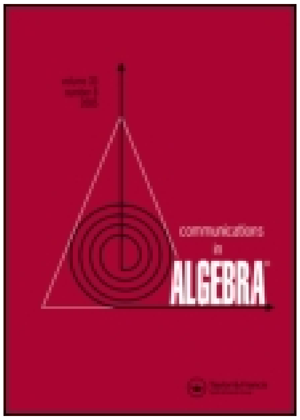

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A ZERO DIVISOR GRAPH DETERMINED BY EQUIVALENCE CLASSES OF ZERO DIVISORS

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We study the zero divisor graph determined by equivalence classes of zero divisors of a commutative Noetherian ring R. We demonstrate how to recover information about R from this structure. In particular, we determine how to identify associated primes from the graph.

Key Words: Annihilator ideal; Associated prime; Zero divisor graph.

2000 Mathematics Subject Classification: Primary 13A15; 13A99; 05C12.

INTRODUCTION

Beck first introduced the notion of a zero divisor graph of a ring R in 1988 [5] from the point of view of colorings. Since then, others have studied and modified these graphs, whose vertices are the zero divisors of R, and found various properties to hold. Inspired by ideas from Mulay in [10, §3], we study the *graph of equivalence classes of zero divisors of a ring R*, which is constructed from classes of zero divisors determined by annihilator ideals, rather than individual zero divisors themselves. It will be denoted by $\Gamma_E(R)$.

This graph has some advantages over the earlier zero divisor graph $\Gamma(R)$ in [2–5], or subsequent zero divisor graph determined by an ideal of R in [8, 11]. In many cases $\Gamma_E(R)$ is finite when $\Gamma(R)$ is infinite. For example, if $S = \mathbb{Z}[X, Y]/(X^3, XY)$, then $\Gamma(S)$ is an infinite graph, while $\Gamma_E(S)$ has only four vertices. To be specific, although $x^2, 2x^2, 3x^2, \ldots,$ are distinct zero divisors, where x denotes the image of X in S, they all have the same annihilator; they are represented by a single vertex in $\Gamma_E(S)$. In addition, there are no complete $\Gamma_E(R)$ graphs with three or more vertices since the graph would collapse to a single point. These are two ways in which $\Gamma_E(R)$ represents a more succinct description of the "zero divisor activity" in R.

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Another important aspect of graphs of equivalence classes of zero divisors is the connection to associated primes of the ring. For example, in the ring S above, the annihilator of x^2 is an associated prime. In general, all of the associated primes of a ring R correspond to distinct vertices in $\Gamma_E(R)$. Moreover, every vertex in a graph either corresponds to an associated prime or is connected to one. The study of the structure of associated primes in $\Gamma_E(R)$ is one of our main motivations.

In Section 1, we compare and contrast $\Gamma_E(R)$ with the more familiar $\Gamma(R)$ defined by D. Anderson and P. Livingston [2]. In Section 2, we consider infinite graphs and star graphs and answer the question of whether or not the Noetherian condition on R is enough to force $\Gamma_E(R)$ to be finite. Section 3 is devoted to the relation between the associated primes of R and the vertices of $\Gamma_E(R)$. In particular, we demonstrate how to identify some elements of $\mathrm{Ass}(R)$.

Throughout, R will denote a commutative Noetherian ring with unity, and all graphs are simple graphs in the sense that there are no loops or double edges.

1. DEFINITIONS AND BASIC RESULTS

Let $Z^*(R)$ denote the zero divisors of R and $Z(R) = Z^*(R) \cup \{0\}$. For $x, y \in R$, we say that $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$. As noted in [10], $\sim$ is an equivalence relation. Furthermore, if $x_1 \sim x_2$ and $x_1 y = 0$, then $y \in \mathrm{ann}(x_1) = \mathrm{ann}(x_2)$ and, hence, $x_2 y = 0$. It follows that multiplication is well defined on the equivalence classes of $\sim$; that is, if $[x]$ denotes the class of x, then the product $[x] \cdot [y] = [xy]$ makes sense. Note that $[0] = \{0\}$ and $[1] = R - Z(R)$; the other equivalence classes form a partition of $Z^*(R)$.

Definition 1.1. The *graph of equivalence classes of zero divisors of a ring* R, denoted $\Gamma_E(R)$, is the graph associated to R whose vertices are the classes of elements in $Z^*(R)$, and with each pair of distinct classes $[x]$, $[y]$ joined by an edge if and only if $[x] \cdot [y] = [0]$.

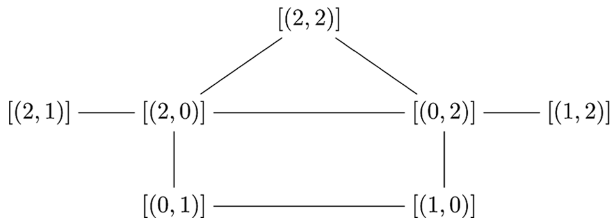
In [10], Mulay first defines $\Gamma_E(R)$ using different terminology and asserts several properties of $\Gamma_E(R)$, in particular, that $\Gamma_E(R)$ is connected. He then uses the automorphism group of $\Gamma_E(R)$ to help describe the automorphism group of the zero divisor graph of R. The graph of equivalence classes of zero divisors has also been studied by several others in different contexts, see, for example, [1, 12]. In addition, the set of equivalence classes forms a semigroup with a zero element. The zero divisor graph of such semigroups have been studied extensively, beginning with [7] and continuing in [6] among many others.

Recall that a prime ideal $\mathfrak{p}$ of R is an *associated prime* if $\mathfrak{p} = \mathrm{ann}(y)$ for some $y \in R$. The set of associated primes is denoted $\mathrm{Ass}(R)$; it is well known that for a Noetherian ring R, $\mathrm{Ass}(R)$ is finite, and any maximal element of the family of ideals $\mathfrak{F} = \{\mathrm{ann}(x) \mid 0 \neq x \in R\}$ is an associated prime. There is a natural injective map from $\mathrm{Ass}(R)$ to the vertex set of $\Gamma_E(R)$ given by $\mathfrak{p} \mapsto [y]$ where $\mathfrak{p} = \mathrm{ann}(y)$. As a result, we will slightly abuse terminology and refer to $[y]$ as an associated prime.

Lemma 1.2. *Any two distinct elements of* $\mathrm{Ass}(R)$ *are connected by an edge. Furthermore, every vertex* $[v]$ *of* $\Gamma_E(R)$ *is either an associated prime or adjacent to an associated prime maximal in* $\mathfrak{F}$.

Proof. The proof of the first statement is essentially the same as in [4, Lemma 2.1]. If $\mathfrak{p} = \text{ann}(x)$ and $\mathfrak{q} = \text{ann}(y)$ are primes, then one can assume that there is an element $r \in \mathfrak{p} \backslash \mathfrak{q}$. Since $rx = 0 \in \mathfrak{q}$, $x \in \text{ann}(y)$, and hence $[x] \cdot [y] = 0$. Next, suppose $[v] \in \Gamma_E(R)$ is not an associated prime. Since v is a zero divisor, $v \in \text{ann}(z)$ for some z maximal in $\mathfrak{F}$; thus, there is an edge between $[v]$ and $[z]$. □

Example 1.3. The associated primes of $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ are $[(2, 0)]$ and $[(0, 2)]$.



$\Gamma_E(R)$

The *degree* of a vertex v in a graph, denoted $\deg v$, is the number of edges incident to v. When $\deg v = 1$, we call v an *end*. A *regular graph* is one in which $\deg v = \deg w$ for every pair of vertices v and w of the graph. A *path of length n* between two vertices u and w is a sequence of distinct vertices v_i of the form $u = v_0 - v_1 - \cdots - v_n = w$ such that $v_{i-1} - v_i$ is an edge for each $i = 1, \ldots, n$. The *distance* between a pair of vertices is the length of the shortest path between them; if no path exists, the distance is infinite. The *diameter* of a graph is the greatest distance between any two distinct vertices.

The next proposition follows from previous work on zero divisor graphs of semigroups (in particular, see [7, Theorem 1.2]). We include a proof specific to $\Gamma_E(R)$ to give the reader a feel for some of the techniques subsequently used in this article. One could also obtain this result by noting that $\Gamma_E(R)$ is a quotient graph of the zero divisor graph of R which has the same properties, see [1, Proposition 4.5] for the special case in which R is a von Neumann regular ring.

Proposition 1.4. *The graph* $\Gamma_E(R)$ *is connected and* $\text{diam}\,\Gamma_E(R) \leq 3$.

Proof. Let $[x]$ and $[y]$ be two non-adjacent vertices. At worst, neither $[x]$ nor $[y]$ is an associated prime. By Lemma 1.2, there are associated primes $[v_1]$ and $[v_2]$ adjacent to $[x]$ and $[y]$, respectively, providing a path $[x] - [v_1] - [v_2] - [y]$, where $[v_1]$ may or may not equal $[v_2]$. In any case, $\Gamma_E(R)$ is connected and $\text{diam}\,\Gamma_E(R) \leq 3$. □

Connectivity and a restricted diameter are two similarities between $\Gamma_E(R)$ and $\Gamma(R)$, [2, Theorem 2.3]. One important difference between $\Gamma_E(R)$ and $\Gamma(R)$ lies in the study of complete graphs. The idea in the next proposition is that complete zero divisor graphs determined by equivalence classes, with the exception of graphs consisting of a single edge (e.g., $\Gamma_E(\mathbb{Z}_{pq})$ and $\Gamma_E(\mathbb{Z}_{p^3})$ where p and q are distinct primes p and q), all collapse to a single point. Each vertex in $\Gamma_E(R)$ is representative of a distinct class of zero divisor activity in R.

Proposition 1.5. *Let R be such that $\Gamma_E(R)$ has at least three vertices. Then $\Gamma_E(R)$ is not complete.*

Proof. Suppose $\Gamma_E(R)$ is a complete graph and $[x]$, $[y]$, and $[z]$ are three distinct vertices. We may assume that $\text{ann}(z)$ is a maximal element of $\mathfrak{F}$ and hence is not contained in either $\text{ann}(x)$ or $\text{ann}(y)$. Then there is an element $w \in \text{ann}(z)$ such that $w \notin \text{ann}(x) \cup \text{ann}(y)$. This immediately contradicts the assumption that the graph is complete. □

Corollary 1.6. *There is only one graph $\Gamma_E(R)$ with exactly three vertices that can be realized as a graph of equivalence classes of zero divisors for some ring R.*

Proof. Since every graph is connected, by Proposition 1.5, we need only note that the graph of $R = \mathbb{Z}_4[X, Y]/(X^2, XY, 2X)$, for example, is $[2]$—$[x]$—$[y]$, where x and y denote the images of X and Y, respectively. □

The simplification, or collapse, of complete $\Gamma_E(R)$ graphs applies also to complete r-partite diagrams, with the exclusion of star graphs $K_{n,1}$, like the one above. (The latter are considered in the next section.) A graph is *complete bipartite* if there is a partition of the vertices into two subsets $\{u_i\}$ and $\{v_j\}$ such that u_i is adjacent to v_j for all pairs i, j, but no two elements of the same subset are adjacent. More generally, a graph is *complete r-partite* if the vertices can be partitioned into r distinct subsets such that each element of a subset is adjacent to every element not in the same subset, but no two elements of the same subset are adjacent.

Proposition 1.7. *Let R be a ring such that $\Gamma_E(R)$ is complete r-partite. Then $r = 2$ and $\Gamma_E(R) = K_{n,1}$ for some $n \geq 1$.*

Proof. First suppose $\Gamma_E(R) = K_{n_1,n_2,\ldots,n_r}$ for some $r \geq 3$. Proposition 1.5 implies that not all $n_i = 1$; without loss of generality assume $n_1 > 1$. Suppose for the moment that $n_2 > 1$. Then there are two pairs of non-adjacent vertices, say $[u_1]$, $[u_2]$ and $[v_1]$, $[v_2]$ such that $u_i v_j = 0$ for $i, j \in \{1, 2\}$. Since $[u_1] \neq [u_2]$, we may assume there is $z \in \text{ann}(u_1)\backslash\text{ann}(u_2)$. But $[u_1]$ and $[u_2]$ have the same set of neighbors, hence $\text{ann}(u_1)\backslash[u_1] = \text{ann}(u_2)\backslash[u_2]$. It follows that $z \in [u_1]$, that is, $\text{ann}(z) = \text{ann}(u_1)$, and so $u_1^2 = 0$. Similarly, $v_1^2 = 0$. This implies that $[u_1 + v_1]$ is adjacent to both $[u_1]$ and $[v_1]$, but not to either $[u_2]$ or $[v_2]$, a contradiction. Thus $n_2 = 1$. A similar argument shows that $n_2 = \cdots = n_r = 1$.

Now let $[u_1]$ and $[u_2]$ be as above. Since we are assuming that $r \geq 3$, let $[v]$ and $[w]$ be two adjacent vertices of $\Gamma_E(R)$ both of which are adjacent to $[u_1]$ and $[u_2]$. If $v^2 = 0$ and $w^2 = 0$, then $\text{ann}(v) = Z^*(R) = \text{ann}(w)$ implying that $[v] = [w]$. Therefore, we may assume that $w^2 \neq 0$. Consider $u_1 + w$, which is a zero divisor since $(u_1 + w)v = 0$. Note that $(u_1 + w)^2 = w^2 \neq 0$, so $[u_1 + w] \neq [u_1]$. However, $[u_1 + w]$ is adjacent to $[u_1]$, but not to $[u_2]$. Since this is impossible, we must have $r = 2$ and hence $\Gamma_E(R) = K_{n,1}$. □

A second look at the above results allows us to deduce some facts about cycle graphs, which are n-gons. An immediate consequence of Proposition 1.4 is that there are no cycle graphs with eight or more vertices. Likewise, Proposition 1.7 tells us that there are no 3- or 4-cycle graphs, i.e., $K_{1,1,1}$ or $K_{2,2}$, respectively.

Proposition 1.8. *For any R, $\Gamma_E(R)$ is not a cycle graph.*

Proof. As mentioned above, it suffices to show that cycle graphs which contain a path of length five are not possible. Suppose $[x] - [y] - [z] - [w] - [v]$ is a path in the graph. Then $[yw]$ is annihilated by $[x]$, $[z]$, and $[v]$, but no such class exists. □

For a vertex v of a simple graph G, the set of vertices adjacent to v is called the *neighborhood of* v and denoted $N_G(v)$, or simply $N(v)$ if the graph is understood.

Lemma 1.9. *Let G be a finite, simple graph with the property that two distinct vertices v and w of G are non-adjacent if and only if $N_G(v) = N_G(w)$. Then G is a complete r-partite graph for some positive integer r.*

Proof. We induct on the number of vertices of G. If G has one vertex, then G is a complete 1-partite graph. So suppose G has more than one vertex. Let $I = \{y_1, y_2, \ldots, y_n\}$ be a maximal set of pairwise non-adjacent vertices in G (that is, I is an independent set of G), and let $N = \{x_1, \ldots, x_d\}$ be the common neighborhood of the elements of I. Then I and N are disjoint and the vertices of G are precisely $I \cup N$. If $N = \emptyset$, then G is 1-partite, and we are done. So assume $N \neq \emptyset$. Let H be the graph induced by N. Note that $N_H(v) = N_G(v) \cap N$ and $N_G(v) = N_H(v) \cup I$ for any vertex v of H, hence two distinct vertices v and w of H are non-adjacent if and only if $N_H(v) = N_H(w)$. Since H has fewer vertices than G, the induction hypothesis implies that H is a complete $(r-1)$-partite graph for some $r \geq 2$. But by the definition of I, this implies that G is a complete r-partite graph. □

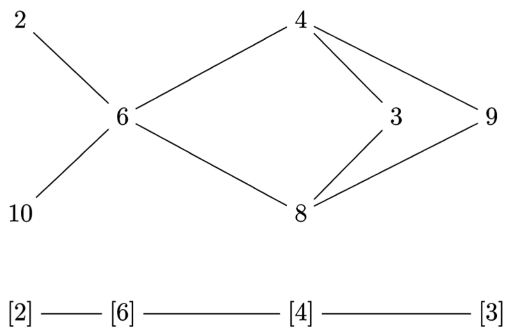
Proposition 1.10. *For any ring R, there are no finite regular graphs $\Gamma_E(R)$ with more than two vertices.*

Proof. Suppose that R is a ring such that $\Gamma_E(R)$ is a regular graph of degree d with at least 3 vertices. By earlier results, $d \geq 2$ and $|N([x])| = d$ for all $[x] \in \Gamma_E(R)$. Since $\Gamma_E(R)$ is not complete, there exists two non-adjacent vertices $[y_1]$ and $[y_2]$. If $N([y_1]) \neq N([y_2])$, then without loss of generality we may assume that there is a vertex $[u] \in N([y_1]) \backslash N([y_2])$. Thus $uy_1 = 0$ but $uy_2 \neq 0$. This implies $[u] \in N([y_1 y_2])$ and so $N([y_2]) \neq N([y_1 y_2])$. But $N([y_2]) \subseteq N([y_1 y_2])$ and, since each set has cardinality d, we must have equality, which leads to a contradiction. Therefore, any two non-adjacent vertices on the graph have the same neighborhood, and clearly the converse is true. Thus by Lemma 1.9 and Proposition 1.7, $\Gamma_E(R) = K_{n,1}$ for some $n \geq 1$. Since $\Gamma_E(R)$ has at least 3 vertices, $n \geq 2$, which contradicts the assumption that $\Gamma_E(R)$ is regular. □

Finally, we give the connection between $\Gamma_E(R)$ and the zero divisor graph $\Gamma(R)$ in [2]. Let Γ be a graph. To each vertex v_i of Γ, assign an element $w_i \in \mathbb{Z}^+ \cup \{\infty\}$, called the weight of v_i, and let $w = (w_1, w_2, \ldots)$. Define a graph $\Gamma^{(w)}$ with vertex set $\{v_{k_i,i} \mid 1 \leq k_i \leq w_i\}$ and edge set is $\{v_{k_i,i} v_{k_j,j} \mid i \neq j \text{ and } v_i v_j \text{ is an edge of } \Gamma\}$. Intuitively, the vertices $v_{k_i,i}$ of $\Gamma^{(w)}$ form a "covering set" of cardinality w_i of the vertex v_i of Γ, and if v_i is connected to v_j in Γ, then every vertex of the covering set of v_i is connected to every vertex of the covering set of v_j.

For each vertex $[v_i]$ of $\Gamma_E(R)$, let $w_i = |[v_i]|$, and let $w = (w_1, w_2, \ldots)$. By the definitions of $\Gamma_E(R)$ and $\Gamma(R)$, it is clear that $\Gamma_E(R)^{(w)}$ is a subgraph of $\Gamma(R)$. In fact, $\Gamma_E(R)^{(w)} = \Gamma(R)$ if and only if whenever R contains a nonzero element x such that $x^2 = 0$, then $|[x]| = 1$. In particular, if R is reduced, then $\Gamma_E(R)^{(w)} = \Gamma(R)$. In general, $\Gamma(R)$ is the graph obtained from $\Gamma_E(R)^{(w)}$ by adding edges between every pair of vertices $v_{k_i,i}$ and $v_{k_i',i}$ if and only if $v_i^2 = 0$.

Example 1.11. The zero divisor graph $\Gamma(\mathbb{Z}_{12})$ is shown "covering" the graph $\Gamma_E(\mathbb{Z}_{12})$. Note that if $w = (2, 1, 2, 2)$, then $\Gamma_E(\mathbb{Z}_{12})^{(w)} = \Gamma(\mathbb{Z}_{12})$.



$\Gamma(\mathbb{Z}_{12})$

$\Gamma_E(\mathbb{Z}_{12})$

2. INFINITE GRAPHS AND STAR GRAPHS

In this section we investigate infinite graphs and star graphs. In particular, one might ask if there exists a ring R such that $\Gamma_E(R)$ is an infinite star graph, or perhaps if the assumption that R is Noetherian forces $\Gamma_E(R)$ to be finite. We begin with a discussion of degrees.

Proposition 2.1. *Let x and y be elements of R. If $0 \neq \text{ann}(x) \subsetneq \text{ann}(y)$, then* $\deg[x] \leq \deg[y]$.

Proof. If $[u] \in \Gamma_E(R)$ such that $ux = 0$, then clearly $uy = 0$. □

Proposition 2.2. *The vertex set of $\Gamma_E(R)$ is infinite if and only if there is some associated prime $[x]$ maximal in $\mathfrak{F}$ such that* $\deg[x] = \infty$.

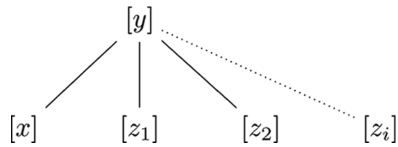
Proof. Clearly, if some vertex has infinite degree, then the vertex set of $\Gamma_E(R)$ is infinite. Conversely, suppose the vertex set of $\Gamma_E(R)$ is infinite. Let $[x_1], \ldots, [x_r]$ be the maximal elements in $\mathfrak{F}$. If $\deg[x_1] < \infty$, then there are infinitely many vertices $[w]$ such that $wx_1 \neq 0$. Now $[w][v] = 0$ for some class $[v] \neq [w]$, and since $[x_1]$ is prime, we must have $v \in \text{ann}(x_1)$. If there are infinitely many distinct vertices $[v]$, then $\deg[x_1] = \infty$, a contradiction. Therefore, the set of $[v]$'s is finite, and hence, $\deg[v] = \infty$ for some v. Either $[v]$ is an associated prime of R and maximal in $\mathfrak{F}$, or $\text{ann}(v) \subsetneq \text{ann}(x_j)$ for some $j \neq 1$. In either case, the result holds. □

Definition 2.3. A *star* graph is a complete bipartite graph $K_{n,1}$, $n \in \mathbb{N} \cup \{\infty\}$. If $n = \infty$, we say the graph is an infinite star graph.

Proposition 2.4. *Any ring R such that $\Gamma_E(R)$ is a star graph with at least four vertices satisfies the following properties*:

(1) $\mathrm{Ass}(R) = \{\mathfrak{p}\}$;
(2) $\mathfrak{p}^3 = 0$;
(3) $char(R) = 2, 4,$ *or* 8.

Proof. Suppose we have a star graph with at least four vertices, as shown below.



Let $[y]$ be the unique vertex with maximal degree. Note that $[y]$ is a maximal element in $\mathfrak{F}$, for if $\mathrm{ann}(y) \subsetneq \mathrm{ann}(w)$ for some w, then by Proposition 2.1 $\Gamma_E(R)$ would contain two vertices of degree larger than 1. Next, in order for the classes $[x], [z_1], \ldots$ to be distinct, we must have $z_i^2 = 0$, for all i. If $i \neq j$, then $z_i z_j$ is nonzero and annihilated by z_i and z_j; hence, the only choice for $[z_i z_j]$ is $[y]$. Moreover, $(z_i z_j)^2 = 0$ implies that every element of $[y]$ is nilpotent of order 2, by [10, (3.5)]. Since $[y]$ is connected to every vertex, it follows that $z_i z_j z_k = z_i z_j x = 0$ for any i, j, k. As a result, we have $[xz_i] = [y]$. Thus, $x^2 z_i = 0$ for each i. Consequently, either $x^2 = 0$ or $[x^2] = [y]$. Since the latter case implies $x^3 = 0$, it follows that every zero divisor is nilpotent of order at most three. This establishes (1) and (2).

For (3), consider $z_i + z_j$, which is annihilated by $z_i z_j$, but not z_i or z_j. Therefore, $[z_i + z_j]$ represents a degree one vertex distinct from $[z_i]$ and $[z_j]$. The same conclusion can be reached for $[z_i - z_j]$. Since $(z_i + z_j)(z_i - z_j) = 0$, this means that $[z_i - z_j] = [z_i + z_j]$, and every element in the class is nilpotent of order two. In particular, $(z_i + z_j)^2 = 0 \Rightarrow 2z_i z_j = 0$. Consequently, either $2 \equiv 0$ or 2 is a zero divisor in R. If $\mathrm{char}(R) \neq 2$, then $[2]$ is somewhere on the graph and either $2^2 = 0$ or $2^3 = 0$. □

Corollary 2.5. *If R is a finite ring and $\Gamma_E(R)$ is a star graph with at least four vertices, then R is a local ring.*

Proof. A finite ring is a product of finite local rings, see, e.g., [9, Theorem VI.2]. Hence the number of associated primes corresponds to the number of factors in the product. □

It is unknown to the authors whether, for each positive integer n, the star graph $K_{n,1}$ can be realized as $\Gamma_E(R)$ for some ring R, or how one would go about the general construction or argument. However, the following example shows that there exist rings R such that $\Gamma_E(R)$ is a star graph with infinitely many ends.

Example 2.6. Let $R = \mathbb{Z}_2[X, Y, Z]/(X^2, Y^2)$. Then $\Gamma_E(R)$ is an infinite star graph. If x and y denote the images of X and Y, respectively, then $\mathrm{Ass}(R) = \{(x, y)\}$, and the corresponding vertex $[xy]$ is the central vertex of the graph. The ends, besides $[x]$,

$[y]$, are vertices of the form $[z^m x + z^n y]$ for each ordered pair (m, n) of nonnegative integers.

Remark 2.7. Example 2.6 shows that the Noetherian condition is not enough to force $\Gamma_E(R)$ to be finite.

3. ASSOCIATED PRIMES

One of the main motivations in studying graphs of equivalence classes of zero divisors is the fact that the associated primes of R, by their very definition, correspond to vertices in $\Gamma_E(R)$. The focus here is the identification of associated primes of R, given $\Gamma_E(R)$.

It is important to note from Proposition 2.1 in the last section that proper containment of annihilator ideals does not translate into a strict inequality of degrees. For instance, the ring $\mathbb{Z}_4[X, Y]/(X^2, XY, 2X)$, shown in Corollary 1.6, satisfies $\text{ann}(y) \subsetneq \text{ann}(2)$, but $\deg[y] = \deg[2] = 1$, where y denotes the image of Y (we adopt this convention for the remainder of the article). In order to achieve strict inequality on degrees, further assumptions on the annihilator ideals are needed. This issue is addressed in this section; but first we make some key observations. The contrapositive of Proposition 2.1 is useful and worth stating:

Proposition 3.1. *Let x and y be elements of R. If $\deg[y] > \deg[x]$ for every $[x]$ in $\Gamma_E(R)$, then $\text{ann}(y)$ is maximal in $\mathfrak{F}$ and hence is an associated prime.*

Of course, we already saw evidence of this fact in Proposition 2.4. In that case, the vertex of maximal degree corresponds to the unique element of $\text{Ass}(R)$. Similarly, if $\Gamma_E(R)$ has exactly three vertices, then the vertex of degree two always corresponds to an associated prime maximal in $\mathfrak{F}$. However, in this case $\text{Ass}(R)$ may consist of more than one ideal. For the example used in Corollary 1.6, the associated primes are $\text{ann}(x) = (2, x, y)$ and $\text{ann}(2) = (2, x)$. This is just one instance where graphs with only two or three vertices prove the exception to the rule.

Proposition 3.2. *If R is a ring such that $|\Gamma_E(R)| > 3$, then no associated prime of R is an end.*

Proof. Suppose $[y]$ is an associated prime of degree one. Then there is one and only one $[x]$ such that $[x] \neq [y]$ and $[x][y] = 0$. By Lemma 1.2, $\text{Ass}(R) = \{\text{ann}(x), \text{ann}(y)\}$. Any additional vertices, of which there are at least two, must be connected to $[x]$. If $[z]$ is another vertex and $\deg[z] \geq 2$, then for some $[w]$, $zw = 0 \in \text{ann}(y)$; hence $z \in \text{ann}(y)$ or $w \in \text{ann}(y)$. Since this is not possible, it must be that $\deg[z] = \deg[w] = 1$. But in order for $[z]$ and $[w]$ to be distinct, we must have $z^2 = 0$ or $w^2 = 0$, which again puts $z \in \text{ann}(y)$ or $w \in \text{ann}(y)$. □

Corollary 3.3. *If R is a ring such that $|\Gamma_E(R)| \geq 3$, then any vertex with an end is an associated prime maximal in $\mathfrak{F}$.*

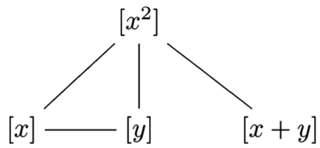
Proof. Proposition 3.1 and Corollary 1.6 take care of the case when there are exactly three vertices. Suppose $|\Gamma_E(R)| > 3$. Since every graph is connected, the result follows from Lemma 1.2 and Proposition 3.2. □

We are now in a position to show a finer relationship between vertices of high degree and associated primes.

Proposition 3.4. *Let $x_1, \ldots, x_r$ be elements of R, with $r \geq 2$, and suppose* $\mathrm{ann}(x_1) \subsetneq \cdots \subsetneq \mathrm{ann}(x_r)$ *is a chain in* $\mathrm{Ass}(R)$. *If* $3 \leq |\Gamma_E(R)| < \infty$, *then* $\deg[x_1] < \cdots < \deg[x_r]$.

Proof. If $|\Gamma_E(R)| = 3$, then $|\mathrm{Ass}(R)| \leq 2$ by Lemma 1.2 and Proposition 1.5, and the result follows immediately from Corollaries 1.6 and 3.1. Thus, we may assume that $|\Gamma_E(R)| > 3$. By Proposition 2.4, $\Gamma_E(R)$ is not a star graph, given the hypotheses. By the preceding proposition, the degrees of $[x_1]$ and $[x_2]$ are at least two. Suppose $\deg[x_1] = n$. Then there are $n-1$ vertices $[u_i]$, distinct from either $[x_1]$ or $[x_2]$, such that $x_1u_i = 0$ for all i. Since $\mathrm{ann}(x_1) \subsetneq \mathrm{ann}(x_2)$, $x_2u_i = 0$ for all i as well. Furthermore, $x_2 \in \mathrm{ann}(x_1)$ by (the proof of) Lemma 1.2, hence $x_2^2 = 0$. If $x_1^2 \neq 0$, then each $\mathrm{ann}(x_1 + u_i)$ contains x_2 but not x_1, making $\deg[x_2] > \deg[x_1]$. If $x_1^2 = 0$, then take $z \in \mathrm{ann}(x_2)\backslash\mathrm{ann}(x_1)$. Since $x_1 \notin \mathrm{ann}(z)$, $[z]$ is distinct from $[x_1]$ and $[x_2]$ and adjacent to $[x_2]$, but not $[x_1]$. Again, $\deg[x_1] < \deg[x_2]$. Moreover, this last argument applies to each pair $\mathrm{ann}(x_i)$, $\mathrm{ann}(x_{i+1})$, for $i \geq 1$, since $x_ix_j = 0$ for $1 \leq i \leq j \leq r$. □

Example 3.5. Let $R = \mathbb{Z}[X, Y]/(X^3, XY)$. Then $(x) \subsetneq (x, y)$ is a chain in $\mathrm{Ass}(R)$, corresponding to the vertices $[y]$ and $[x^2]$, respectively, with $\deg[y] = 2$ and $\deg[x^2] = 3$.



$\Gamma_E(R)$

We now make use of Proposition 1.10 to establish a correspondence between associated primes and vertices of relatively large degree.

Theorem 3.6. *Let R be such that $2 < |\Gamma_E(R)| < \infty$. Then any vertex of maximal degree is maximal in $\mathfrak{F}$ and hence is an associated prime.*

Proof. Let d denote the maximal degree of $\Gamma_E(R)$. If there is only one vertex of degree d, then Proposition 3.1 yields the desired result. Therefore, we may assume that $|\Gamma_E(R)| > 3$ and that $\Gamma_E(R)$ has at least two vertices of degree d. Suppose that $[y_1]$ is a vertex of degree d. Then $\mathrm{ann}(y_1) \subseteq \mathrm{ann}(y_2)$ for some y_2 such that $\mathrm{ann}(y_2)$ is maximal in $\mathfrak{F}$, and so $[y_2]$ is an associated prime. We aim to show that, in fact, $[y_1] = [y_2]$. By Proposition 2.1, $d = \deg[y_1] \leq \deg[y_2] \leq d$, so $\deg[y_2] = \deg[y_1]$. Since $\mathrm{ann}(y_1) \subseteq \mathrm{ann}(y_2)$ yields $(N([y_1])\backslash\{[y_2]\}) \subseteq (N([y_2]) \setminus \{[y_1]\})$, the equality of degrees implies that $(N([y_1])\backslash\{[y_2]\}) = (N([y_2])\backslash\{[y_1]\})$; denote this set by $\mathcal{N}$. The connectivity of $\Gamma_E(R)$ implies that $d \geq 2$ and so $\mathcal{N} \neq \varnothing$.

To get the desired result, we must show that $\mathrm{ann}(y_1) = \mathrm{ann}(y_2)$, that is, $\mathrm{ann}(y_2)\backslash\mathrm{ann}(y_1) = \varnothing$. Suppose this is not the case and let $z \in \mathrm{ann}(y_2)\backslash\mathrm{ann}(y_1)$. If $[z]$ is distinct from $[y_2]$ and $[y_1]$, then $[z] \in (N([y_2])\backslash\{[y_1]\})$ but $[z] \notin (N([y_1])\backslash\{[y_2]\})$, a contradiction. Suppose $[z] = [y_2]$, in which case $y_2y_1 \neq 0$, $y_2^2 = 0$, and $y_1^2 \neq 0$. Let

$[w] \in \mathcal{N}$ and consider $w + y_2$, which is annihilated by y_2, but not y_1. Then $[w + y_2] \notin \mathcal{N}$ and thus $[w + y_2] = [y_2]$. So if $[v] \in \mathcal{N}$, then $0 = (w + y_2)v = wv$. Since $[w]$ was chosen arbitrarily, it follows that the vertices in $\mathcal{N}$ form a complete subgraph. Hence $(\mathcal{N}\backslash\{[w]\}) \cup \{[y_2], [y_1]\} \subseteq N([w])$ for any $[w] \in \mathcal{N}$, implying that $\deg[w] \geq \deg[y_2] + 1$, contradicting the maximality of $\deg[y_2]$.

Suppose $[z] = [y_1]$ and let $[x] \in \mathcal{N}$. If $\deg[x] < d$, then there exists some $u \in \text{ann}(y_1)\backslash\text{ann}(x)$ such that $[u]$, $[y_1]$, and $[x]$ are distinct. Note that $x(u + y_1) = xu \neq 0$, so $[u + y_1] \neq [y_i]$ for $i = 1, 2$. However, $y_2(u + y_1) = 0$ which implies $[u + y_1] \in \mathcal{N}$, but $y_1(u + y_1) = y_1^2 \neq 0$, contradicting that $[u + y_1] \in \mathcal{N}$. Therefore, $\deg[x] = d$. Furthermore, since $y_2(x + y_1) = 0$ but $y_1(x + y_1) = y_1^2 \neq 0$, then $[x + y_1] \in N([y_2])\backslash\mathcal{N}$, hence $[x + y_1] = [y_1]$. Since $[x]$ was chosen arbitrarily in $\mathcal{N}$, it follows that $x_1x_2 = 0$ for distinct $[x_1], [x_2] \in \mathcal{N}$. Therefore, the subgraph induced by $\mathcal{N} \cup \{[y_2], [y_1]\}$ is a complete graph. By Proposition 1.10 there exists a vertex $[w] \in \Gamma_E(R)$ such that $\deg[w] < \deg[y_1]$, so $[w] \notin \mathcal{N} \cup \{[y_2], [y_1]\}$. However, since $[y_2]$ is an associated prime, then by Lemma 1.2, $\text{Ass}(R) \subseteq \mathcal{N} \cup \{[y_2], [y_1]\}$ and $[w]$ is adjacent to some vertex $[v] \in \mathcal{N} \cup \{[y_2], [y_1]\}$. This implies that $\deg[v] > d$, a contradiction. □

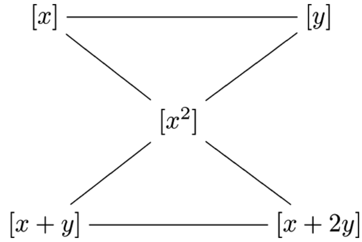
Example 3.7. Recall that $\Gamma_E(\mathbb{Z}_4 \times \mathbb{Z}_4)$ in Example 1.3 had two vertices, namely, $[(2, 0)]$ and $[(0, 2)]$, of maximal degree. Their annihilators are the associated primes of the ring.

Remark 3.8. We collect some comments regarding Theorem 3.6:

(1) The converse is false: For example, if p and q are distinct primes, then in $\mathbb{Z}_{p^2q^3}$, $[pq^3]$ and $[p^2q^2]$ correspond to associated primes (p) and (q), respectively, each maximal in $\mathfrak{F}$, but $\deg[pq^3] = 6$ and $\deg[p^2q^2] = 7$.
(2) The assumption of finiteness is necessary. The graph of the ring $R = \mathbb{Z}[X, Y]/(X^2, Y^2, Z^2)$ has four vertices with infinite degree, namely, $[xy], [xz]$, $[yz]$, and $[xyz]$, but only $\text{ann}(xyz)$ is an associated prime of R.

Finally, since each of our examples possesses an end, one might ask if Corollary 3.3 supersedes Theorem 3.6; that is, if every graph must contain an end. Based on the results in section one, the first case where such an example can occur is when the graph has exactly five vertices.

Example 3.9. Let $R = \mathbb{Z}_3[[X, Y]]/(XY, X^3, Y^3, X^2 - Y^2)$.



$\Gamma_E(R)$

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