three dimensional case is shown below.

Example 4.1. Let $A = \mathbb{R}[[X, Y, Z]]/(XY^2 - Z^9)$ and $I = (x, y, z)A$ as in [13, Example 3.1]. Set $J_1 = (x, z^2)$ and $J_2 = (y^2, z^3)$. The topological closure of the cone generated by the triples (m_1, m_2, n) satisfying $J_1^{m_1} J_2^{m_2} \subseteq I^n$ is shown from two different angles.

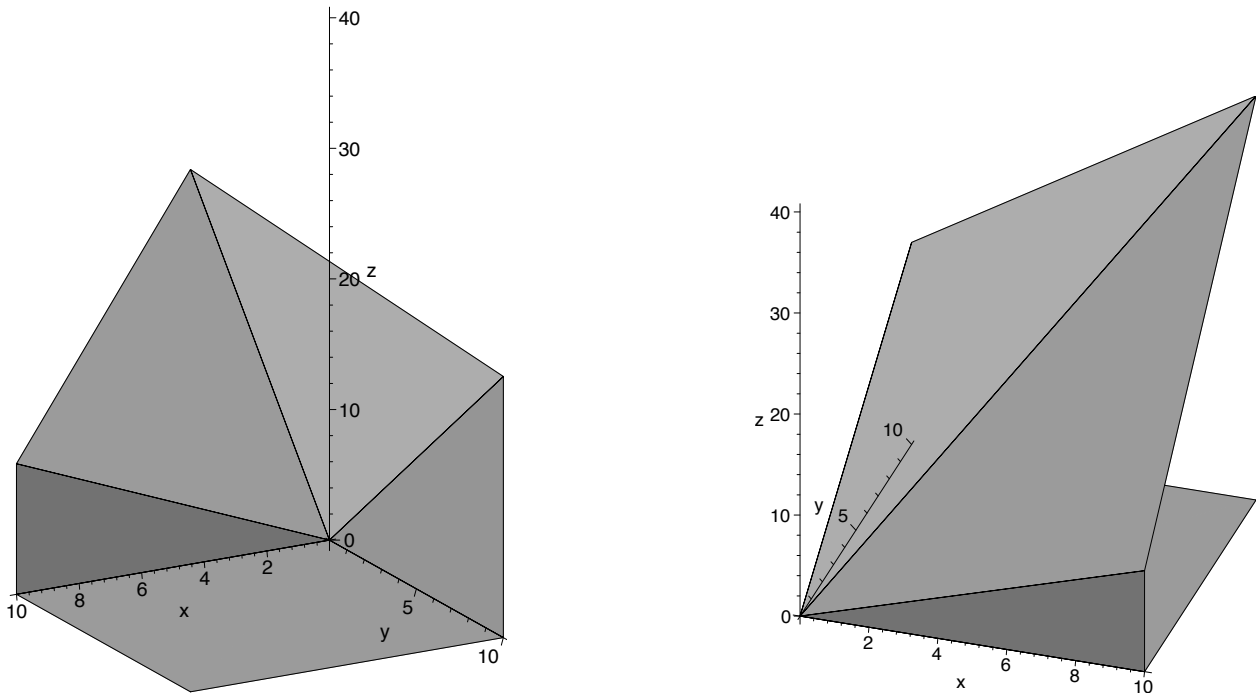


FIGURE 1. View into the cone and rotated 90° ctr-clockwise around the z -axis.

We also consider the question of whether the sequence $\{v_I(J, m)\}_m$ increases eventually in a periodic way; that is, whether or not there exists a positive integer t such that

$$v(m+t) - v(m+t-1) = v(m) - v(m-1)$$

for large enough m . We are motivated by Nagata [17, Theorem 8], since his result implies that there exists a positive constant c such that $0 \leq v(m+t) - v(m) - v(t) < c$ for all m, t .

Proposition 4.1. *Let I, J be ideals in a Noetherian local ring A such that conditions (i)-(iii) above are satisfied. Assume that J is principal and the ring $\bigoplus_{m,n} J^m \cap I^n$ is Noetherian. Then there exists a positive integer t such that $v(m+t) = v(m) + v(t)$ for all $m \geq t$.*

One possibility for future work on this topic involves cofinal filtrations. Much of Samuel's work with power filtrations extends to cofinal ones, but there are some important differences, like the fact that limits are no longer necessarily rational. Given two cofinal filtrations \mathcal{F} and \mathcal{G} , one can ask questions such as (a) if $l_{\mathcal{F}}(\mathcal{G})$ is rational, is there a power filtration $I = \{I^n\}$ such that $l_I(\mathcal{G}) = l_{\mathcal{F}}(\mathcal{G})$, and (b) if $l_{\mathcal{F}}(I)$ and $l_I(\mathcal{F})$ are both rational, then does there exist a power filtration $\{J^m\}$ equivalent to \mathcal{F} ? In other words, the aim is to understand when general filtrations can be approximated by power filtrations.

5. ZERO DIVISOR GRAPHS

This is joint work with Cameron Wickham at Missouri State University and elaborates on an idea first introduced by Mulay [16]. The *graph of equivalence classes of zero-divisors of a ring R* , which we will denote by $\text{EG}(R)$, is constructed from classes of zero divisors, rather than individual zero divisors themselves. (However, an edge between vertices x and y still represents the relation $x \cdot y = 0$.) This has many advantages over the zero-divisor graph $G(R)$, first introduced by Beck [1], or zero-ideal graphs. For example, in many cases $\text{EG}(R)$ is finite when $G(R)$ is infinite. The ring $S = \mathbb{Z}[X, Y]/(X^3, XY)$, in particular, has an infinite zero divisor graph, while $\text{EG}(S)$ has only four vertices. More specifically, although xy and nxy , where $n \in \mathbb{Z}$, are distinct zero divisors in S , they are all annihilated by the ideal (x, y) ; hence their "zero divisor activity" can be represented by a single vertex in $\text{EG}(S)$.

This analysis of the zero divisor activity of the ring is supported by another difference between the two graphs, namely the existence of complete diagrams. While much study has been made of complete zero divisor graphs, there are no complete equivalence class graphs with three or more vertices, since in that case every node would collapse to a single point.

A second important aspect of graphs of equivalence classes of zero divisors is the connection to associated primes of the ring. In the ring above, for example, (x, y) is an associated prime. More generally, in $\text{EG}(R)$ all of the associated primes are represented by distinct vertices. Furthermore, every vertex is connected to an associated prime of the ring. The study of associated primes in $\text{EG}(R)$ is one of our main motivations.

Questions about equivalence classes of zero divisors range from the very easy to the difficult, and consequently provide many opportunities for student research at the various levels. Some preliminary results are listed below:

Proposition 5.1. *Each of the following holds for $\text{EG}(R)$:*

- (1) *Any two associated primes are connected by an edge. Furthermore, every vertex is either an associated prime or a neighbor of an associated prime.*
- (2) *Every graph is connected with diameter less than or equal to three.*
- (3) *There are no complete or cycle graphs with three or more vertices.*

Theorem 5.2. *Let R be a Noetherian ring. If $\text{EG}(R)$ is finite with three or more vertices and the set of classes $\{[y_1], \dots, [y_r]\}$ is such that $\deg[y_i] \geq \deg[x]$ for all $[x] \in \text{EG}(R)$, then $[y_i]$ is an associated prime for each i .*

Proposition 5.3. *Let R be a Noetherian ring. Suppose $\text{ann}(x_0) \subsetneq \text{ann}(x_1) \subsetneq \dots \subsetneq \text{ann}(x_r)$ is a (necessarily finite) saturated chain of associated primes. If $\text{EG}(R)$ has three or more vertices and $\deg[x_r] < \infty$, then $\deg[x_0] < \deg[x_1] < \dots < \deg[x_r]$.*

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