AN ALGEBRAIC PROOF OF THE COMMUTATIVITY OF INTERSECTION WITH DIVISORS

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Abstract. We present a purely algebraic proof of the commutativity of the operation defined by intersection with divisors on the Chow group of a local Noetherian domain.

Introduction

The operation given by intersecting with a Cartier divisor is one of the basic ideas of Intersection Theory, and the fact that it defines a commutative operation in the Chow group is fundamental in making the theory work. If the intersection is proper, that is, if one intersects a divisor D with a variety W not contained in D, this concept is quite simple. However, if W is contained in D, then even the basic definition is considerably more complicated. A classical approach to this question is to use a "moving lemma" to move D to another divisor which meets W properly, while a newer method, introduced by Fulton ([1]), is to use a theory of "pseudo-divisors". However, in the case of a local noetherian ring, such an intersection must always be zero, and one can give a simple definition in general. In spite of this, there has been no proof of the crucial property that this operation is commutative in the case of local rings that did not involve the general definition, as well as a considerable amount of machinery from Algebraic Geometry. Proofs of this fact can be found in [1] and [6]; they use a pullback to the blow-up of an ideal, the general theory for the resulting divisors, and properties of proper morphisms of schemes.

Our aim is to give an algebraic proof for this purely algebraic statement. In the next section we give some background information as well as precise definitions and a statement of our main result. The following sections reduce the problem to normal domains and give a more detailed statement of the theorem. We then prove the theorem in a special case, and finally give a proof of the general theorem, inducting on the number of height one primes contained in the intersection. (If the intersection has codimension two, the proof of the result is easy).

In our proof we give an explicit formula for the difference between the intersections with two divisors taken in different orders as a sum of divisors of rational functions (see Theorem 3.1 in Section 3). This formula has been discovered previously in different contexts. First, it has an interpretation in K-theory. Basically, the formula given in (6) below amounts to the assertion that the composition of the tame symbol and the div map in the Gersten complex is zero. More specifically, for any

1991 *Mathematics Subject Classification.* Primary 14C17, 14C20.
The first author was supported in part by NSF grant 0500588.

Noetherian domain R with field of fractions K, there is a complex [5]

$$K_2(K) \to \sum_{\mathrm{ht}(\mathfrak{p})=1} \kappa(\mathfrak{p})^\times \to \sum_{\mathrm{ht}(\mathfrak{q})=2} \mathbb{Z},$$

and when R is normal the first map is the tame symbol [2]

$$\{\alpha, \beta\} \mapsto \sum_{\mathfrak{p}} (-1)^{\nu_\mathfrak{p}(\alpha)\nu_\mathfrak{p}(\beta)} \cdot \frac{\alpha^{\nu_\mathfrak{p}(\beta)}}{\beta^{\nu_\mathfrak{p}(\alpha)}}$$

and the second map is div. That the Gersten complex is exact when R is the localization of a finite type smooth k-algebra at a prime [5] leads to Bloch's formula (where $d = \dim(R)$)

$$H^2_{Zar}(X, \tilde{K_2}) \cong \mathrm{CH}_{d-2}(X).$$

The formula of Theorem 3.1 was also used by Kresch [3] to give a more canonical geometric proof of the commutativity that we prove here by algebraic means.

1. Preliminaries

We assume throughout that A is a Noetherian ring. In order to make intersection theory work it is necessary to assume a few further properties that hold in most situations that arise naturally. First, we assume that there is a good definition of dimension; that is, for all prime ideals $\mathfrak{p}$ the dimension of $A/\mathfrak{p}$ is defined and that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals such that $\mathfrak{p} \subset \mathfrak{q}$ and there are no prime ideals strictly between $\mathfrak{p}$ and $\mathfrak{q}$, then $\dim A/\mathfrak{p} = \dim A/\mathfrak{q} + 1$. The other condition we assume is that for all $\mathfrak{p}$, the normalization of $A/\mathfrak{p}$ in its quotient field is a finitely generated $A/\mathfrak{p}$-module. In particular, an excellent ring satisfies these properties. For more details on these assumptions, we refer to [1, Ch. 2] and [6, Ch. 8].

If M is a module of finite length, we denote its length $\ell(M)$. Let $Z_i(A)$ be the free abelian group with basis consisting of all prime ideals $\mathfrak{q}$ such that the dimension of $A/\mathfrak{q}$ is i. The elements of $Z_i(A)$ are called *cycles* of dimension i, and the basis element corresponding to $A/\mathfrak{q}$ is denoted $[A/\mathfrak{q}]$.

Definition 1.1. Let $\mathfrak{p}$ be a prime ideal such that $\dim A/\mathfrak{p} = i+1$ and x an element of A which is not in $\mathfrak{p}$. The cycle

$$\sum \ell_{A_\mathfrak{q}}(A_\mathfrak{q}/(\mathfrak{p}, x)A_\mathfrak{q})[A/\mathfrak{q}],$$

where the sum is taken over all $\mathfrak{q} \in \mathrm{Spec}(A)$ such that $\dim(A/\mathfrak{q}) = i$, is denoted by **div($\mathfrak{p}$, x)**, or occasionally $\mathrm{div}(A/\mathfrak{p}, x)$.

Definition 1.2. **Rational equivalence** is the equivalence relation on $Z_i(A)$ generated by setting $\mathrm{div}(\mathfrak{p}, x) = 0$ for all such primes $\mathfrak{p}$ and elements x. We remark that if x and y are not in $\mathfrak{p}$, then $\mathrm{div}(\mathfrak{p}, xy) = \mathrm{div}(\mathfrak{p}, x) + \mathrm{div}(\mathfrak{p}, y)$, and thus for any element x/y in the fraction field of $A/\mathfrak{p}$, we can define $\mathrm{div}(\mathfrak{p}, x/y) = \mathrm{div}(\mathfrak{p}, x) - \mathrm{div}(\mathfrak{p}, y)$.

Definition 1.3. The i-th component of the Chow group of A, denoted by $\mathrm{CH}_i(A)$, is $Z_i(A)$ modulo rational equivalence. The **Chow group** of A, denoted by $\mathrm{CH}_*(A)$, is obtained by taking the direct sum of $\mathrm{CH}_i(A)$ for all i. Similarly, the group of cycles $Z_*(A)$ is the direct sum of the $Z_i(A)$.

Definition 1.4. The intersection of a principal divisor (u), where u is an element in A, is a map $Z_*(A) \to Z_*(A/uA)$. It is denoted by $(u) \cap -$ and referred to as **intersection with (u)**. On a basis element $[A/\mathfrak{p}]$ it is defined by

$$(u) \cap [A/\mathfrak{p}] = \begin{cases} 0 & \text{if } u \in \mathfrak{p} \\ \\ \operatorname{div}(\mathfrak{p}, u) & \text{if } u \notin \mathfrak{p} \end{cases}$$

If $\alpha = \sum n_i [A/\mathfrak{p}_i]$ is an arbitrary cycle, it follows from the above definitions that

$$(u) \cap \alpha = \sum_{u \notin \mathfrak{p}_i} n_i \operatorname{div}(\mathfrak{p}_i, u).$$

We note that if $u \notin \mathfrak{p}$, then $(u) \cap [A/\mathfrak{p}]$ is by definition rationally equivalent to zero in the Chow group of A, but it is generally not rationally equivalent to zero in the Chow group of A/uA.

Our main theorem is the following:

Theorem 1.5. *Let u and v be elements of the ring A, and let $\alpha \in Z_i(A)$. Then the cycles $(u) \cap (v) \cap \alpha$ and $(v) \cap (u) \cap \alpha$ are rationally equivalent in $Z_{i-2}(A/(u, v))$.*

One of the main consequences of the theorem is that intersection with (u) defines an operation from the Chow group of A to the Chow group of A/uA.

Corollary 1.6. *The mapping on cycles that sends α to $(u) \cap \alpha$ induces a mapping from $CH_*(A)$ to $CH_*(A/uA)$.*

Proof. We must show that for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and any $x \notin \mathfrak{p}$, the cycle $(u) \cap \operatorname{div}(\mathfrak{p}, x)$ is rationally equivalent to zero as a cycle in $\operatorname{Spec}(A/uA)$. By Theorem 1.5, we have

$$(u) \cap \operatorname{div}(\mathfrak{p}, x) = (u) \cap (x) \cap [A/\mathfrak{p}] = (x) \cap (u) \cap [A/\mathfrak{p}].$$

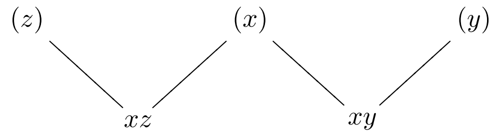
Let $(u) \cap [A/\mathfrak{p}] = \sum n_i [A/\mathfrak{q}_i]$. Then each $\mathfrak{q}_i$ contains u, so we may consider the $\mathfrak{q}_i$ to be prime ideals in A/uA. We thus have

$$(x) \cap (u) \cap [A/\mathfrak{p}] = \sum_{x \notin \mathfrak{q}_i} n_i \operatorname{div}(\mathfrak{q}_i, x),$$

which is clearly rationally equivalent to zero in the Chow group of A/uA. □

We remark that Theorem 1.5 is very easy to prove when the ideal generated by u and v in $A/\mathfrak{p}$ has height two and $\alpha = [A/\mathfrak{p}]$; in this case the two cycles are in fact equal, not just rationally equivalent. To illustrate the general situation, we give an example where two elements intersect in codimension one.

Example 1.1. Let $A = k[x, y, z]$, where k is a field. We consider the intersections with the divisors defined by the elements xz and xy. The following diagram shows the height one prime ideals that contain these elements.



By Definition 1.4,

$$(xz) \cap (xy) \cap [A] = (xz) \cap ([A/xA] + [A/yA]) = [A/(x,y)] + [A/(y,z)],$$

and

$$(xy) \cap (xz) \cap [A] = (xy) \cap ([A/xA] + [A/zA]) = [A/(x,z)] + [A/(y,z)].$$

Clearly these cycles are not equal. However, $(xz) \cap (xy) \cap [A] - (xy) \cap (xz) \cap [A] = \operatorname{div}((x), y/z)$, so they are rationally equivalent in $Z_1(A/(xy,xz))$.

In closing this section, we provide a statement of the Approximation Theorem [4, 12.6] since it is instrumental to our calculations.

Approximation Theorem: *Let K be the field of fractions of a Krull domain A. Given any set of height one primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in$ Spec(A) with corresponding discrete valuations $v_{\mathfrak{p}_i}$, and given integers $n_1, \ldots, n_r$, there is an element $x \in K^*$ such that $v_{\mathfrak{p}_i}(x) = n_i$ with $v_{\mathfrak{q}}(x) \geq 0$ for all $\mathfrak{q} \neq \mathfrak{p}_i$.*

2. Reduction to the case of a two-dimensional normal domain

We first note that since we are proving a result for elements of the group of cycles, we can assume our element is a generator; that is, a cycle $[A/\mathfrak{p}]$ for some prime ideal $\mathfrak{p}$. Since the support of the cycles under consideration lie in $\operatorname{Spec}(A/\mathfrak{p})$ we can then assume that $\mathfrak{p} = 0$ and we are dealing with $[A]$ for an integral domain A.

As a first step in reducing to the case in which A is a two-dimensional local domain, we prove the following lemma.

Lemma 2.1. *Let A be a one-dimensional local domain with maximal ideal $\mathfrak{m}$, and let x be a nonzero element of A. For a finitely generated A-module M, let $\chi(M) = \ell(M/xM) - \ell({}_xM)$, where ${}_xM = \{m \in M | xm = 0\}$. Then*

$$\chi(M) = \ell(A/xA)(rank(M)).$$

Proof. The lengths involved are finite, and both sides of the equation are additive on short exact sequences. Thus, by taking a filtration of M, we can reduce to the cases where $M = A$ or $M = A/\mathfrak{m}$. For $M = A$ both sides are equal to the length of A/xA, and for $M = A/\mathfrak{m}$ both sides are zero. □

Now let A and B be integral domains, let B be a finite extension of A, and let Φ be the induced map from $\operatorname{Spec}(B)$ to $\operatorname{Spec}(A)$. We define a map Φ_* from cycles on B to cycles on A by letting

$$\Phi_*([B/\mathfrak{P}]) = [\kappa(\mathfrak{P}) : \kappa(\mathfrak{p})][A/\mathfrak{p}],$$

where $\mathfrak{p} = A \cap \mathfrak{P}$. Here $[\kappa(\mathfrak{P}) : \kappa(\mathfrak{p})]$ denotes the degree of the extension of residue fields, which is finite since B is a finite extension of A. The next lemma is a special case of the projection formula in intersection theory.

Lemma 2.2. *Let $A \subset B$ be as above, and let u be a nonzero element of A (and thus also of B). Then for any cycle η on B, the cycles $\Phi_*((u)\cap\eta)$ and $(u)\cap(\Phi_*(\eta))$ are equal.*

Proof. It suffices to prove the result for a cycle of the form $[B/\mathfrak{P}]$, and in addition we may assume that $\mathfrak{P} = 0$. (If $u \in \mathfrak{P}$, then both cycles are zero.) The cycle $\Phi_*([B])$ is $r[A]$, where r is the rank of B as an A-module. Thus if $\mathfrak{q}$ is a height one prime of A containing u, the coefficient of $[A/\mathfrak{q}]$ in $(u) \cap (\Phi_*([B]))$ is $\ell(A_\mathfrak{q}/uA_\mathfrak{q})$ times r, and r is also the rank of $B_\mathfrak{q}$ over $A_\mathfrak{q}$. By Lemma 2.1, this is equal to the length of $B_\mathfrak{q}/uB_\mathfrak{q}$ as an $A_\mathfrak{q}$ module (since in this case there are no nonzero elements annihilated by u). By taking a filtration of $B_\mathfrak{q}/uB_\mathfrak{q}$ with quotients $B_\mathfrak{q}/\mathfrak{Q}B_\mathfrak{q}$ for primes $\mathfrak{Q}$ containing u, we get

$$\ell_{A_\mathfrak{q}}(B_\mathfrak{q}/uB_\mathfrak{q}) = \sum_{\mathfrak{Q}}[\kappa(\mathfrak{Q}) : \kappa(\mathfrak{q})]\ell(B_\mathfrak{Q}/uB_\mathfrak{Q}).$$

The right hand side of this equation is the coefficient of $[A/\mathfrak{q}]$ in $\Phi_*((u) \cap [B])$, so this proves the lemma. □

We also need the following result, which is a special case of "proper push-forward" of cycles. If the field L is a finite extension of a field K, we denote the norm from L to K by $N_{L/K}$; recall that $N_{L/K}(x)$ is the determinant of the map given by multiplication by x on L considered as a vector space over K.

Lemma 2.3. *Let A be a local one-dimensional domain.*

(1) *Let M be a finitely generated torsion-free A-module, and let ϕ be an A-endomorphism of M such that $Coker(\phi)$ has finite length. Let K be the quotient field of A, and let $k = a/b$ be the determinant of the induced endomorphism on $M \otimes K$, where a and b are in A. Then*

$$\ell(Coker(\phi)) = \ell(A/aA) - \ell(A/bA).$$

(2) *Let B be an integral domain containing A that is a finitely generated A-module, and set L and K to be their quotient fields, respectively. Let k be an element of L, and let Φ_* be defined as above. Then*

$$\Phi_*(\mathrm{div}(B, k)) = \mathrm{div}(A, N_{L/K}(k)).$$

Proof. To prove (1), let $\overline{A}$ be the integral closure of A in K, which we are assuming is a finitely generated A-module, and let $\overline{M}$ be the $\overline{A}$-module generated by M in $M \otimes_A K$. Then ϕ extends to an endomorphism of $\overline{M}$ and thus also to an endomorphism of $\overline{M}/M$, which has finite length. An application of the Snake Lemma shows that the length of the cokernel of ϕ on M is equal to the length of the cokernel of its extension to $\overline{M}$ (we note that since Coker(ϕ) has finite length and M is torsion-free, ϕ is injective). Similarly, the lengths of A/aA and A/bA are equal to the lengths of $\overline{A}/a\overline{A}$ and $\overline{A}/b\overline{A}$. Thus we may assume that A is integrally closed in its quotient field so is a semi-local Dedekind domain. In this case A is a principal ideal domain, so we can diagonalize ϕ and the result is clear.

It suffices to prove (2) for $k = b \in B$, and from part (1) it suffices to show that for $\mathfrak{p} \in \mathrm{Spec}(A)$ of height one, the length of $B_{\mathfrak{p}}/bB_{\mathfrak{p}}$ is equal to

$$\sum_{\mathfrak{P}} [\kappa(\mathfrak{P}) : \kappa(\mathfrak{p})] \ell_{B_{\mathfrak{P}}}(B_{\mathfrak{P}}/bB_{\mathfrak{P}}),$$

where the sum is taken over all $\mathfrak{P}$ lying over $\mathfrak{p}$. This formula follows immediately from taking a filtration of $B_{\mathfrak{p}}/bB_{\mathfrak{p}}$ with quotients of the form $B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$. □

Theorem 2.4. *(Reduction to the normal case). Let u, v be elements of an integral domain A of dimension d, and let B be the normalization of A in its quotient field. If $(u) \cap (v) \cap [B]$ and $(v) \cap (u) \cap [B]$ are rationally equivalent in $Z_{d-2}(B/(u, v))$, then $(u) \cap (v) \cap [A]$ and $(v) \cap (u) \cap [A]$ are rationally equivalent in $Z_{d-2}(A/(u, v))$.*

Proof. Let $\mathfrak{P}_i$ be the height one prime ideals of B in the support of (u, v), and let $\mathfrak{p}_i$ be their intersections with A; we note that the $\mathfrak{p}_i$ are exactly the height one primes of A that contain (u, v). Let k_i be rational functions on $B/\mathfrak{P}_i$ such that we have an equality of cycles

$$(u) \cap (v) \cap [B] - (v) \cap (u) \cap [B] = \sum \mathrm{div}(\mathfrak{P}_i, k_i). \tag{5}$$

Now from Lemma 2.2, we have

$$\Phi_*((u) \cap (v) \cap [B]) = (u) \cap \Phi_*((v) \cap [B]) = (u) \cap (v) \cap \Phi_*([B]),$$

and similarly

$$\Phi_*((v) \cap (u) \cap [B]) = (v) \cap (u) \cap \Phi_*([B]).$$

Since $\Phi_*([B]) = [A]$ (as B is finitely-generated over A), applying Φ_* to the left hand side of equation (5) gives

$$(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A].$$

On the other hand, if we apply Φ_* to the right hand side, by Lemma 2.3 we obtain

$$\sum_{\mathfrak{P}_i} \mathrm{div}(\mathfrak{p}_i, N_{\kappa(\mathfrak{P}_i)/\kappa(\mathfrak{p}_i)}(k_i)).$$

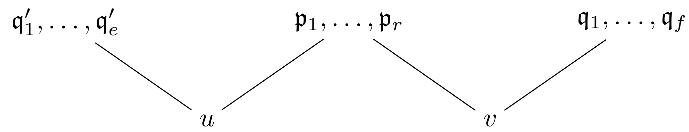
This shows that $(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A]$ is rationally equivalent to zero in $Z_{d-2}(A/(u, v))$. □

In summary, we may assume that A is a normal domain and that the cycle we are intersecting is $[A]$. The reduction to dimension two occurs in the next section.

3. A formula for the cycle $(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A]$

We begin by setting up the general situation we will be considering and then give a formula for the difference of the cycles in terms of elements of the form $\mathrm{div}(\mathfrak{p}_i, k_i)$, for rational functions k_i on $A/\mathfrak{p}_i$. The remainder of the paper is devoted to proving the formula.

Our situation is depicted below:



All of the prime ideals shown are height one primes of A, and the $\mathfrak{q}'_k$, $\mathfrak{p}_i$, and $\mathfrak{q}_l$ are those primes that contain only u, both u and v, and only v, respectively. Since A is a normal domain, the localization at every height one prime $\mathfrak{p}$ is a discrete valuation ring and defines a valuation $\nu_{\mathfrak{p}}$. We let the orders of u and v at the primes displayed above be as follows:

$$\nu_{\mathfrak{q}'_k}(u) = s_k \qquad \nu_{\mathfrak{p}_i}(u) = n_i \qquad \nu_{\mathfrak{p}_i}(v) = m_i \qquad \nu_{\mathfrak{q}_l}(v) = t_l$$

If A has dimension d, the prime ideals $\mathfrak{p}$ with $\dim(A/\mathfrak{p}) = d-1$ in $A/(u,v)$ are the images of the $\mathfrak{p}_i$. Hence to show that the cycle is rationally equivalent to zero, we must show that it is a sum of cycles of the form $\operatorname{div}(\mathfrak{p}_i, k_i)$. Our main theorem (a more detailed statement of Theorem 1.5) gives the rational functions that make this work.

Theorem 3.1. *Let $\mathfrak{p}_i$ be the height one prime ideals of a Noetherian normal domain A containing $u, v \in A$ as above. If, for each i between 1 and r, the pair of elements $a_i, b_i \in A$ is not in $\mathfrak{p}_i$ and satisfies*

$$\frac{a_i}{b_i} = \frac{v^{n_i}}{u^{m_i}},$$

then there is an equality of cycles

$$(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A] = \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, a_i/b_i). \tag{6}$$

By the Approximation Theorem, there always exists elements a_i and b_i in $A \backslash \mathfrak{p}_i$ such that $a_i/b_i = v^{n_i}/u^{m_i}$. In the course of the proof we will give a particular choice of a_i and b_i, but we note that the cycle on the right is independent of the choice as long as the elements satisfy the hypotheses of the theorem; i.e., $\operatorname{div}(\mathfrak{p}_i, a_i/b_i) = \operatorname{div}(\mathfrak{p}_i, c_i/d_i)$ whenever $a_i/b_i = c_i/d_i$ and $a_i, b_i, c_i, d_i \notin \mathfrak{p}_i$.

We also note that since this is an equality of cycles, it is enough to check that the coefficient of $[A/\mathfrak{m}]$ is the same for both sides of the equation for every prime ideal $\mathfrak{m}$ of height two. Thus, by localizing we may assume that A is a local normal domain of dimension two and that $\mathfrak{m}$ is its maximal ideal.

In summary then, to establish (6) we show the following equality:

$$\boxed{\sum_{l=1}^{f} t_l \ell(A/(\mathfrak{q}_l, u))[A/\mathfrak{m}] - \sum_{k=1}^{e} s_k \ell(A/(\mathfrak{q}'_k, v))[A/\mathfrak{m}] = \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, a_i/b_i),}$$

where we will often omit writing the basis element $[A/\mathfrak{m}]$ on the left hand side and use $\operatorname{div}(\mathfrak{p}_i, a_i/b_i)$ to denote the coefficient of $[A/\mathfrak{m}]$ on the right hand side.

4. First Step in the Induction Argument

From this point on, we assume that A is a local normal domain of dimension two and that the elements u, v of A intersect in codimension one. We remark that in the case where u and v generate a height two ideal, since we are assuming that A is a normal domain, u, v form a regular sequence and hence both $(u) \cap (v) \cap [A]$ and $(v) \cap (u) \cap [A]$ give the length of $A/(u, v)$. As a result, the right hand side of equation (6) is zero. In the case we are considering, where u and v generate a height one ideal, $A/(u, v)$ no longer has finite length, but this quotient, or more precisely a subquotient, is still the starting point for the computation.

In this section we prove the special case where $m_i = n_i$ for each i, which, as we show below, implies the case of a single prime. We recall that $m_i = \nu_{\mathfrak{p}_i}(v)$ and $n_i = \nu_{\mathfrak{p}_i}(u)$, so the assumption says that u and v have the same order for each $\mathfrak{p}_i$. As a result, only one pair of elements $a, b \in A$ is necessary. In the next section we will prove the general case by using this one.

Theorem 4.1. *Let $\mathfrak{p}_i$ and u, v be as in 3.1. If $n_i = m_i$ for all i, and a and b are elements of A not in any of the $\mathfrak{p}_i$ such that $a/b = v/u$, then we have an equality of cycles*

$$(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A] = \sum_{i=1}^{r} \operatorname{div}\left(\mathfrak{p}_i, \frac{a^{n_i}}{b^{n_i}}\right).$$

Proof. Let $P = \cap_{i=1}^{r} \mathfrak{p}_i^{(n_i)}$. Then u and v are in the ideal P and, since $\nu_{\mathfrak{p}}(u) = \nu_{\mathfrak{p}}(P)$ or $\nu_{\mathfrak{p}}(v) = \nu_{\mathfrak{p}}(P)$ for all height one prime ideals $\mathfrak{p}$ of A, $P/(u, v)A$ is a module of finite length. Our proof consists of expressing the length of this module in different ways.

Let $Q = \cap_{l=1}^{f} \mathfrak{q}_l^{(t_l)}$ and $Q' = \cap_{k=1}^{e} {\mathfrak{q}'_k}^{(s_k)}$. We note that $Q \cap P = vA$ and $Q' \cap P = uA$.

We claim that we have a short exact sequence

$$0 \to A/(Q + P) \xrightarrow{u} P/(vA + uP) \to P/(u, v)A \to 0.$$

To see this, we note that if $a \in Q$, then $ua \in vA$, so multiplication by u does take $Q + P$ to $vA + uP$. Conversely, if $ua = va' + up$, for $p \in P$, then $u(a - p) \in vA$. This happens exactly when $a - p \in Q$, which implies that $a \in Q + P$. It is clear that the image of this map is $(u, v)A/(vA + uP)$, so exactness at the other places holds.

Interchanging u and v yields a similar short exact sequence. Combining these, we deduce that

$$\ell(P/(vA + uP)) - \ell(A/(Q + P)) = \ell(P/(uA + vP)) - \ell(A/(Q' + P)).$$

Consider the term $\ell(P/(vA + uP))$. The height one prime ideals in the support of P/vA are the $\mathfrak{q}_l$. Since u is not contained in any of these, we determine that multiplication by u on P/vA is injective; its cokernel is $P/(vA+uP)$. Furthermore, since P/vA has a filtration with quotients $A/\mathfrak{q}_l$ of multiplicity t_l, we obtain

$$\ell(P/(vA + uP)) = \sum_{l=1}^{f} t_l \ell(A/(\mathfrak{q}_l, u)).$$

Similarly, we have

$$\ell(P/(uA+vP)) = \sum_{k=1}^{e} s_k \ell(A/(\mathfrak{q}'_k, v)).$$

Combining these terms, we obtain

$$\sum_{l=1}^{f} t_l \ell(A/(\mathfrak{q}_l, u)) - \sum_{k=1}^{e} s_k \ell(A/(\mathfrak{q}'_k, v)) = \ell(P/(vA+uP)) - \ell(P/(uA+vP)),$$

and from the previous equation this difference is equal to

$$\ell(A/(Q+P)) - \ell(A/(Q'+P)).$$

It now remains to prove that if we have a and b not in $\mathfrak{p}_i$ for any i with $a/b = v/u$, then

$$\ell(A/(Q+P)) - \ell(A/(Q'+P)) = \sum_{i=1}^{r} \operatorname{div}\left(\mathfrak{p}_i, \frac{a^{n_i}}{b^{n_i}}\right).$$

From the Approximation Theorem, we can find an element $a \in A$ such that a avoids all the $\mathfrak{p}_i$ and $\mathfrak{q}'_k$, but $\nu_{\mathfrak{q}_l}(a) = \nu_{\mathfrak{q}_l}(v) = t_l$ for all l. Additionally, we might have $a \in J_h$ for a finite collection of height one primes J_h. Let λ_h be the order of a in J_h. Set $b = ua/v$. Then $b \in A$. In particular, b avoids every $\mathfrak{p}_i$ and $\mathfrak{q}_l$, $\nu_{\mathfrak{q}'_k}(b) = \nu_{\mathfrak{q}'_k}(u) = s_k$ for every k, and $\nu_{J_h}(b) = \lambda_h$ for all h.

Let K be the quotient field of A. We next consider the composition of multiplications,

$$P^{(-1)}/A \xrightarrow{u} A/P \xrightarrow{a} A/P,$$

where $P^{(-1)} = \cap_{i=1}^{r} \mathfrak{p}_i^{(-n_i)}$ and $\mathfrak{p}_i^{(-n_i)} = \{x \in K : \nu_{\mathfrak{p}_i}(x) \geq -n_i\}$.

The kernel-cokernel exact sequence gives us a short exact sequence

$$0 \to \operatorname{Coker}(u) \to \operatorname{Coker}(ua) \to \operatorname{Coker}(a) \to 0.$$

The first cokernel is $A/(P + uP^{(-1)}) = A/(P+Q')$. The length of the cokernel of multiplication by a on A/P is, by looking at a filtration of A/P with quotients of the form $A/\mathfrak{p}_i$,

$$\sum_{i=1}^{r} n_i \ell(A/(\mathfrak{p}_i, a)) = \sum_{i=1}^{r} n_i \operatorname{div}(\mathfrak{p}_i, a).$$

Thus the above short exact sequence gives

$$\ell(\operatorname{Coker}(ua)) = \ell(A/(P+Q')) + \sum_{i=1}^{r} n_i \operatorname{div}(\mathfrak{p}_i, a) = \ell(A/(P+Q')) + \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, a^{n_i}).$$

A similar computation gives

$$\ell(\operatorname{Coker}(vb)) = \ell(A/(P+Q)) + \sum_{i=1}^{r} n_i \operatorname{div}(\mathfrak{p}_i, b) = \ell(A/(P+Q)) + \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, b^{n_i}).$$

Since $ua = vb$, we obtain

$$\ell(A/(P+Q)) - \ell(A/(P+Q')) = \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, a^{n_i}) - \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, b^{n_i}) = \sum_{i=1}^{r} \operatorname{div}(\mathfrak{p}_i, \frac{a^{n_i}}{b^{n_i}}),$$

which proves the theorem. □

As mentioned, the above argument implies the case in which there is only one height one prime $\mathfrak{p}$ over (u, v). This will establish the first step in the induction argument.

Corollary 4.2. *With the same hypotheses of 3.1 and $r = 1$, we have an equality of cycles*

$$(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A] = \operatorname{div}\left(\mathfrak{p}, \frac{a}{b}\right).$$

We apply the previous argument to u^m and v^n, where $\nu_{\mathfrak{p}}(u) = n$ and $\nu_{\mathfrak{p}}(v) = m$. In this case, $P = \mathfrak{p}^{(mn)}$, $Q = \cap_{l=1}^{f} \mathfrak{q}_l^{(mt_l)}$, $Q' = \cap_{k=1}^{e} {\mathfrak{q}'_k}^{(ns_k)}$, and $a/b = v^n/u^m$. The resulting equality of cycles is

$$(u^m) \cap (v^n) \cap [A] - (v^n) \cap (u^m) \cap [A] = \operatorname{div}\left(\mathfrak{p}, \frac{a^{mn}}{b^{mn}}\right),$$

which simplifies to the one shown.

There is another important application of Theorem 4.1. With the notation as above, the roles of the pairs $\{u, v\}$ and $\{a, b\}$ can be interchanged. Of course, as a result the ideals $\mathfrak{p}_i$ and J_h must also swap roles.

Corollary 4.3. *With the same notation as in the proof of 4.1, we have*

$$(b) \cap (a) \cap [A] - (a) \cap (b) \cap [A] = \sum_h \operatorname{div}\left(J_h, \frac{v^{\lambda_h}}{u^{\lambda_h}}\right).$$

5. The General Induction Argument

We are now in a position to prove the general result. In the previous section we established this result in two cases, and the condition that made these proofs possible was that the ratios n_i/m_i were the same for all i, or in the case of Corollary 4.2, that there was only one i. In the general case this will not hold. The general proof is by induction on the number of primes of height one containing (u, v). Since the ratios n_i/m_i and n_j/m_j are not necessarily the same for different i and j, the numbers $n_i m_j - m_i n_j$ will not all be zero, and this will effect our choice of a_i and b_i.

We now prove our theorem in general.

Proof. Assume that $r \geq 2$ and that the result holds when there are $r - 1$ primes. Specifically, our induction hypothesis is: *Given a pair of elements x and y in A that intersect in some* proper *subset $\mathcal{S}$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, there is an equality of cycles*

$$(x) \cap (y) \cap [A] - (y) \cap (x) \cap [A] = \sum_{\mathfrak{p}_i \in \mathcal{S}} \operatorname{div}\left(\mathfrak{p}_i, \frac{c_i}{d_i}\right),$$

for elements c_i, d_i not in $\mathfrak{p}_i$ such that $c_i/d_i = y^{\nu_{\mathfrak{p}_i}(x)}/x^{\nu_{\mathfrak{p}_i}(y)}$.

As in Corollary 4.2, we want to use the Approximation Theorem to find elements $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ such that for each i,

$$\frac{a_i}{b_i} = \frac{v^{n_i}}{u^{m_i}}.$$

After a possible reordering of the primes $\mathfrak{p}_i$, we may assume that

$$n_1/m_1 \geq n_2/m_2 \geq \cdots \geq n_r/m_r.$$

Let G be the integer such that $n_1/m_1 = n_2/m_2 = \cdots = n_G/m_G > n_{G+1}/m_{G+1}$; then $1 \leq G < r$. For $j \geq 1$, set $\alpha_j = n_1 m_j - m_1 n_j$. We have $\alpha_1 = \cdots = \alpha_G = 0$, and $\alpha_j > 0$ for $j \geq G+1$.

Using the Approximation Theorem, choose a_1 such that

$$\operatorname{div}(a_1) = \sum_{j=G+1}^{r} \alpha_j [A/\mathfrak{p}_j] + \sum_{l=1}^{f} n_1 t_l [A/\mathfrak{q}_l] + \sum_h \lambda_h [A/J_h],$$

where the J_h are a finite number of height one primes of A and $\lambda_h > 0$. Set $b_1 = \dfrac{u^{m_1} a_1}{v^{n_1}}$. Then $\operatorname{div}(b_1) = \sum_{k=1}^{e} m_1 s_k [A/\mathfrak{q}'_k] + \sum_h \lambda_h [A/J_h]$. It is important to note that a_1 and b_1 do not intersect on any of the original primes $\mathfrak{p}_j, \mathfrak{q}_l, \mathfrak{q}'_k$; they only intersect on the primes J_h and their orders are equal for each h. This is exactly the scenario of Corollary 4.3, using the relation $v^{n_1}/u^{m_1} = a_1/b_1$. The explicit formula from Corollary 4.3 is shown below.

$$(b_1) \cap (a_1) \cap [A] - (a_1) \cap (b_1) \cap [A] = \sum_h \operatorname{div}\left(J_h, \frac{v^{n_1 \lambda_h}}{u^{m_1 \lambda_h}}\right) \tag{7}$$

Moreover, a direct calculation shows it is also true that

$$(v^{n_1}) \cap (a_1) \cap [A] - (u^{m_1}) \cap (b_1) \cap [A] = \sum_h \operatorname{div}\left(J_h, \frac{v^{n_1 \lambda_h}}{u^{m_1 \lambda_h}}\right). \tag{8}$$

(Note that, on the left-hand side of (8), if we first intersect with a_1 or b_1, both of which are contained in some subset of the $\mathfrak{p}_j, \mathfrak{q}_l, \mathfrak{q}'_k$, and J_h, followed by intersection with v or u, the only elements that do not map to zero are the $[A/J_h]$.)

Lemma 5.1. $(u^{m_1}) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (u^{m_1}) \cap [A] =$

$$\begin{aligned} &(u^{m_1}) \cap (a_1) \cap [A] - (a_1) \cap (u^{m_1}) \cap [A] \\ &+(b_1) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (b_1) \cap [A] \\ &+(a_1) \cap (v^{n_1}) \cap [A] - (b_1) \cap (u^{m_1}) \cap [A]. \end{aligned}$$

Proof. We will use the following fact, for $x, y \in A : (x) \cap (y) \cap [A] = (x) \cap \operatorname{div}(y/x)$.

$$\begin{aligned}
&(u^{m_1}) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (u^{m_1}) \cap [A] \\
&= (u^{m_1}) \cap \operatorname{div}(v^{n_1}/u^{m_1}) - (v^{n_1}) \cap \operatorname{div}(u^{m_1}/v^{n_1}) \\
&= (u^{m_1}) \cap \operatorname{div}(a_1/b_1) - (v^{n_1}) \cap \operatorname{div}(b_1/a_1) \\
&= (u^{m_1}) \cap \operatorname{div}(a_1) - \underbrace{(u^{m_1}) \cap \operatorname{div}(b_1) + (v^{n_1}) \cap \operatorname{div}(a_1)}_{\text{equations (7),(8)}} - (v^{n_1}) \cap \operatorname{div}(b_1) \\
&= (u^{m_1}) \cap \operatorname{div}(a_1) + (b_1) \cap \operatorname{div}(a_1) - (a_1) \cap \operatorname{div}(b_1) - (v^{n_1}) \cap \operatorname{div}(b_1) \\
&= (u^{m_1}) \cap \operatorname{div}(a_1) + (b_1) \cap \operatorname{div}(a_1/b_1) - (a_1) \cap \operatorname{div}(b_1/a_1) - (v^{n_1}) \cap \operatorname{div}(b_1) \\
&= (u^{m_1}) \cap \operatorname{div}(a_1) + (b_1) \cap \operatorname{div}(v^{n_1}/u^{m_1}) - (a_1) \cap \operatorname{div}(u^{m_1}/v^{n_1}) - (v^{n_1}) \cap \operatorname{div}(b_1) \\
&= (u^{m_1}) \cap (a_1) \cap [A] - (a_1) \cap (u^{m_1}) \cap [A] + (b_1) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (b_1) \cap [A]
\end{aligned}$$

$$+(a_1)\cap(v^{n_1})\cap[A]-(b_1)\cap(u^{m_1})\cap[A].$$

□

Lemma 5.1 represents the difference $(u^{m_1})\cap(v^{n_1})\cap[A]-(v^{n_1})\cap(u^{m_1})\cap[A]$ as a sum of three terms, each of which is itself a difference of two terms. We will establish our theorem by computing each of these three differences and combining the results.

First, we need to find the remaining elements a_i, b_i for $2\le i\le r$. Again we use the Approximation Theorem, and always we set $b_i=\dfrac{u^{m_i}a_i}{v^{n_i}}$ once we have chosen a_i. The basic idea is that the element a_i will always be chosen in the $\mathfrak{q}_l$'s but never in the $\mathfrak{q}'_k$'s, and the pair a_i, b_i will never be contained in the same $\mathfrak{p}_j$'s. To be specific, we choose a_2 so that it is contained in $\mathfrak{p}_{G+1},\dots,\mathfrak{p}_r$, with (positive) orders $n_2m_{G+1}-m_2n_{G+1},\dots,n_2m_r-m_2n_r$, but is *not* contained in (1) $\mathfrak{p}_1,\dots,\mathfrak{p}_G$, (2) any of the $\mathfrak{q}'_k$, or (3) any of the J_h. In $\mathfrak{q}_l$ it will have order n_2t_l. We follow the same process for $a_3,\dots,a_G$, and note that none of $b_3,\dots,b_G$ is contained in any $\mathfrak{p}_j$. At the next step, the distribution of the $\mathfrak{p}_j$ will change: we choose a_{G+1} such that

$$\operatorname{div}(a_{G+1})=\sum_{j=G+2}^{r}(n_{G+1}m_j-m_{G+1}n_j)[A/\mathfrak{p}_j]+\sum_{l=1}^{f}n_{G+1}t_l[A/\mathfrak{q}_l]+\sum_h\mu_h[A/I_h],$$

where $n_{G+1}m_j-m_{G+1}n_j\ge 0$ and where the I_h are a finite number of height one primes of A different from all previous collections of height one primes. Note that

$$\operatorname{div}(b_{G+1})=\sum_{j=1}^{G}(m_{G+1}n_j-n_{G+1}m_j)[A/\mathfrak{p}_j]+\sum_{k=1}^{e}m_{G+1}s_k[A/\mathfrak{q}'_k]+\sum_h\mu_h[A/I_h],$$

where $m_{G+1}n_j-n_{G+1}m_j>0$. Note that b_{G+1} is contained in $\mathfrak{p}_1,\dots,\mathfrak{p}_G$ and no other $\mathfrak{p}_j$, while a_{G+1} is contained in some subset of $\mathfrak{p}_{G+2},\dots,\mathfrak{p}_r$. From here we continue in this way to obtain all of the elements a_i and b_i. (Note that a_r will not be contained in any of the $\mathfrak{p}_j$.)

It follows directly from the definitions of the b_j that for every j we have

$$\frac{b_1^{n_j}a_j^{n_1}}{b_j^{n_1}}=\frac{a_1^{n_j}}{u^{\alpha_j}}.$$

The first term in Lemma 5.1 involves the pair a_1, u^{m_1}, which intersects on the primes $\mathfrak{p}_{G+1},\dots,\mathfrak{p}_r$. Therefore, by the induction hypothesis,

$$\begin{aligned}(u^{m_1})\cap(a_1)\cap[A]-(a_1)\cap(u^{m_1})\cap[A]&=\sum_{j=G+1}^{r}m_1\operatorname{div}\left(\mathfrak{p}_j,\frac{a_1^{n_j}}{u^{\alpha_j}}\right)\\&=\sum_{j=G+1}^{r}m_1\operatorname{div}\left(\mathfrak{p}_j,\frac{b_1^{n_j}a_j^{n_1}}{b_j^{n_1}}\right)\\&=\sum_{j=G+1}^{r}m_1n_1\operatorname{div}\left(\mathfrak{p}_j,\frac{a_j}{b_j}\right)+\sum_{j=G+1}^{r}m_1n_j\operatorname{div}(\mathfrak{p}_j,b_1).\end{aligned}$$

To compute the second term in Lemma 5.1, we note that the pair v^{n_1}, b_1 is a regular sequence, so $(b_1)\cap(v^{n_1})\cap[A]-(v^{n_1})\cap(b_1)\cap[A]=0$.

In the third term of Lemma 5.1, we have

$$(a_1) \cap (v^{n_1}) \cap [A] = \sum_{j=1}^{G} m_j n_1 \operatorname{div}(\mathfrak{p}_j, a_1),$$

since $a_1 \in \mathfrak{p}_j$ for $j = G+1, \ldots, r$. We also have

$$(b_1) \cap (u^{m_1}) \cap [A] = \sum_{j=1}^{r} m_1 n_j \operatorname{div}(\mathfrak{p}_j, b_1)$$

since $b_1 \notin \mathfrak{p}_j$ for any j. Thus

$$(a_1) \cap (v^{n_1}) \cap [A] - (b_1) \cap (u^{m_1}) \cap [A] = \sum_{j=1}^{G} m_j n_1 \operatorname{div}(\mathfrak{p}_1, a_1) - \sum_{j=1}^{r} m_1 n_j \operatorname{div}(\mathfrak{p}_j, b_1).$$

Putting the three terms together, we have $(u^{m_1}) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (u^{m_1}) \cap [A] =$

$$\sum_{j=G+1}^{r} m_1 n_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_j}{b_j}\right) + \sum_{j=G+1}^{r} m_1 n_j \operatorname{div}\left(\mathfrak{p}_j, b_1\right)$$

$$+0$$

$$+\sum_{j=1}^{G} m_j n_1 \operatorname{div}(\mathfrak{p}_j, a_1) - \sum_{j=1}^{r} m_1 n_j \operatorname{div}(\mathfrak{p}_j, b_1).$$

The first sum in this expression is in the form we want. The remaining three sums combine to give

$$\sum_{j=1}^{G} (m_j n_1 \operatorname{div}(\mathfrak{p}_j, a_1) - m_1 n_j \operatorname{div}(\mathfrak{p}_j, b_1)). \tag{9}$$

We recall that we have $n_1 m_j = m_1 n_j$ for each $j = 1, \ldots, G$. Consequently, the expression in (9) can be written as

$$= \sum_{j=1}^{G} (m_j n_1 \operatorname{div}(\mathfrak{p}_j, a_1) - m_j n_1 \operatorname{div}(\mathfrak{p}_j, b_1))$$

$$= \sum_{j=1}^{G} m_j n_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_1}{b_1}\right).$$

In addition, it follows that

$$\left(\frac{v^{n_1}}{u^{m_1}}\right)^{m_j} = \left(\frac{v^{n_j}}{u^{m_j}}\right)^{m_1}$$

for each $j = 1, \ldots, G$. Since $a_j/b_j = v^{n_j}/u^{m_j}$ for each j, this implies that $(a_1/b_1)^{m_j} = (a_j/b_j)^{m_1}$, so

$$m_j \operatorname{div}\left(\mathfrak{p}_j, \frac{a_1}{b_1}\right) = m_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_j}{b_j}\right).$$

Thus we have

$$\sum_{j=1}^{G} m_j n_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_1}{b_1}\right) = \sum_{j=1}^{G} m_1 n_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_j}{b_j}\right).$$

Putting this together with the first term finally gives

$$(u^{m_1}) \cap (v^{n_1}) \cap [A] - (v^{n_1}) \cap (u^{m_1}) \cap [A] = \sum_{j=1}^{r} m_1 n_1 \operatorname{div}\left(\mathfrak{p}_j, \frac{a_j}{b_j}\right).$$

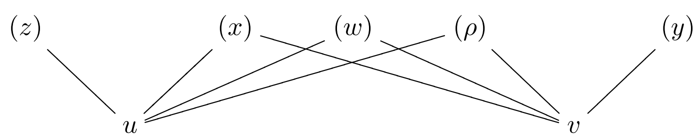
Dividing both sides of this equation by $m_1 n_1$ now gives

$$(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A] = \sum_{j=1}^{r} \operatorname{div}\left(\mathfrak{p}_j, \frac{a_j}{b_j}\right).$$

□

We close with an example which demonstrates our choice of a_i and b_i and the cancelation that occurs. In this instance, we have $r = 3$.

Example 5.1. Let $A = k[x, w, \rho, y, z]$, where k is a field. Let $u = x^2 w^3 \rho z^2$ and $v = x^4 w^6 \rho^3 y$ and set $\mathfrak{p}_1 = (x), \mathfrak{p}_2 = (w), \mathfrak{p}_3 = (\rho), \mathfrak{q}' = (z)$, and $\mathfrak{q} = (y)$.



Using Definition 1.4, one can calculate that

$$(u) \cap (v) \cap [A] = 2[A/(x, y)] + 3[A/(y, w)] + [A/(\rho, y)] + 2[A/(y, z)],$$

and

$$(v) \cap (u) \cap [A] = 8[A/(x, z)] + 12[A/(w, z)] + 6[A/(\rho, z)] + 2[A/(y, z)].$$

Then, $(u) \cap (v) \cap [A] - (v) \cap (u) \cap [A]$

$$= \operatorname{div}\left((x), y^2/z^8\right) + \operatorname{div}\left((w), y^3/z^{12}\right) + \operatorname{div}\left((\rho), y/z^6\right). \tag{10}$$

Using the ratios v^2/u^4, v^3/u^6, and v/u^3, choose $a_1 = \rho^2 y^2, b_1 = z^8, a_2 = \rho^3 y^3, b_2 = z^{12}, a_3 = y$, and $b_3 = x^2 w^3 z^6$. (In this case, no ideals J_h come into play; i.e., the pair a_1, b_1 is a system of parameters. Note that $\alpha_1 = \alpha_2 = 0$, and $\alpha_3 = 2$.) One can check that the expression in equation (10) is equal to

$$= \operatorname{div}((x), a_1/b_1) + \operatorname{div}((w), a_2/b_2) + \operatorname{div}\left((\rho), a_3/b_3\right).$$

Acknowledgement. *The authors would like to thank Mark Walker for pointing out the connection to K-theory and Bill Fulton for bringing to our attention the paper by Andrew Kresch. We would also like to thank the referee for suggesting some simplifications in the argument in section 5.*

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