

# DIVISOR CLASS GROUPS OF GRADED HYPERSURFACES

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ABSTRACT. We demonstrate how some classical computations of divisor class groups can be obtained using the theory of rational coefficient Weil divisors and related results of Watanabe.

## 1. INTRODUCTION

The purpose of this note is to provide a simple technique to compute divisor class groups of affine normal hypersurfaces of the form

$$k[z, x_1, \dots, x_d]/(z^n - g),$$

where  $g$  is a weighted homogeneous polynomial in  $x_1, \dots, x_d$  of degree relatively prime to  $n$ . We use the theory of rational coefficient Weil divisors due to Demazure [3] and related results of Watanabe [14]. This provides an alternative approach to various classical examples found in Samuel's influential lecture notes [10], as well as to computations due to Lang [6] and Scheja and Storch [11]. While the computations we present here are subsumed by those of [11], our techniques are different. A key point in our approach is that the projective variety defined by a hypersurface as above is weighted projective space over  $k$ , and this makes for straightforward, elementary calculations.

Watanabe [14, page 206] pointed out that  $\mathbb{Q}$ -divisor techniques can be used to recover the classification of graded factorial domains of dimension two, originally due to Mori [8]. Robbiano has applied similar methods to a study of factorial and almost factorial schemes in weighted projective space [9].

## 2. $\mathbb{Q}$ -DIVISORS

We review some material from [3] and [14]. Let  $k$  be a field, and let  $X$  be a normal irreducible projective variety over  $k$ , with rational function field  $k(X)$ .

A *rational coefficient Weil divisor* or a  $\mathbb{Q}$ -*divisor* on  $X$  is a  $\mathbb{Q}$ -linear combination of irreducible subvarieties of  $X$  of codimension one. Let  $D = \sum n_i V_i$  be a  $\mathbb{Q}$ -divisor, where  $V_i$  are distinct. Then  $\lfloor D \rfloor$  is defined as

$$\lfloor D \rfloor = \sum \lfloor n_i \rfloor V_i,$$

where  $\lfloor n \rfloor$  denotes the greatest integer less than or equal to  $n$ . We set

$$\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor).$$

If each coefficient  $n_i$  occurring in  $D$  is nonnegative, we say that  $D \geq 0$ .

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2000 *Mathematics Subject Classification.* Primary 13C20, Secondary 14C20.

The first author was supported by the NSF under grants DMS 0300600 and DMS 0600819.

A  $\mathbb{Q}$ -divisor  $D$  is *ample* if  $nD$  is an ample Cartier divisor for some  $n \in \mathbb{N}$ . In this case, the *generalized section ring* corresponding to  $D$  is the ring

$$R(X, D) = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jD)).$$

If  $R = R(X, D)$ , then the  $n$ -th Veronese subring of  $R = R(X, D)$  is the ring

$$R^{(n)} = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jnD)) = R(X, nD).$$

The following theorem, due to Demazure, implies that a normal  $\mathbb{N}$ -graded ring  $R$  is determined by a  $\mathbb{Q}$ -divisor on  $\text{Proj } R$ .

**Theorem 2.1.** [3, 3.5]. *Let  $R$  be an  $\mathbb{N}$ -graded normal domain, finitely generated over a field  $R_0$ . Let  $T$  be a homogeneous element of degree 1 in the fraction field of  $R$ . Then there exists a unique ample  $\mathbb{Q}$ -divisor  $D$  on  $X = \text{Proj } R$  such that*

$$R = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jD))T^j.$$

We next recall a result of Watanabe, which expresses the divisor class group of  $R$  in terms of the divisor class group of  $X$  and a  $\mathbb{Q}$ -divisor corresponding to  $R$ .

**Theorem 2.2.** [14, Theorem 1.6] *Let  $X$  be a normal irreducible projective variety over a field. Assume  $\dim X \geq 1$  and let  $D = \sum_{i=1}^r (p_i/q_i)V_i$  be a  $\mathbb{Q}$ -divisor on  $X$  where  $V_i$  are distinct irreducible subvarieties,  $p_i, q_i \in \mathbb{Z}$  are relatively prime, and  $q_i > 0$ . Set*

$$R = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jD))T^j.$$

Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\theta} \text{Cl}(X) \xrightarrow{\alpha} \text{Cl}(R) \longrightarrow \text{coker } \alpha \longrightarrow 0,$$

where  $\theta(1) = \text{lcm}(q_i) \cdot D$ , and  $\alpha: \mathbb{Z} \rightarrow \bigoplus_{i=1}^r \mathbb{Z}/q_i\mathbb{Z}$  is the map  $1 \mapsto (p_i \pmod{q_i})_i$ .

In the exact sequence above,  $\text{coker } \alpha$  is always a finite group. Moreover, if  $X$  is projective space, a Grassmannian variety, or a smooth complete intersection in  $\mathbb{P}^n$  of dimension at least three, then  $\text{Cl}(X) = \mathbb{Z}$ . It follows that, in these cases, the divisor class group of  $R(X, D)$  is finite for any ample  $\mathbb{Q}$ -divisor  $D$  on  $X$ , and hence that  $R(X, D)$  is *almost factorial* in the sense of Storch [12].

Lipman proved that the divisor class group of a two-dimensional normal local ring  $R$  with rational singularities is finite, [7, Theorem 17.4]. While this is a hard result, the analogous statement for graded rings is a straightforward application of Theorem 2.2. Indeed, let  $R$  be an  $\mathbb{N}$ -graded normal ring of dimension two, finitely generated over an algebraically closed field  $R_0$ , such that  $R$  has rational singularities. Then  $R$  has a negative  $a$ -invariant by [14, Theorem 3.3], so  $H^1(X, \mathcal{O}_X) = 0$  where  $X = \text{Proj } R$ . But then  $X$  is a curve of genus 0 so it must be  $\mathbb{P}^1$ , and it follows that the divisor class group of  $R$  is finite.

*Remark 2.3.* We note some aspects of Watanabe's proof of Theorem 2.2. Let  $\text{Div}(X)$  be the group of Weil divisors on  $X$ , and let

$$\text{Div}(X, \mathbb{Q}) = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the group of  $\mathbb{Q}$ -divisors. For  $D$  as in Theorem 2.2, set  $\text{Div}(X, D)$  to be the subgroup of  $\text{Div}(X, \mathbb{Q})$  generated by  $\text{Div}(X)$  and the divisors

$$\frac{1}{q_1}V_1, \dots, \frac{1}{q_r}V_r.$$

Each element  $E \in \text{Div}(X, D)$  gives a divisorial ideal

$$\bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(E + jD))T^j$$

of  $R$ , and hence an element of  $\text{Cl}(R)$ . The map  $\text{Div}(X, D) \rightarrow \text{Cl}(R)$  induces a surjective homomorphism

$$\text{Div}(X, D)/\text{Div}(X) \rightarrow \text{Cl}(R)/\text{image}(\text{Cl}(X)).$$

### 3. COMPUTING DIVISOR CLASS GROUPS

The divisor class groups of affine surfaces of characteristic  $p$  defined by equations of the form  $z^{p^n} = g(x, y)$  have been studied in considerable detail; such surfaces are sometimes called *Zariski surfaces*. In [6] Lang computed the divisor class group of hypersurfaces of the form  $z^{p^n} = g(x_1, \dots, x_d)$  where  $g$  is a homogeneous polynomial of degree relatively prime to  $p$ . The proposition below recovers [6, Proposition 3.11].

Let  $A = k[x_1, \dots, x_d]$  be a polynomial ring over a field. We say  $g \in A$  is a *weighted homogeneous polynomial* if there exists an  $\mathbb{N}$ -grading on  $A$ , with  $A_0 = k$ , for which  $g$  is a homogeneous element.

**Proposition 3.1.** *Let  $R = k[z, x_1, \dots, x_d]/(z^n - g)$  be a normal hypersurface over a field  $k$ , where  $g \in k[x_1, \dots, x_d]$  is a weighted homogeneous polynomial with degree relatively prime to  $n$ . Let  $g = h_1 \cdots h_r$ , where  $h_i \in k[x_1, \dots, x_d]$  are irreducible polynomials. Then*

$$\text{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{r-1},$$

and the images of  $(z, h_1), \dots, (z, h_{r-1})$  form a minimal generating set for  $\text{Cl}(R)$ .

Note that if  $n \geq 2$ , then the hypothesis that  $R$  is normal forces  $h_1, \dots, h_r$  to be pairwise coprime irreducible polynomials.

*Proof of Proposition 3.1.* The polynomial ring  $k[x_1, \dots, x_d]$  has a grading under which  $\deg x_i = c_i$  for  $c_i \in \mathbb{N}$ , and the degree of  $g$  is an integer  $m$  relatively prime to  $n$ . We assume, without any loss of generality, that  $\gcd(c_1, \dots, c_d) = 1$ . Consider the  $\mathbb{N}$ -grading on  $R$  where  $\deg x_i = nc_i$  and  $\deg z = m$ . Note that under this grading  $\deg g = \sum \deg h_i = mn$ . The  $n$ -th Veronese subring of  $R$  is

$$R^{(n)} = k[z^n, x_1, \dots, x_d]/(z^n - g) = k[x_1, \dots, x_d],$$

which is a polynomial ring in  $x_1, \dots, x_d$ . Let  $X = \text{Proj } R^{(n)} = \text{Proj } R$ .

There exist integers  $s_i$ ,  $a$ , and  $b$  such that  $\sum_{i=1}^d s_i c_i = 1$  and  $am + bn = 1$ . Consider the  $\mathbb{Q}$ -divisor on  $X$  given by

$$D = b \text{div}(\mathbf{x}) + \frac{a}{n} \text{div}(g) = b \sum_{i=1}^d s_i V(x_i) + \frac{a}{n} \sum_{i=1}^r V(h_i),$$

where  $\mathbf{x} = x_1^{s_1} \cdots x_d^{s_d}$ . We claim that

$$(3.1.1) \quad R = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jD))T^j,$$

where  $T = z^a \mathbf{x}^b$  is a homogeneous degree 1 element of the fraction field of  $R$ . First note that  $\lfloor am/n \rfloor = \lfloor (1 - bn)/n \rfloor = -b$ , so

$$\lfloor mD \rfloor = bm \text{div}(\mathbf{x}) + \left\lfloor \frac{am}{n} \right\rfloor \text{div}(g) = bm \text{div}(\mathbf{x}) - b \text{div}(g).$$

Consequently  $\deg[mD] = 0$ , and  $H^0(X, \mathcal{O}_X(mD))T^m$  is the  $k$ -vector space spanned by the element

$$\mathbf{x}^{-bm} g^b T^m = \mathbf{x}^{-bm} (z^n)^b (z^a \mathbf{x}^b)^m = z^{bn+am} = z.$$

Let  $c = c_t$  for an integer  $1 \leq t \leq d$ . Then  $ncD = bnc \operatorname{div}(\mathbf{x}) + ac \operatorname{div}(g)$  has degree  $nc$ , and  $H^0(X, \mathcal{O}_X(ncD))T^{nc}$  contains the element

$$x_t \mathbf{x}^{-bnc} g^{-ac} T^{nc} = x_t \mathbf{x}^{-bnc} (z^n)^{-ac} (z^a \mathbf{x}^b)^{nc} = x_t.$$

To prove the claim (3.1.1), it remains to verify that  $z, x_1, \dots, x_d$  are  $k$ -algebra generators for the ring  $\bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(jD))T^j$ . An arbitrary positive integer  $j$  can be written as  $um + vn$  for  $0 \leq u \leq n - 1$ . We then have

$$\begin{aligned} [jD] &= b(um + vn) \operatorname{div}(\mathbf{x}) + \left\lfloor \frac{a(um + vn)}{n} \right\rfloor \operatorname{div}(g) \\ &= b(um + vn) \operatorname{div}(\mathbf{x}) + (va - ub) \operatorname{div}(g), \end{aligned}$$

which has degree  $vn$ . Consequently  $H^0(X, \mathcal{O}_X(jD))T^j$  vanishes if  $v$  is negative, and for nonnegative  $v$ , it is spanned by elements

$$\mu \mathbf{x}^{-b(um+vn)} g^{-va+ub} T^{um+vn} = \mu z^u,$$

for monomials  $\mu$  in  $x_i$  of degree  $v$ . This completes the proof of (3.1.1).

Since  $nD$  has integer coefficients, the exact sequence of Theorem 2.2 for the divisor  $nD$  and corresponding ring  $R^{(n)}$  reduces to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\theta} \operatorname{Cl}(X) \xrightarrow{\alpha} \operatorname{Cl}(R^{(n)}) \longrightarrow 0,$$

where  $\theta(1) = nD$ . Since  $R^{(n)}$  is a polynomial ring, and hence factorial, it follows that  $nD$  generates  $\operatorname{Cl}(X)$ . Next, consider the exact sequence applied to the divisor  $D$  and corresponding ring  $R$ , i.e., the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\theta} \operatorname{Cl}(X) \xrightarrow{\alpha} \operatorname{Cl}(R) \longrightarrow \operatorname{coker} \alpha \longrightarrow 0.$$

The lcm of the denominators occurring in  $D$  is  $n$ , so we once again have  $\theta(1) = nD$ . Consequently  $\theta$  is an isomorphism and  $\operatorname{Cl}(R) = \operatorname{coker} \alpha$ , where

$$\alpha: \mathbb{Z} \longrightarrow \bigoplus_1^r \mathbb{Z}/n\mathbb{Z} \quad \text{with} \quad \alpha(1) = (a, \dots, a).$$

Since  $a$  and  $n$  are relatively prime, it follows that

$$\operatorname{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{r-1}.$$

We next determine explicit generators for  $\operatorname{Cl}(R)$  by Remark 2.3. The  $\mathbb{Q}$ -divisors

$$E_t = -\frac{1}{n} V(h_t) \quad \text{for } 1 \leq t \leq r$$

give a generating set for  $\operatorname{Div}(X, D)/\operatorname{Div}(X)$  which surjects onto  $\operatorname{Cl}(R)$ . Hence the divisorial ideals

$$\mathfrak{p}_t = \bigoplus_{j \geq 0} H^0(X, \mathcal{O}_X(E_t + jD))T^j \quad \text{where } 1 \leq t \leq d,$$

generate  $\text{Cl}(R)$ . The computation of  $\mathfrak{p}_t$  is straightforward, and we give a brief sketch. First note that

$$\begin{aligned} [E_t + mD] &= bm \operatorname{div}(x) + \left\lfloor \frac{am-1}{n} \right\rfloor V(h_t) + \sum_{i \neq t} \left\lfloor \frac{am}{n} \right\rfloor V(h_i) \\ &= bm \operatorname{div}(\mathbf{x}) - b \operatorname{div}(g), \end{aligned}$$

so  $H^0(X, \mathcal{O}_X(E_t + mD))T^m$  is the  $k$ -vector space spanned by

$$\mathbf{x}^{-bm} g^b T^m = z.$$

Since the degree of each  $x_i$  is a multiple of  $n$ , we have  $\deg h_t = n\gamma$  for some integer  $\gamma$ . We next compute the component of  $\mathfrak{p}_t$  in degree  $n\gamma$ . Note that

$$[E_t + n\gamma D] = -V(h_t) + bn\gamma \operatorname{div}(\mathbf{x}) + a\gamma \operatorname{div}(g),$$

so  $H^0(X, \mathcal{O}_X(E_t + n\gamma D))T^{n\gamma}$  is the  $k$ -vector space spanned by

$$h_t \mathbf{x}^{-bn\gamma} g^{-a\gamma} T^{n\gamma} = h_t.$$

It is now a routine verification that  $z, h_t$  are generators for the ideal  $\mathfrak{p}_t$ , which, we note, is a height one prime of  $R$ . Consequently  $\text{Cl}(R)$  is generated by  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ . Using  $\sim$  to denote linear equivalence, we have

$$nE_t + n\gamma D \sim 0 \quad \text{and} \quad \sum_{i=1}^r E_i + mD \sim 0,$$

implying that  $n[\mathfrak{p}_t] = 0$  and  $\sum_i [\mathfrak{p}_i] = 0$  in  $\text{Cl}(R)$ . These correspond to the calculations with divisorial ideals,

$$\mathfrak{p}_t^{(n)} = h_t R \quad \text{and} \quad \bigcap_{i=1}^r \mathfrak{p}_i = zR,$$

and imply, in particular, that  $[\mathfrak{p}_1], \dots, [\mathfrak{p}_{r-1}]$  is a generating set for  $\text{Cl}(R)$ .  $\square$

*Example 3.2.* We use Proposition 3.1 to compute the divisor class group of diagonal hypersurfaces

$$R = k[z, x_1, \dots, x_d] / (z^n - x_1^{m_1} - \dots - x_d^{m_d})$$

where  $n$  is relatively prime to  $m_i$  for  $1 \leq i \leq d$ , and  $k$  is a field of characteristic zero, or of characteristic not dividing each  $m_i$ .

By the Jacobian criterion,  $R$  has an isolated singularity at the homogeneous maximal ideal  $\mathfrak{m}$ . Hence if  $d \geq 4$ , then  $R$ , as well as its  $\mathfrak{m}$ -adic completion  $\widehat{R}$ , are factorial by Grothendieck's parafactoriality theorem [5]; see [2] for a simple proof of Grothendieck's theorem.

*Case  $d = 3$ .* The polynomial  $g = x_1^{m_1} + x_2^{m_2} + x_3^{m_3}$  is irreducible since  $k[x_1, x_2, x_3]/(g)$  is a normal domain by the Jacobian criterion. We set  $\deg x_i$  to be  $m_1 m_2 m_3 / m_i$ . Then  $g$  is a weighted homogeneous polynomial of degree  $m_1 m_2 m_3$ , which is relatively prime to  $n$ , so Proposition 3.1 implies that  $R$  is factorial. Since  $R$  satisfies the Serre conditions  $(R_2)$  and  $(S_3)$ , the completion  $\widehat{R}$  is factorial as well by [4, Korollar 1.5]. The divisor class groups of rational three-dimensional Brieskorn singularities are computed in [1, Chapter IV]; see also [13].

*Case  $d = 2$ .* Let  $g = x_1^{m_1} + x_2^{m_2}$ . If  $c = \gcd(m_1, m_2)$ , let  $m_1 = ac$  and  $m_2 = bc$ , and set  $\deg x_1 = b$  and  $\deg x_2 = a$ . Let  $f$  be an irreducible factor of  $g$ . Then  $f$  is homogeneous, and hence has the form  $\sum a_{ij} x_1^i x_2^j$  where  $a_{ij} \in k$  and  $bi + cj = \deg f$

for each term occurring in the summation. Since  $x_1$  and  $x_2$  do not divide  $g$ , we see that  $f$  must contain nonzero terms of the form  $a_{0j}x_2^j$  and  $a_{i0}x_1^i$ . Hence  $\deg f$  is a multiple of  $ab$ , and it follows that  $f$  is a polynomial in  $x_1^a$  and  $x_2^b$ . Consequently the number of factors of  $g$  in  $k[x_1, x_2]$  is the number of factors of  $s^c + t^c$  in  $k[s, t]$  or, equivalently, the number of factors of  $1 + t^c$  in  $k[t]$ .

In particular, if  $m_1$  and  $m_2$  are relatively prime, then  $\widehat{g}$  is irreducible and Proposition 3.1 implies that  $R$  is factorial. As is well-known,  $\widehat{R}$  need not be factorial; see for example, [10, Theorem III.5.2].

If  $k$  is algebraically closed, then  $g$  is a product of  $c$  irreducible factors, and so Proposition 3.1 implies that

$$\mathrm{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{c-1}.$$

*Remark 3.3.* The condition that the degree of  $g$  is relatively prime to  $n$  is certainly crucial in Proposition 3.1. In the absence of this,  $\mathrm{Cl}(R)$  need not be finite, for example  $\mathbb{C}[z, x_1, x_2, x_3]/(z^3 - x_1^3 - x_2^3 - x_3^3)$  has divisor class group  $\mathbb{Z}^6$ . However, one can drop the relatively prime condition when considering hypersurfaces of the form  $z^n - x_0g(x_1, \dots, x_d)$ , see also [6, Proposition 3.12]:

**Corollary 3.4.** *Let  $R = k[z, x_0, \dots, x_d]/(z^n - x_0g)$  be a normal hypersurface over a field  $k$ , where  $g$  is a weighted homogeneous polynomial in  $x_1, \dots, x_d$ . Let  $g = h_1 \cdots h_r$ , where  $h_i \in k[x_1, \dots, x_d]$  are irreducible. Then*

$$\mathrm{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^r,$$

and the images of  $(z, h_1), \dots, (z, h_r)$  form a minimal generating set for  $\mathrm{Cl}(R)$ .

*Proof.* We may choose the degree of  $x_0$  such that  $\deg(x_0g)$  is relatively prime to  $n$ . The result then follows from Proposition 3.1.  $\square$

We conclude with the following example.

*Example 3.5.* Let  $k$  be a field. Corollary 3.4 implies that the divisor class group of the ring  $R = k[xy, x^n, y^n]$  is  $\mathbb{Z}/n\mathbb{Z}$ , since  $R$  is isomorphic to the hypersurface

$$k[z, x_0, x_1]/(z^n - x_0x_1).$$

In [10, Chapter III], the divisor class group of  $R$  is computed by Galois descent if  $n$  is relatively prime to the characteristic of  $k$ , and by using derivations if  $n$  equals the characteristic of  $k$ .

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