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# Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori

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## Abstract

We construct a rigorous renormalization scheme for two-degree-of-freedom analytic Hamiltonians associated with Diophantine frequency vectors. We prove the existence of an attracting, integrable limit set of the renormalization. As an application of this renormalization scheme, we give a proof of a KAM theorem. We construct analytic invariant tori with Diophantine frequency vectors for near-integrable Hamiltonians in the domain of attraction of this limit set.

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## 1. Introduction and summary of the results

Half a century ago, Kolmogorov [24] stated a theorem concerning the persistence of quasi-periodic motion (invariant tori) with sufficiently incommensurate frequency vectors in the phase space of a (non-degenerate) integrable Hamiltonian system under small analytic perturbations. Several years later, detailed proofs of the theorem were given, and the requirement of analyticity was weakened, by Arnold [1] and Moser [35]. It was the beginning of a series of rigorous results and methods that are collectively referred to as KAM theory (see also [26]).

Almost at the same time, the simple and novel idea of renormalization originated in theoretical physics. It was introduced first in the quantum field theory by Stueckelberg and Petermann [39] and later in statistical mechanics by Kadanoff [17]. In the theory of dynamical systems, renormalization group techniques were first introduced by Feigenbaum [7, 8] and Couillet and Tresser [5] in studying the universality of period-doubling sequences in one-parameter families of one-dimensional maps. Since then, renormalization group ideas have

been widely applied in the investigation of a variety of dynamical phenomena. In discrete-time dynamical systems these include self-similarity of period-doubling bifurcations in two- and higher-dimensional mappings, bifurcations in circle maps and bifurcations of invariant KAM curves in area-preserving maps [9, 16, 18, 19, 25, 30–34, 36–38].

In the framework of the breakup of invariant tori in Hamiltonian flows, renormalization group ideas were first introduced by Escande and Doveil [6]. Some related ideas concerning the renormalization of Hamiltonian flows, due to MacKay and others, can be found in [14, 34]. A rigorous renormalization scheme for analytic Hamiltonians, associated to frequency vectors of a specific type, was formulated by Koch [20, 21]. References such as [10, 11, 27, 29] contain some later related work.

Here, we construct a rigorous renormalization scheme for analytic Hamiltonian functions of two-degree-of-freedom systems which applies to the problem of stability of invariant tori with Diophantine frequency vectors. The renormalization scheme is an extension of Koch's renormalization, from quadratic irrational to a set of full Lebesgue measure Diophantine frequency ratios, in the case of two-degree-of-freedom Hamiltonians. Recently, such a generalization was done by Lopes-Dias for vector fields on a torus of dimension two [28]. It is also related to the approximate renormalization schemes of MacKay [32] and Chandre and Moussa [3] in the framework of the break-up of invariant tori in Hamiltonian systems (see also [4]).

Further, we apply our renormalization scheme to give a proof of a KAM theorem. We construct analytic invariant tori with Diophantine frequency vectors for near-integrable Hamiltonians. A similar method has been used in [23] to prove the existence of non-differentiable invariant tori with a golden mean winding number for Hamiltonians attracted to the critical fixed point of the renormalization [22]. A different method for the construction of invariant tori, based on the analogy with quantum field theory and the summation of the (generically divergent) Lindstedt series, was used by Gallavotti *et al* [12, 13] and Bricmont *et al* [2]. A related result for discrete-time systems concerning the existence of invariant graphs for pairs of commuting maps, for Diophantine rotation numbers, has been obtained by Haydn [16].

We define a sequence of renormalization operators  $\mathcal{R}_n$ ,  $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , between Banach spaces of the analytic two-degree-of-freedom Hamiltonians. The Hamiltonians are functions on complex neighbourhoods of  $\mathbb{T}^2 \times \{0\}$ , where  $\mathbb{T}^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2$  and  $0$  is the zero vector in  $\mathbb{R}^2$ . We call these neighbourhoods the phase space. We refer to the coordinates of this space,  $q$  and  $p$ , as angles and momenta, respectively.

Each of the Banach spaces, indexed by  $n$ , contains an integrable Hamiltonian  $H_n^0$  with an invariant torus of a given frequency  $\omega_n \in \mathbb{R}^2 \setminus \{0\}$  at  $p = 0$  and frequencies of nearby tori *twisted* in the direction of  $\Omega_n \in \mathbb{R}^2 \setminus \{0\}$ . The vectors  $\omega_n$  and  $\Omega_n$  are uniquely determined given a pair of vectors  $(\omega, \Omega) = (\omega_0, \Omega_0)$ . The domain of the  $n$ th-step renormalization operator  $\mathcal{R}_n$  will be restricted to a small neighbourhood of the integrable Hamiltonian  $H_n^0$ , keeping the analysis within the scope of classical KAM theory. The  $n$ th-step renormalization operator  $\mathcal{R}_n$  is a function from that neighbourhood into another that maps a Hamiltonian  $H_n$ , close to  $H_n^0$ , into  $H_{n+1} = \mathcal{R}_n(H_n)$ , close to  $H_{n+1}^0 = \mathcal{R}_n(H_n^0)$ . On the winding ratio  $\alpha_n = \omega_{n2}/\omega_{n1}$ ,  $\omega_{n1} \neq 0$ , of the frequency vector  $\omega_n = (\omega_{n1}, \omega_{n2})^*$ , the operator  $\mathcal{R}_n$  acts as a shift of its continued fraction expansion. The renormalization of a Hamiltonian  $H_0$  consists of successive application of the operators  $\mathcal{R}_n$ ,  $n \in \mathbb{N}_0$ . Given a Hamiltonian  $H_0$ , we call the sequence of Hamiltonians  $H_n$ ,  $n \in \mathbb{N}_0$ , consisting of  $H_0$  and its images  $H_{n+1} = \mathcal{R}_n \circ \dots \circ \mathcal{R}_0(H_0)$ , the orbit of the Hamiltonian  $H_0$ .

Renormalization techniques in dynamical systems are designed to study systems on progressively smaller spatial scales and longer time scales. The renormalization scheme

for Hamiltonians is essentially a transformation of classical Hamiltonian functions generated by a scaling of the phase space, modulo a set of transformations that preserve the topological characteristics of the orbits of the Hamiltonian flows. This set includes canonical transformations homotopic to the identity, scaling of the momenta, time and energy. The  $n$ th-step renormalization operator basically consists of time-rescaling, composed with a nonlinear diffeomorphism homotopic to the identity, and a linear scaling of the (lifted) phase space.

The linear scaling transformation is intended to enlarge a region around the orbits of the integrable Hamiltonian flow while keeping the periodicity of the angle coordinates. It is a composition of a linear scaling of the momentum space and a linear canonical transformation of the phase space generated by a point transformation in  $GL(2, \mathbb{Z})$ .

The growth of the Fourier modes of the Hamiltonian in the direction of the dominant flow is prevented by eliminating these modes in each step. This is achieved by a nonlinear canonical transformation homotopic to the identity. The process of elimination (of ‘irrelevant’ modes of a Hamiltonian) and rescaling (of the Fourier lattice) is similar in spirit to block spin transformations in statistical mechanics—a standard tool in the theory of critical phenomena.

Additionally, time-rescaling, a nonlinear scaling of the momenta and a translation in momentum space are included. The latter transformation prevents the renormalization operators from having an expanding eigendirection. These transformations are also homotopic to the identity and do not change the winding number of the orbits of the Hamiltonian flow.

**Main results.** *The results of this paper concerning the constructed renormalization scheme for analytic Hamiltonians and its applications are:*

- (i) *the  $n$ th-step renormalization operator  $\mathcal{R}_n$ ,  $n \in \mathbb{N}_0$ , is a well-defined analytic map on an open ball of analytic Hamiltonians around  $H_n^0$  for a generic frequency vector  $\omega_n \in \mathbb{R}^2 \setminus \{0\}$  (theorem 4.4);*
- (ii) *for a set of Diophantine frequency vectors  $\omega_0$  of full Lebesgue measure, the renormalization orbits of all Hamiltonians in a neighbourhood of an integrable Hamiltonian  $H_0^0$  associated to  $\omega_0$  approach the orbit of the integrable Hamiltonian  $H_0^0$  (theorem 7.2);*
- (iii) *every Hamiltonian  $H_0$  sufficiently close to such a  $H_0^0$  has an invariant torus on which the motion is conjugate to a linear flow of Diophantine frequency vector  $\omega_0$  (theorem 9.5);*
- (iv) *the invariant torus is an analytic function (theorem 10.4).*

The above-mentioned set of Diophantine frequency vectors for which the convergence result has been obtained and invariant tori have been constructed contains those with Diophantine exponent  $\beta < (\sqrt{161} - 11)/10$ . Though this set is not optimal, it is of full Lebesgue measure. We believe that this is not a restriction of the renormalization method and that the set can be extended to other Diophantine frequency vectors.

This paper is organized as follows. In the next section, we define the Banach spaces of Hamiltonians that will be renormalized. In section 3, we describe the performed linear scaling of the phase space. Section 4 contains the construction of the  $n$ th-step renormalization operator. The existence of a canonical transformation that eliminates the non-resonant modes of a near-integrable Hamiltonian is proved in section 5. In section 6, we recall the definition of Diophantine frequency vectors and obtain some bounds that will be used in section 7 to prove the convergence of the renormalization dynamics to an integrable limit set for  $\omega$  satisfying a Diophantine condition. The remaining sections contain an application of the previously constructed renormalization scheme. Section 8 contains the definition of invariant tori. In section 9, we construct the invariant tori with Diophantine frequency vectors for near-integrable Hamiltonians. Finally, in section 10, we prove that the constructed invariant tori can be extended to analytic functions.

## 2. The spaces of Hamiltonians

Let us start by defining the spaces of Hamiltonians that we will consider. Given  $n \in \mathbb{N}_0$ , let  $\omega_n, \Omega_n \in \mathbb{R}^2 \setminus \{0\}$  be two vectors not parallel to each other. We introduce the normalized vector  $\hat{\omega}_n = \omega_n / \|\omega_n\|$ . Here and in what follows  $\|\cdot\|$  denotes  $\ell^1$ -norm of a vector. Define  $\omega'_n$  and  $\Omega'_n$  in  $\mathbb{R}^2$  by the following relations:  $\omega'_n \cdot \Omega_n = 0$ ,  $\omega'_n \cdot \hat{\omega}_n = 1$ ,  $\Omega'_n \cdot \hat{\omega}_n = 0$  and  $\Omega'_n \cdot \Omega_n = 1$ , where ‘ $\cdot$ ’ stands for the usual scalar product of the vectors in  $\mathbb{R}^2$ .

**Definition 2.1.** Given a pair of positive numbers  $\rho = (\rho_1, \rho_2)$ , define

$$\begin{aligned} \mathcal{D}_{n,1}(\rho_1) &= \{q \in \mathbb{C}^2 : |\operatorname{Im} \omega'_n \cdot q| < \rho_1, |\operatorname{Im} \Omega'_n \cdot q| < \rho_1\}, \\ \mathcal{D}_{n,2}(\rho_2) &= \{p \in \mathbb{C}^2 : |\hat{\omega}_n \cdot p| < \rho_2, |\Omega_n \cdot p| < \rho_2\} \end{aligned} \tag{1}$$

and let  $\mathcal{D}_n(\rho) = \mathcal{D}_{n,1}(\rho_1) \times \mathcal{D}_{n,2}(\rho_2)$ .

The Hamiltonians are analytic functions  $H_n : \mathcal{D}_n(\rho) \rightarrow \mathbb{C}$ ,  $2\pi$ -periodic in both components of  $q$ . These functions can be expanded in Fourier–Taylor series

$$H_n(q, p) = \sum_{(v,k) \in I} (H_n)_{v,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq \cdot v}, \tag{2}$$

where  $k = (k_1, k_2)$  and  $I = \mathbb{Z}^2 \times \mathbb{N}_0^2$ . We will refer to each term in this sum as a mode of the Hamiltonian  $H_n$ .

**Definition 2.2.** Given  $\rho > 0$ , componentwise, define  $\mathcal{A}_n(\rho)$  to be the Banach space of functions  $H_n$  that are analytic on  $\mathcal{D}_n(\rho)$ , extend continuously to the boundary of  $\mathcal{D}_n(\rho)$  and have finite norm

$$\|H_n\|_{n,\rho} = \sum_{(v,k) \in I} |(H_n)_{v,k}| \rho_2^{\|k\|} e^{\rho_1(|\hat{\omega}_n \cdot v| + |\Omega_n \cdot v|)}. \tag{3}$$

Let us also define the projection operators  $\mathbb{P}_n^k$  on  $\mathcal{A}_n(\rho)$  by

$$\mathbb{P}_n^k H_n = (H_n)_{0,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} \tag{4}$$

and let  $\mathbb{E} = \sum_{k \in \mathbb{N}_0^2} \mathbb{P}_n^k$  be the projection operator onto the subspace of  $q$ -independent Hamiltonians. Define the functionals  $\mathfrak{p}_n^k : \mathcal{A}_n(\rho) \rightarrow \mathbb{C}$  by  $\mathfrak{p}_n^k H_n = (H_n)_{0,k}$ .

The analysis of this paper is focused on Hamiltonians of the form  $H_n = H_n^0 + h_n$ , where

$$H_n^0 = \omega_n \cdot p + \frac{1}{2} (\Omega_n \cdot p)^2 \tag{5}$$

is an integrable Hamiltonian and  $h_n \in \mathcal{A}_n(\rho)$  is a perturbation. The integrable Hamiltonians  $H_n^0$  are degenerate, in the sense that they do not satisfy Kolmogorov’s non-degeneracy condition, but they do have a *twist* in the  $\Omega_n$  direction, i.e. they satisfy

$$\det \left[ \frac{\partial^2 H_n^0}{\partial p_i \partial p_j} \right] = 0, \quad \left| \frac{\partial^2 H_n^0}{\partial (\Omega_n \cdot p)^2} \right| = 1 \neq 0. \tag{6}$$

Hamiltonians satisfying this weaker (than Kolmogorov’s original) non-degeneracy condition have been included in the improved versions of the KAM theory.

The sequence of renormalization transformations is associated to the sequence of vector pairs  $(\omega_n, \Omega_n)$ ,  $n \in \mathbb{N}_0$ . This sequence has been constructed from a pair of vectors  $\omega, \Omega \in \mathbb{R}^2 \setminus \{0\}$ . We assume that  $\omega \in \mathbb{R}^2$  is of the form  $\omega = \ell(1, \alpha)^*$ , where  $\ell \in \mathbb{R}^+$  and  $\alpha > 1$  is an irrational number. Here ‘ $*$ ’ stands for ‘transpose’.

The unique continued fraction expansion [15] of such an irrational  $\alpha \in \mathbb{R}$  is given by

$$\alpha = a_0 + \frac{1}{a_1 + (1/(a_2 + \dots))}, \tag{7}$$

where  $a_n \in \mathbb{Z}^+, n \geq 0$ , are called partial quotients. We also write  $\alpha = [a_0, a_1, \dots]$ . The vectors  $\omega_n$  are constructed from the continued fraction expansion of the winding ratio  $\alpha$  of the frequency vector  $\omega$ . We define  $\omega_n = \ell(1, \alpha_n)^*$ , where  $\alpha_n = [a_n, a_{n+1}, \dots]$ .

We choose a vector  $\Omega \in \mathbb{R}^2$  which is suitable in the sense of the following definition.

**Definition 2.3.**  $\Omega \in \mathbb{R}^2$  will be called suitable if it is of the form  $\Omega = (\Omega^1, \Omega^2)^*$ , with  $|\Omega^1| \geq |\Omega^2| \geq 0, \Omega^1 \Omega^2 \leq 0$  and  $\|\Omega\| = 1$ .

Notice that a suitable vector  $\Omega$  cannot be arbitrary close to the direction of  $\omega$ . The smaller (positive) angle between these vectors is larger than  $\pi/4$  and smaller than  $3\pi/4$ .

### 3. Scaling of the phase space

The renormalization scheme for  $d$ -degree-of-freedom Hamiltonians introduced by Koch is associated to a frequency vector  $\omega$  whose components span an algebraic number field of degree  $d \in \mathbb{N}$ . For two-degree-of-freedom systems, this scheme can be applied to Hamiltonians associated to a frequency vector with a quadratic irrational slope. In that case, one can find a hyperbolic matrix  $T \in GL(2, \mathbb{Z})$  with determinant  $\pm 1$ , for which  $\omega$  is an expanding eigenvector (lemma 4.1 in [20]). That matrix is then used to perform linear scaling of the phase space at each renormalization step.

In order to construct a renormalization scheme which applies to a larger set of frequency vectors, we use a different linear scaling transformation at each step. We generate the  $n$ th-step scaling transformation of the phase space using the matrix  $T_n \in GL(2, \mathbb{Z})$  defined by

$$T_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}. \tag{8}$$

More precisely, at the  $n$ th renormalization step, the scaling of the (lifted) phase space is performed with a linear map  $\mathcal{T}_n^{-1}$  on  $\mathbb{C}^2 \times \mathbb{C}^2$ , defined by  $\mathcal{T}_n(q, p) = (T_n q, \mu_n T_n^{*-1} p)$ , where  $\mu_n = \alpha_{n+1}^{-1} \|T_n^{-1} \Omega_n\|^{-2}$  is a positive number. This map generates the transformation of Hamiltonians  $H_n \mapsto H_n \circ \mathcal{T}_n$ . Notice that the vectors  $\omega_n$  could also be defined recursively using this matrix, via  $\omega_{n+1} = \alpha_{n+1} T_n^{-1} \omega_n$ , given  $\omega = \omega_0$ . Similarly, given a suitable vector  $\Omega = \Omega_0$ , the vectors  $\Omega_n$  are defined recursively by  $\Omega_n = T_{n-1}^{-1} \Omega_{n-1} / \|T_{n-1}^{-1} \Omega_{n-1}\|$ , for  $n \in \mathbb{N}$ . Thus, under this scaling, the integrable Hamiltonian  $H_n^0$  is mapped into  $\alpha_{n+1}^{-1} \mu_n H_{n+1}^0, n \in \mathbb{N}_0$ . A time rescaling, then, normalizes the latter to  $H_{n+1}^0$ .

The matrix  $T_n$  has the following properties. If  $\alpha > 1$ , then, for all  $n \geq 0, a_n \geq 1$  and  $T_n$  is a hyperbolic matrix with  $\det(T_n) = -1$  and eigenvalues  $\lambda_n$  and  $-1/\lambda_n$ , where

$$\lambda_n = \frac{a_n + \sqrt{a_n^2 + 4}}{2}. \tag{9}$$

The expanding and contracting eigendirections corresponding to these eigenvalues are given by  $(1, \lambda_n)^*$  and  $(1, -1/\lambda_n)^*$ , respectively. The expanding eigenvector is close to  $\omega_n$ , in the sense that the absolute value of the angle between it and  $\omega_n$  is smaller than  $\pi/4$ .

The map  $\omega_n \mapsto \omega_{n+1}$  is related with the Gauss map of the fractional part of  $\alpha_n$ . Given  $\alpha \in \mathbb{R}$ , let  $[\alpha]$  be the integer part of  $\alpha$ , i.e.  $[\alpha] = \max\{k \in \mathbb{Z} : k \leq \alpha\}$ . Also, let  $\{\alpha\} = \alpha - [\alpha]$  be the fractional part of  $\alpha$ .

For  $x > 0$ , the Gauss map is defined as

$$G : x \mapsto \left\{ \frac{1}{x} \right\}. \tag{10}$$

If we define  $x_n = \alpha_n - a_n$ , for  $n \geq 0$ , we obtain  $x_{n+1} = G(x_n)$ . Thus, the shift  $[a_n, a_{n+1}, \dots] \mapsto [a_{n+1}, a_{n+2}, \dots]$  of the continued fraction expansion of  $\alpha_n$  corresponds to the Gauss map of its fractional part  $x_n$ .

The transformation  $\alpha_n \mapsto \alpha_{n+1} = (\alpha_n - a_n)^{-1}$  is a special case of a modular transformation,

$$\alpha \mapsto \frac{T_{21} + \alpha T_{22}}{T_{11} + \alpha T_{12}}, \tag{11}$$

generated by the action of an integer matrix  $T = (T_{ij}) \in GL(2, \mathbb{Z})$  of determinant  $\pm 1$ . Numbers related by such a transformation are called equivalent. Two numbers are equivalent if and only if they have the same tail in their continued fraction expansions (theorem 175 in [15]). The matrix  $T_n$  generates an equivalence relation between  $\alpha_{n+1}$  and  $\alpha_n$ .

The linear coordinate change  $\mathcal{T}_n^{-1}$  maps the orbit of  $H_n^0$  with rotation number  $\alpha_n$  into the orbit of  $H_{n+1}^0$  with the rotation number  $\alpha_{n+1}$ . We have made the particular choice of  $T_n$  in order to provide the desired ‘scale change’ in the sense explained below.

Let us consider the sequence of periodic orbits approximating an invariant torus with frequency ratio  $\alpha$ . Recursively, define the sequence of convergent matrices associated to  $\alpha$ ,

$$P_n = \begin{bmatrix} q_{n-1} & p_{n-1} \\ q_n & p_n \end{bmatrix}, \tag{12}$$

for  $n \geq 0$ , via  $P_n = T_n P_{n-1}$ , with  $P_{-1}$  being the identity matrix in  $GL(2, \mathbb{Z})$ . Thus,  $q_n$  and  $p_n$  are determined by the following recursion relations

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2}, \\ p_n &= a_n p_{n-1} + p_{n-2} \end{aligned} \tag{13}$$

and  $q_{-2} = 1, p_{-2} = 0, q_{-1} = 0, p_{-1} = 1$ . The convergent matrices define a sequence of convergents  $p_n/q_n = [a_0, \dots, a_n]$  that approaches  $\alpha$  as  $n \rightarrow \infty$ .

The values of  $q_n$  and  $p_n$  are associated to  $\alpha$ . In the following, we will stress this fact by writing explicitly  $q_n(\alpha)$  and  $p_n(\alpha)$ . Consider the sequence of periodic orbits of frequency ratios  $p_k(\alpha_n)/q_k(\alpha_n), k \in \mathbb{N}_0$ . The matrix  $T_n^{-1}$  produces the following ‘scale change’

$$\begin{pmatrix} -a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_k(\alpha_n) \\ p_k(\alpha_n) \end{pmatrix} = \begin{pmatrix} q_{k-1}(\alpha_{n+1}) \\ p_{k-1}(\alpha_{n+1}) \end{pmatrix} \tag{14}$$

motivating its use. A periodic orbit labelled by  $k$  that approximates an invariant torus with frequency ratio  $\alpha_n$  of  $H_n^0$  is mapped into a periodic orbit labelled by  $k - 1$  that approximates an invariant torus with frequency ratio  $\alpha_{n+1}$  of  $H_{n+1}^0$ .

The composition  $H_n \circ \mathcal{T}_n$  represents a singular operator, as the matrix  $T_n$  has an expanding eigendirection. The composition is, however, harmless when the domain of the operator is restricted to Hamiltonians that contain only the modes for which  $\|k\|$  is large or  $\nu$  is almost perpendicular to  $\omega$ . These modes are called *resonant*. These are the modes that produce small denominators in KAM theory.

**Definition 3.1.** Given  $\sigma, \kappa > 0$  and vectors  $\omega_n, \Omega_n \in \mathbb{R}^2$ , we define the non-resonant index set,

$$I_n^- = \{(v, k) \in I : |\omega_n \cdot v| > \sigma |\Omega_n \cdot v|, |\omega_n \cdot v| > \kappa \|k\|\}. \tag{15}$$

The resonant index set is defined as its complement,  $I_n^+ = I \setminus I_n^-$ . The corresponding projection operators on  $\mathcal{A}_n(\rho), \mathbb{I}_n^-$  and  $\mathbb{I}_n^+ = \mathbb{I} - \mathbb{I}_n^-$ , are defined by setting

$$(\mathbb{I}_n^- H_n)(q, p) = \sum_{(v,k) \in I_n^-} (H_n)_{v,k} (\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} e^{iq \cdot v}. \tag{16}$$

Hamiltonians consisting only of resonant modes will be called resonant. On the space of resonant Hamiltonians, the map  $H_n \mapsto H_n \circ \mathcal{T}_n$  is analyticity improving.

**Proposition 3.2.** Let  $0 < \rho' < \rho$  with  $3\rho' > 2\rho$ , componentwise. If  $\sigma$  and  $\kappa$  are sufficiently small constants, then every Hamiltonian  $H_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$ ,  $n \in \mathbb{N}_0$ , has an analytic extension to  $\mathcal{T}_n \mathcal{D}_{n+1}(\rho)$ . The linear map from  $\mathbb{I}_n^+ \mathcal{A}_n(\rho')$  to  $\mathcal{A}_{n+1}(\rho)$ , given by  $H_n \mapsto H_n \circ \mathcal{T}_n$ , is compact.

**Proof.** As

$$H_n \circ \mathcal{T}_n(q, p) = \sum_{(v,k) \in I_n^+} (H_n)_{v,k} (\mu_n T_n^{-1} \hat{\omega}_n \cdot p)^{k_1} (\mu_n T_n^{-1} \Omega_n \cdot p)^{k_2} e^{iq \cdot T_n^* v}, \tag{17}$$

one can obtain

$$\begin{aligned} \|H_n \circ \mathcal{T}_n\|_{n+1, \rho} &= \sum_{(v,k) \in I_n^+} |(H_n)_{v,k}| \left( \frac{\mu_n \|\omega_{n+1}\|}{\alpha_{n+1} \|\omega_n\|} \rho_2 \right)^{k_1} (\mu_n \|T_n^{-1} \Omega_n\| \rho_2)^{k_2} \\ &\quad \times \exp \left( \rho_1 \left( \frac{\alpha_{n+1}}{\|\omega_{n+1}\|} |\omega_n \cdot v| + \frac{|\Omega_n \cdot v|}{\|T_n^{-1} \Omega_n\|} \right) \right) \\ &= \sum_{(v,k) \in I_n^+} |(H_n)_{v,k}| \left( \frac{\mu_n \|\omega_{n+1}\| \rho_2}{\alpha_{n+1} \|\omega_n\| \rho_2'} \right)^{k_1} \left( \frac{\mu_n \|T_n^{-1} \Omega_n\| \rho_2}{\rho_2'} \right)^{k_2} \rho_2'^{|k|} \\ &\quad \times \exp \left( \left( \frac{\rho_1 \alpha_{n+1} \|\omega_n\|}{\|\omega_{n+1}\|} - \rho_1' \right) |\hat{\omega}_n \cdot v| \right. \\ &\quad \left. + \left( \frac{\rho_1}{\|T_n^{-1} \Omega_n\|} - \rho_1' \right) |\Omega_n \cdot v| \right) e^{\rho_1' (|\hat{\omega}_n \cdot v| + |\Omega_n \cdot v|)} \\ &\leq \|H_n\|_{n, \rho'} \end{aligned} \tag{18}$$

provided that all of the modes contract. The terms with  $v$  obeying the inequality  $|\omega_n \cdot v| \leq \sigma |\Omega_n \cdot v|$  contract for sufficiently small  $\sigma$  satisfying the second part of the double inequality

$$\frac{\sigma (\alpha_{n+1} / \|\omega_{n+1}\|) + (1 / \|T_n^{-1} \Omega_n\|)}{1 + (\sigma / \|\omega_n\|)} \leq \frac{\sigma}{\ell} + \frac{2}{3} < \frac{\rho_1'}{\rho_1}. \tag{19}$$

Given  $\sigma > 0$ , the first part is satisfied for any  $n \in \mathbb{N}_0$ . The other conditions are trivially satisfied as  $\mu_n \|T_n^{-1} \Omega_n\| \leq 2/3$ .

The modes indexed by  $(v, k)$  satisfying  $|\omega_n \cdot v| \leq \kappa \|k\|$  also contract if  $\kappa$  satisfies the second part of the double inequality

$$\mu_n \|T_n^{-1} \Omega_n\| \exp \left( \kappa \left( \frac{\rho_1 \alpha_{n+1}}{\|\omega_{n+1}\|} - \frac{\rho_1'}{\|\omega_n\|} \right) \right) \leq \frac{2}{3} e^{\kappa \rho_1 / \ell} < \frac{\rho_2'}{\rho_2}. \tag{20}$$

Given  $\kappa > 0$ , the first part is satisfied for any  $n \in \mathbb{N}_0$ .

These estimates show that  $H_n \circ \mathcal{T}_n$  is analytic in  $\mathcal{D}_{n+1}(\rho)$ . Now, given  $\sigma, \kappa > 0$  satisfying the second parts of the double inequalities (19) and (20), one can find  $r > \rho$  satisfying  $3\rho' > 2r$  componentwise, such that  $H_n \circ \mathcal{T}_n$  is also analytic and bounded in  $\mathcal{D}_{n+1}(r)$ . The assertion now follows from the fact that the inclusion map from  $\mathcal{A}_{n+1}(r)$  to  $\mathcal{A}_{n+1}(\rho)$  is compact.  $\square$

#### 4. The $n$ th-step renormalization operator

We will restrict the domain of the  $n$ th-step renormalization operator to resonant Hamiltonians. The composition of a resonant Hamiltonian with  $\mathcal{T}_n$  produces, in general, non-resonant modes. In the  $n$ th renormalization step, we completely eliminate these modes such that the renormalized Hamiltonians are also resonant. We also include a translation in the variable  $\Omega_{n+1} \cdot p$  to prevent the  $n$ th-step renormalization operator from having an expanding eigendirection. The existence of the non-zero quadratic (in the components of  $p$ ) part of the integrable Hamiltonian  $H_n^0$  is essential for the construction of such a transformation. Finally, we perform an additional, nonlinear, scaling of the action (momentum) variables and time, in order to fix the coefficients of the  $(\omega_{n+1} \cdot p)$  and  $(\Omega_{n+1} \cdot p)^2$  modes of the renormalized Hamiltonians to 1 and  $1/2$ , respectively.



**Definition 4.1.** The  $n$ th-step renormalization operator  $\mathcal{R}_n$  is defined (formally) on an open ball in  $\mathbb{I}_n^+ \mathcal{A}_n(\rho')$ , with  $\rho' > 0$ , componentwise, by the following action on a Hamiltonian  $H_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$ ,

$$\mathcal{R}_n(H_n) = \frac{\theta'_n}{\mu'_n} (H_n \circ \Lambda_n - \mathbb{P}_{n+1}^{(0,0)} H_n \circ \Lambda_n), \tag{21}$$

where  $\Lambda_n = \mathcal{T}_n \circ V_{H'_{n+1}} \circ U_{H''_{n+1}} \circ S_{\tilde{H}_{n+1}}$ . Here  $H'_{n+1} = (\theta_n/\mu_n)H_n \circ \mathcal{T}_n$ , with  $\theta_n = \alpha_{n+1}$ ,  $H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}}$  and  $\tilde{H}_{n+1} = H''_{n+1} \circ U_{H''_{n+1}}$ . The transformation  $V_{H'_{n+1}} : \mathcal{D}_{n+1}(\varrho) \rightarrow \mathcal{D}_{n+1}(\rho)$ , where  $3/2\rho' > \rho > \varrho > \varrho' > \rho'$ , componentwise, represents the translation  $V_{H'_{n+1}}(q, p) = (q, p - v_{H'_{n+1}}\Omega'_{n+1})$ , with  $v_{H'_{n+1}} \in \mathbb{C}$  determined by the equation  $\mathfrak{p}_{n+1}^{(0,1)} H'_{n+1} \circ V_{H'_{n+1}} = 0$ . The map  $U_{H''_{n+1}} : \mathcal{D}_{n+1}(\varrho') \rightarrow \mathcal{D}_{n+1}(\varrho)$  is a canonical transformation that satisfies  $\mathbb{I}_{n+1}^-(H''_{n+1} \circ U_{H''_{n+1}}) = 0$ . The transformation  $S_{\tilde{H}_{n+1}} : \mathcal{D}_{n+1}(\rho') \rightarrow \mathcal{D}_{n+1}(\varrho')$  represents a scaling  $S_{\tilde{H}_{n+1}}(q, p) = (q, z_{\tilde{H}_{n+1}} p)$  of the action variables. The scaling parameters  $\theta'_n$  and  $\mu'_n$  take the values  $\theta'_n = \theta_n \cdot \tau_{\tilde{H}_{n+1}}$  and  $\mu'_n = \mu_n \cdot z_{\tilde{H}_{n+1}}$ . The parameters  $z_{\tilde{H}_{n+1}}, \tau_{\tilde{H}_{n+1}} \in \mathbb{C}$  are determined such that  $\mathfrak{p}_{n+1}^{(0,2)} H_{n+1} = 1/2$  and  $\mathfrak{p}_{n+1}^{(1,0)} H_{n+1} = 1$ , where  $H_{n+1} = \mathcal{R}_n(H_n)$ .

In the following, we show that the translations and scalings included in the  $n$ th-step renormalization operator are well defined on a sufficiently small open ball around  $H_{n+1}^0$ . Notice first that  $H_{n+1}^0 = \tilde{H}_{n+1}^0 = H_{n+1}^{0'} = H_{n+1}^{0''} = \mathcal{R}_n(H_n^0)$ . For  $n \in \mathbb{N}_0$ ,  $b > 0$  and  $\rho > 0$ , componentwise, define  $B_{n,\rho}(b)$  to be the open ball of radius  $b$ , in  $\mathcal{A}_n(\rho)$ , centred at  $H_n^0$ . Define also  $B_{n,\rho}^+(b)$  to be the open ball in  $\mathbb{I}_n^+ \mathcal{A}_n(\rho)$  of radius  $b$ , centred at  $H_n^0$ .

**Proposition 4.2.** Given  $\rho_2 > \rho_2 > 0$  and  $\rho_1 = \rho_1 > 0$ , the following holds for a sufficiently small constant  $b > 0$ . For every Hamiltonian  $H_{n+1} \in B_{n+1,\rho}(b)$ ,  $n \in \mathbb{N}_0$ , there exists  $v_{H_{n+1}} \in \mathbb{C}$ , such that the translation map  $V_{H_{n+1}} : \mathcal{D}_{n+1}(\varrho) \rightarrow \mathcal{D}_{n+1}(\rho)$  is well defined by  $V_{H_{n+1}}(q, p) = (q, p - v_{H_{n+1}}\Omega'_{n+1})$ , where  $\mathfrak{p}_{n+1}^{(0,1)} H_{n+1} \circ V_{H_{n+1}} = 0$ . The derivative of the map  $\mathcal{V}_{H_{n+1}} : \mathcal{A}_{n+1}(\rho) \rightarrow \mathcal{A}_{n+1}(\varrho)$ , defined by  $\mathcal{V}_{H_{n+1}}(H_{n+1}) = H_{n+1} \circ V_{H_{n+1}}$ , at  $H_{n+1}^0$ , is the linear map  $D\mathcal{V}_{n+1}(H_{n+1}^0) = \mathbb{I} - \mathbb{P}_{n+1}^{(0,1)}$ .

**Proof.** Define the function  $F : \mathcal{A}_{n+1}(\rho) \times \mathbb{C} \rightarrow \mathbb{C}$ , by setting  $F(H_{n+1}, v) = \mathfrak{p}_{n+1}^{(0,1)} H_{n+1} \circ V$ , where  $V(q, p) = (q, p - v\Omega'_{n+1})$ . The implicit equation  $F(H_{n+1}^0, v) = 0$  has a unique solution  $v = v_{H_{n+1}^0} = 0$ . Moreover,  $D_2 F(H_{n+1}^0, v)|_{v=0} = -1 \neq 0$ , where  $D_2$  is the derivative with respect to the second entry. We can use this fact to solve the implicit equation,  $F(H_{n+1}, v) = 0$ , for a given Hamiltonian  $H_{n+1}$  in a ball  $B_{n+1,\rho}(b)$  of sufficiently small,  $n$ -independent radius  $b > 0$ .

The problem of the existence of a solution  $v = v_{H_{n+1}}$  of the implicit equation  $F(H_{n+1}, v) = 0$ , for a given Hamiltonian  $H_{n+1} \in B_{n+1,\rho}(b)$ , is equivalent to the problem of the existence of a fixed point of the function  $G_{H_{n+1}} : v \mapsto v + F(H_{n+1}, v)$ . Notice that  $|G_{H_{n+1}}(0)| = |(h_{n+1})_{0,(0,1)}| < b/\rho_2$ . We will show that, given  $\lambda > 1$ , for sufficiently small  $b > 0$ ,  $G_{H_{n+1}}$  is a contraction on a ball of radius  $\lambda b/\rho_2$ .

Let  $|v| \leq \lambda b/\rho_2$ ,  $\lambda > 1$ . The norms of  $G_{H_{n+1}}(v)$  and  $G'_{H_{n+1}}(v)$  can be bounded by

$$|G_{H_{n+1}}(v)| \leq \sum_{k_2=1}^{\infty} |(h_{n+1})_{0,(0,k_2)}| k_2 |v|^{k_2-1} < \frac{b}{\rho_2(1 - \lambda b/\rho_2^2)^2} = \tilde{b}$$

and

$$|G'_{H_{n+1}}(v)| = |1 + D_2 F(H_{n+1}, v)| \leq \sum_{k_2=2}^{\infty} |(h_{n+1})_{0,(0,k_2)}| k_2(k_2 - 1) |v|^{k_2-2} < \frac{2b}{\rho_2^2(1 - \lambda b/\rho_2^2)^3},$$

respectively. If  $b > 0$  is sufficiently small, then

$$\lambda b/\rho_2 < \rho_2 - \varrho_2, \quad \frac{1}{(1 - \lambda b/\rho_2^2)^2} < \lambda \quad \text{and} \quad \frac{2b}{\rho_2^2(1 - \lambda b/\rho_2^2)^3} < 1 - \frac{1}{\lambda} < 1. \tag{22}$$

These bounds show that for sufficiently small  $b > 0$ ,  $G_{H_{n+1}}$  is a contraction on a closed ball of radius  $\tilde{b}$  and, thus, has a unique fixed point in that ball. As the bounds (22) are independent of  $n$ , so is  $b$ . The first of the bounds (22) shows that the translation map is well defined from  $\mathcal{D}_{n+1}(\varrho)$  to  $\mathcal{D}_{n+1}(\rho)$  and that  $H_{n+1} \circ V_{H_{n+1}}$  belongs to  $\mathcal{A}_{n+1}(\varrho)$ .  $\square$

The map  $H''_{n+1} \mapsto H''_{n+1} \circ U_{H''_{n+1}}$  is well defined on a sufficiently small ball centred at  $H_{n+1}^0$ . Theorem 5.6, proved in the next section, guarantees that given  $\varrho > \varrho' > 0$ , componentwise, for every Hamiltonian  $H''_{n+1} \in \mathcal{A}_{n+1}(\varrho)$ , sufficiently close to  $H_{n+1}^0$ , there exists a canonical map  $U_{H''_{n+1}} : \mathcal{D}_{n+1}(\varrho') \rightarrow \mathcal{D}_{n+1}(\varrho)$  that satisfies the equation  $\mathbb{I}_{n+1}(H''_{n+1} \circ U_{H''_{n+1}}) = 0$ .

**Proposition 4.3.** *Let  $\varrho'_2 > \rho'_2 > 0$  and  $\varrho'_1 = \rho'_1 > 0$ . For sufficiently small constant  $b > 0$  and for every Hamiltonian  $H_{n+1} \in B_{n+1,\varrho'}(b)$ ,  $n \in \mathbb{N}_0$ , there exist  $z_{H_{n+1}}, \tau_{H_{n+1}} \in \mathbb{C}$  such that the map  $\mathfrak{S}_{H_{n+1}} : H_{n+1} \mapsto (\tau_{H_{n+1}}/z_{H_{n+1}})(H_{n+1} \circ S_{H_{n+1}} - \mathbb{P}_{n+1}^{(0,0)} H_{n+1} \circ S_{H_{n+1}})$ , with  $S_{H_{n+1}}(q, p) = (q, z_{H_{n+1}} p)$ , is well defined from  $\mathcal{A}_{n+1}(\varrho')$  to  $\mathcal{A}_{n+1}(\rho')$  and satisfies  $\mathfrak{p}_{n+1}^{(0,2)} \mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1/2$  and  $\mathfrak{p}_{n+1}^{(1,0)} \mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1$ . The map  $S_{H_{n+1}}$  maps  $\mathcal{D}_{n+1}(\rho')$  into  $\mathcal{D}_{n+1}(\varrho')$ . The derivative of the map  $\mathfrak{S}_{H_{n+1}}$  at the point  $H_{n+1}^0$  is given by  $D\mathfrak{S}_{n+1}(H_{n+1}^0) = \mathbb{I} - \mathbb{P}_{n+1}^{(0,0)} - \mathbb{P}_{n+1}^{(1,0)} - \mathbb{P}_{n+1}^{(0,2)}$ .*

**Proof.** Let  $H_{n+1} \in B_{n+1,\varrho'}(b)$ . For sufficiently small  $b > 0$ , the scaling parameters  $\tau_{H_{n+1}}$  and  $z_{H_{n+1}}$  that satisfy the equations  $\mathfrak{p}_{n+1}^{(0,2)} \mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1/2$  and  $\mathfrak{p}_{n+1}^{(1,0)} \mathfrak{S}_{H_{n+1}}(H_{n+1}) = 1$  are given by  $\tau_{H_{n+1}} = 1/(1 + (h_{n+1})_{0,(1,0)})$  and  $z_{H_{n+1}} = (1 + (h_{n+1})_{0,(1,0)})/(1 + 2(h_{n+1})_{0,(0,2)})$ . We have the bound

$$|z_{H_{n+1}}| \leq |z_{H_{n+1}} - 1| + 1 \leq \frac{|(h_{n+1})_{0,(1,0)} - 2(h_{n+1})_{0,(0,2)}|}{1 - 2|(h_{n+1})_{0,(0,2)}|} + 1 \leq \frac{1 + b/\varrho'_2}{1 - 2b/\varrho_2^2} < \frac{\varrho'_2}{\rho'_2},$$

where the last inequality is satisfied for sufficiently small  $n$ -independent constant  $b > 0$ . The scaling map is well defined from  $\mathcal{D}_{n+1}(\rho')$  to  $\mathcal{D}_{n+1}(\varrho')$  and the resulting Hamiltonian belongs to  $\mathcal{A}_{n+1}(\rho')$ .  $\square$

We can now show that the  $n$ th-step renormalization operator is well defined on an open ball around  $H_n^0$ .

**Theorem 4.4.** *Given  $\rho'_1 > 0$ , for sufficiently small  $\sigma, \kappa > 0$  and  $\rho'_2 > 0$  satisfying  $3\rho'_2/2 < \sigma < \ell/3$ , there exists a constant  $C' > 0$ , such that the  $n$ th-step renormalization operator  $\mathcal{R}_n$  is a well-defined analytic map from an open ball  $B_{n,\rho'}^+(\zeta_n)$ , of radius  $\zeta_n = C'/(\alpha_n \alpha_{n+1})^2$ , into  $\mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho')$ . Also,  $\|\mathcal{R}_n(H_n^0 + h_n) - H_{n+1}^0\|_{n+1,\rho'} \leq \zeta_n^{-1} \|h_n\|_{n,\rho'}$  and*

$$\|\mathcal{R}_n(H_n^0 + h_n) - H_{n+1}^0 - D\mathcal{R}_n(H_n^0)h_n\|_{n+1,\rho'} \leq [\zeta_n(\zeta_n - \|h_n\|_{n,\rho'})]^{-1} \|h_n\|_{n,\rho'}^2. \tag{23}$$

**Proof.** Let  $3\rho'/2 > 3\rho'/(2+9\rho_2^2/(2\ell)) > \rho > \varrho > \varrho' > \rho'$ , componentwise. The bound (18) implies that there exists a constant  $b_1 > 0$ , such that  $(\theta_n/\mu_n)B_{n,\rho'}^+(\zeta_n) \circ \mathcal{T}_n \subset B_{n+1,\rho}(b_1 C')$ , where  $\zeta_n = C'/(\alpha_n \alpha_{n+1})^2$  and  $C' > 0$ . Proposition 4.2 guarantees that for sufficiently small  $C' > 0$ ,

$$\{H''_{n+1} : H''_{n+1} = H'_{n+1} \circ V_{H'_{n+1}}, H'_{n+1} \in B_{n+1,\rho}(b_1 C')\} \subset B_{n+1,\varrho}(b_1 b_2 C'),$$

with  $b_2 > 0$ . If  $C' > 0$  is chosen sufficiently small, there exists (by theorem 5.6) a canonical transformation  $U_{H''_{n+1}}$  for Hamiltonians  $H''_{n+1}$  in a neighbourhood of  $H^0_{n+1}$  containing  $B_{n+1,\varrho}(b_1b_2C')$  that satisfies the equation  $\mathbb{I}^-_{n+1}(H''_{n+1} \circ U_{H''_{n+1}}) = 0$ . Furthermore,

$$\{\tilde{H}_{n+1} : \tilde{H}_{n+1} = H''_{n+1} \circ U_{H''_{n+1}}, H''_{n+1} \in B_{n+1,\varrho}(b_1b_2C')\} \subset B^+_{n+1,\varrho'}(b_1b_2b_3C'),$$

where  $b_3 > 0$  is a constant (dependent on  $\sigma, \kappa$  and  $\varrho$ ). Finally, proposition 4.3 guarantees that for sufficiently small  $C' > 0$ ,

$$\{\mathfrak{S}_{\tilde{H}_{n+1}}(\tilde{H}_{n+1}) : \tilde{H}_{n+1} \in B^+_{n+1,\varrho'}(b_1b_2b_3C')\} \subset B^+_{n+1,\rho'}(b_1b_2b_3b_4C'),$$

with  $b_4 > 0$ . This shows that for sufficiently small  $C' > 0$ , the  $n$ th-step renormalization operator is well defined from  $B^+_{n,\rho'}(\zeta_n)$  to  $\mathbb{I}^+_{n+1}\mathcal{A}_{n+1}(\rho')$ . As a composition of analytic maps, it is an analytic map itself.

Define the map  $g : z \mapsto \mathcal{R}_n(H_n^0 + zh_n) - H^0_{n+1}$ , where  $h_n \in \mathbb{I}^+_n\mathcal{A}_n(\rho')$  is given such that  $H_n = H_n^0 + h_n \in B^+_{n,\rho'}(\zeta_n)$ . This map is analytic from an open ball in  $\mathbb{C}$ , of radius  $\zeta_n/\|h_n\|_{n,\rho'} > 1$ , into  $\mathbb{I}^+_{n+1}\mathcal{A}_{n+1}(\rho')$ . As  $g(0) = 0$ , we have, by the maximum principle, the bound

$$\|g(1)\|_{n+1,\rho'} \leq \sup_{\|\bar{h}_n\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H_n^0 + \bar{h}_n) - H^0_{n+1}\|_{n+1,\rho'} \frac{\|h_n\|_{n,\rho'}}{\zeta_n}.$$

From the construction of the renormalization operator,  $\sup_{\|\bar{h}_n\|_{n,\rho'}=\zeta_n} \|\mathcal{R}_n(H_n^0 + \bar{h}_n) - H^0_{n+1}\|_{n+1,\rho'}$  can be bounded by a constant  $b_1b_2b_3b_4C'$ , less than or equal to 1, if  $C' > 0$  is sufficiently small. Then, we have  $\|\mathcal{R}_n(H_n^0 + h_n) - H^0_{n+1}\|_{n+1,\rho'} \leq \|h_n\|_{n,\rho'}/\zeta_n$ . Cauchy's formula gives an estimate on the norm of the second-order remainder of the Taylor expansion of  $\mathcal{R}_n$  about  $H_n^0$ ,

$$\begin{aligned} F_n^2 = \|g(1) - g(0) - g'(0)\|_{n+1,\rho'} &\leq \frac{1}{2\pi} \oint_{|z|=\zeta_n/\|h_n\|_{n,\rho'}} \frac{\|g(z)\|_{n+1,\rho'}}{|z^2(z-1)|} dz \\ &\leq \frac{\|h_n\|_{n,\rho'}^2}{\zeta_n(\zeta_n - \|h_n\|_{n,\rho'})}, \end{aligned} \tag{24}$$

for sufficiently small  $C' > 0$ . This provides the second desired bound. □

**Remark 4.5.** The renormalization operator  $\mathcal{R}_n$  is actually analyticity improving and can be defined from an open ball in  $\mathbb{I}^+_n\mathcal{A}_n(\rho'')$  into  $\mathbb{I}^+_n\mathcal{A}_n(\rho')$ , with  $\rho' > \rho''$ , componentwise. The loss of analyticity in the transformations close to the identity can be reduced by restricting the domain of the renormalization operator to a smaller ball.

**Remark 4.6.** The  $n$ th-step renormalization operator is well defined on a space of resonant Hamiltonians. For a given  $n \in \mathbb{N}$ , this is not a restriction as, by construction, the renormalized Hamiltonians are always resonant. In order to apply the 0th-step renormalization operator to Hamiltonians containing non-resonant modes also, one can include a pre-renormalization step consisting of a canonical transformation that eliminates the non-resonant modes of a Hamiltonian.

### 5. Elimination of non-resonant modes

In this section, we construct a canonical transformation that eliminates non-resonant modes of a near-integrable Hamiltonian. The whole construction is associated to a single renormalization step. The index of the renormalization step will be suppressed in this section, in order to simplify the notation. For the construction of the canonical transformation, we follow an

approach as in [21]. Our Hamiltonians are, however, assumed to be close to an integrable Hamiltonian that contains a term quadratic in momenta. Technically, in the present scheme, one also needs to assure that the elimination of non-resonant modes is possible at each renormalization step. In that context, we emphasize that the constants that appear in this section will be chosen independently of the renormalization step.

We begin by making a canonical change of coordinates  $(q, p) \rightarrow (x, y)$  with  $x_1 = \omega' \cdot q$ ,  $x_2 = \Omega' \cdot q$ ,  $y_1 = \hat{\omega} \cdot p$  and  $y_2 = \Omega \cdot p$ . We will simplify the notation further, by writing  $H(x, y)$  instead of  $H(q(x, y), p(x, y))$ . Moreover, some of the symbols in this section will have different meanings in other sections. The use of those symbols should be restricted to this section.

In the new coordinates, the Fourier–Taylor series and the norm of a function  $H \in \mathcal{A}(\rho)$ , analytic on

$$\mathcal{D}(\rho) = \{x \in \mathbb{C}^2 : |\operatorname{Im} x_1| < \rho_1, |\operatorname{Im} x_2| < \rho_1\} \times \{y \in \mathbb{C}^2 : |y_1| < \rho_2, |y_2| < \rho_2\}, \quad (25)$$

where  $\rho = (\rho_1, \rho_2) > 0$ , componentwise, are given by

$$H(x, y) = \sum_{(v,k) \in I} H_{v,k} y_1^{k_1} y_2^{k_2} e^{i v \cdot x}, \quad \|H\|_\rho = \sum_{(v,k) \in I} |H_{v,k}| \rho_2^{k_1+k_2} e^{\rho_1(|v_1|+|v_2|)}. \quad (26)$$

Here  $v = (v_1, v_2)$  and  $I = \mathbb{M}^2 \times \mathbb{N}_0^2$ , where  $\mathbb{M}^2 = \{(\hat{\omega} \cdot v, \Omega \cdot v) \in \mathbb{R}^2 : v \in \mathbb{Z}^2\} \subset \mathbb{R}^2$  is a set bijective to  $\mathbb{Z}^2$  that can be determined from the vectors  $\omega$  and  $\Omega$ .

The non-resonant index set is defined as

$$I^- = \left\{ (v, k) \in I : |v_1| > \frac{\sigma}{\|\omega\|} |v_2|, |v_1| > \frac{\kappa}{\|\omega\|} \|k\| \right\}. \quad (27)$$

The resonant index set is its complement  $I^+ = I \setminus I^-$ .

We state without proof the following technical proposition. In what follows, the norm of the functions  $X = (X_1, X_2) \in \mathcal{A}^2(\rho)$  is defined as  $\|X\|_\rho = \max\{\|X_1\|_\rho, \|X_2\|_\rho\}$ . We denote by  $\partial_i H$ , for  $i = 1, 2$ , the partial derivatives of  $H(x, y)$  with respect to  $x_1$  and  $x_2$ , and for  $i = 3, 4$ , the partial derivatives of the same function with respect to  $y_1$  and  $y_2$ , respectively.

**Proposition 5.1.** *Let  $\rho = (\rho_1, \rho_2)$  and  $\delta = (\delta_1, \delta_2)$  be given pairs of positive numbers and let  $0 < \delta < \rho$ , componentwise. If  $f, g, h \in \mathcal{A}(\rho)$  and  $X, Y \in \mathcal{A}^2(\rho)$  satisfy  $\|X\|_\rho \leq \delta_1$  and  $\|Y\|_\rho \leq \delta_2$ , and  $U : (x, y) \mapsto (x + X, y + Y)$  is a given change of variables, then*

- (i)  $|f(x, y)| \leq \|f\|_\rho, \forall (x, y) \in \mathcal{D}(\rho)$ ,
- (ii)  $fg \in \mathcal{A}(\rho)$  and  $\|fg\|_\rho \leq \|f\|_\rho \|g\|_\rho$ ,
- (iii)  $\|\partial_i h\|_{\rho - (\delta_1, 0)} \leq \delta_1^{-1} \|h\|_\rho$  for  $i = 1, 2$ ,
- (iv)  $\|\partial_j h\|_{\rho - (0, \delta_2)} \leq \delta_2^{-1} \|h\|_\rho$  for  $j = 3, 4$ ,
- (v)  $\|h \circ U\|_\rho \leq \|h\|_{\rho + \delta}$ .

Given a Hamiltonian  $H$ , close to  $H^0$ , our goal is to construct a canonical transformation  $U_H$  that satisfies the equation  $\mathbb{I}^- H \circ U_H = 0$ . Such a transformation is close to the identity as the Hamiltonian  $H$  is close to integrable. We would like to perform first a canonical transformation  $U : (x, y) \mapsto (x', y')$ , generated by a function  $\phi$  as

$$x' = x + \nabla_{y'} \phi(x, y'), \quad y' = y - \nabla_x \phi(x, y'), \quad (28)$$

where  $\nabla_x = (\partial_1, \partial_2)$  and  $\nabla_y = (\partial_3, \partial_4)$ , which satisfies the linearized version of the above equation, i.e.  $\mathbb{I}^-(H + \{H, \phi\}) = 0$ . Here,  $\{H, \phi\}$  denotes the Poisson bracket of the functions  $H$  and  $\phi$ , defined by  $\{H, \phi\} = \nabla_x H \cdot \nabla_y \phi - \nabla_x \phi \cdot \nabla_y H$ .

We introduce  $\psi = \partial_1 \phi$  and define the operators  $\mathcal{D}_i = \partial_i \partial_1^{-1}$ , for  $i = 1, 2, 3, 4$ , on  $\mathbb{I}^- \mathcal{A}(\rho)$ ,  $\rho > 0$ , componentwise.

**Proposition 5.2.**  $\mathcal{D}_2, \mathcal{D}_3$  and  $\mathcal{D}_4$  are bounded linear operators on  $\mathbb{I}^- \mathcal{A}(\rho)$ , with operator norms satisfying

$$\|\mathcal{D}_2\| \leq \frac{\|\omega\|}{\sigma}, \quad \|\mathcal{D}_3\| \leq \frac{\|\omega\|}{\kappa\rho_2}, \quad \|\mathcal{D}_4\| \leq \frac{\|\omega\|}{\kappa\rho_2}. \tag{29}$$

**Proof.** Consider first the action of the operators  $\mathcal{D}_2, \mathcal{D}_3$  and  $\mathcal{D}_4$  on  $H_{v,k}(x, y) = y_1^{k_1} y_2^{k_2} e^{iv \cdot x}$  with  $(v, k)$  belonging to  $I^-$ . We find that  $\|\mathcal{D}_2 H_{v,k}\|_\rho \leq (\|\omega\|/\sigma)\|H_{v,k}\|_\rho, \|\mathcal{D}_3 + \mathcal{D}_4\|_\rho \leq (\|\omega\|/\kappa\rho_2)\|H_{v,k}\|_\rho, \|\mathcal{D}_3 H_{v,k}\|_\rho \leq (\|\omega\|/\kappa\rho_2)\|H_{v,k}\|_\rho$  and  $\|\mathcal{D}_4 H_{v,k}\|_\rho \leq (\|\omega\|/\kappa\rho_2)\|H_{v,k}\|_\rho$ . These bounds extend by linearity to the whole  $\mathbb{I}^- \mathcal{A}(\rho)$ .  $\square$

Let  $H = H^0 + h$ , where  $H^0 = \|\omega\|y_1 + y_2^2/2$  and  $h \in \mathcal{A}(\varrho)$ , with  $\varrho > \eta > 0$ , componentwise. On  $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$  define the operator  $L(h)$  by the following action on an arbitrary  $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$ ,

$$L(h)\psi = \frac{1}{\|\omega\|}(-y_2\mathcal{D}_2\psi + \partial_1 h\mathcal{D}_3\psi + \partial_2 h\mathcal{D}_4\psi - \partial_3 h\mathcal{D}_1\psi - \partial_4 h\mathcal{D}_2\psi). \tag{30}$$

Using the bounds obtained in propositions 5.1 and 5.2, we find that

$$\|L(h)\psi\|_{\varrho-\eta} \leq \left[ \frac{\varrho_2 - \eta_2}{\sigma} + \left( \frac{2}{\kappa\eta_1(\varrho_2 - \eta_2)} + \frac{1}{\sigma\eta_2} + \frac{1}{\|\omega\|\eta_2} \right) \|h\|_\varrho \right] \|\psi\|_{\varrho-\eta}. \tag{31}$$

As  $\|\omega\| \geq 2\ell$ , if  $\varrho_2 - \eta_2 < \sigma$  and  $\|h\|_\varrho$  is sufficiently small, then  $\|L(h)\psi\|_{\varrho-\eta} \leq A\|\psi\|_{\varrho-\eta}$  for every  $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$  and some positive constant  $A < 1$ . Therefore, the operator norm  $\|L(h)\| \leq A < 1$ .

This enables us to solve the equation  $\mathbb{I}^-(H + \{H, \phi\}) = 0$  for the generating function of the canonical transformation  $U$ .

**Proposition 5.3.** Let  $H$  belong to  $\mathcal{A}(\varrho)$ , where  $\varrho > \eta > 0$ , componentwise, and  $\varrho_2 - \eta_2 < \sigma$ . If  $\|h\|_\varrho$  is sufficiently small such that, by the inequality (31),  $L(h)$  is an operator on  $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$  bounded in norm by a positive constant  $A < 1$ , then the equation

$$\mathbb{I}^-(H + \{H, \phi\}) = 0, \quad \mathbb{I}^+\phi = 0 \tag{32}$$

has a unique solution  $\phi$ , such that  $\psi = \partial_1\phi$  belongs to  $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$ , and satisfies

$$\|\psi\|_{\varrho-\eta} \leq (1 - A)^{-1} \frac{\|\mathbb{I}^- H\|_{\varrho-\eta}}{\|\omega\|}, \quad \|\{H, \phi\}\|_{\varrho-\eta} \leq (1 - A)^{-1} \|\mathbb{I}^- H\|_{\varrho-\eta}. \tag{33}$$

**Proof.** Let  $L^\pm = \mathbb{I}^\pm L(h)\mathbb{I}^\mp$ . Equation (32) can now be written in the form  $(\mathbb{I} - L^-)\psi = \mathbb{I}^- h/\|\omega\|$ . The assumptions guarantee that the operator norms of  $L^\pm$  satisfy  $\|L^\pm\| \leq A < 1$ . Equation (32) can then be solved by inverting the operator  $\mathbb{I} - L^-$  by means of a Neumann series. The solution  $\psi$  satisfies the first of the bounds (33). As  $\{H, \phi\} = (L(h) - \mathbb{I})\psi\|\omega\| = L^+\psi\|\omega\| - \mathbb{I}^- h$ , using this bound one obtains the second one.  $\square$

The next proposition shows that the function  $\phi$  generates an explicit canonical transformation.

**Proposition 5.4.** Let  $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \epsilon r_2 = \delta_2' < \varrho_2 - \eta_2$  and  $\sigma < 2\ell$ . Let also  $\delta' = (0, \delta_2')$ . Define  $B$  to be the closed ball of radius  $\delta_2''/2$  in  $\mathcal{A}(\varrho - \eta - \delta')$  centred at zero. If  $\psi \in \mathbb{I}^- \mathcal{A}(\varrho - \eta)$  satisfies  $\|\psi\|_{\varrho-\eta} \leq \epsilon^3/\|\omega\|$ , the equation

$$K(g) = g, \quad K(g) = \nabla_x \phi(x, y - g), \quad g = g(x, y) = y - y' \tag{34}$$

has a unique solution  $g \in B^2 = B \times B$  and  $\|g\|_{\varrho-\eta-\delta'} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$ .

**Proof.** By assumption, for every  $g \in B^2$ ,  $\|g\|_{\varrho-\eta-\delta'} \leq \delta_2''/2$ . Using the bounds obtained in propositions 5.1 and 5.2, we find that for every  $g, g' \in B^2$ , there exists  $g^* \in B^2$ , such that

$$\|K(g)\|_{\varrho-\eta-\delta'} \leq \|\nabla_x \phi\|_{\varrho-\eta} \leq \max\{1, \sigma^{-1}\|\omega\|\}\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \leq \sigma^{-1}\epsilon^3 \quad (35)$$

and

$$\begin{aligned} \|K(g') - K(g)\|_{\varrho-\eta-\delta'} &\leq \|\nabla_g K(g^*)\|_{\varrho-\eta-\delta'} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq \|\nabla_{y'} \nabla_x \phi(x, y')\|_{\varrho-\eta-\delta'} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq \|\nabla_y \nabla_x \phi\|_{\varrho-\eta-\delta'/2} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq 2\delta_2'^{-1} \|\nabla_x \phi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq 2\delta_2'^{-1} \max\{1, \sigma^{-1}\|\omega\|\}\|\psi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq 2\delta_2'^{-1} \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} \|g' - g\|_{\varrho-\eta-\delta'} \\ &\leq 2\delta_2'^{-1} \sigma^{-1}\epsilon^3 \|g' - g\|_{\varrho-\eta-\delta'}. \end{aligned}$$

As, by assumption,  $2\delta_2'^{-1}\sigma^{-1}\epsilon^3 < 1$ , these inequalities show that  $K$  is a contraction on  $B^2$  and thus has a unique fixed point  $g \in B^2$ . The inequality (35) provides the desired bound on the norm of  $g$ .  $\square$

**Proposition 5.5.** Let  $\varrho - \eta - 2\delta' > 0$ ,  $\eta > 0$  and  $\delta' = (\delta_1', \delta_2') = \epsilon r > 0$ , componentwise, with  $\delta_1' = \sigma\delta_2'/[\kappa(\varrho_2 - \eta_2)]$ . Also let  $0 < \delta_2'' = 2\sigma^{-1}\epsilon^3 < \delta_2' = \epsilon r_2 < \varrho_2 - \eta_2 < \sigma < 2\ell$ . Assume further that  $H \in \mathcal{A}(\varrho - \eta)$  satisfies  $\|H - H^0\|_{\varrho-\eta} < b$  and that  $b > 0$  is small enough such that the linear operator  $L(H - H^0)$  is bounded on  $\mathbb{I}^- \mathcal{A}(\varrho - \eta)$  by  $\|L(H - H^0)\| \leq A < A' < 1$ . If  $\|\mathbb{I}^- H\|_{\varrho-\eta} \leq (1 - A')\epsilon^3$ , the canonical transformation  $U$  exists and maps  $\mathcal{D}(\varrho - \eta - 2\delta')$  into  $\mathcal{D}(\varrho - \eta)$ . The function  $H \circ U$  belongs to  $\mathcal{A}(\varrho - \eta - 2\delta')$  and  $L(H \circ U - H^0)$  is a bounded operator on  $\mathbb{I}^- \mathcal{A}(\varrho - \eta - 2\delta')$ . They satisfy the bounds

$$\begin{aligned} \|\mathbb{I}^- (H \circ U)\|_{\varrho-\eta-2\delta'} &\leq C_2(\epsilon)\epsilon^4, \\ \|H \circ U - H\|_{\varrho-\eta-2\delta'} &\leq (1 + \epsilon C_2(\epsilon))\epsilon^3 = \Delta b(\epsilon), \\ \|L(H \circ U - H^0)\| &\leq \|L(H - H^0)\| + C_3(\epsilon)(1 + \epsilon C_1(\epsilon))\epsilon^2 = A + \Delta A(\epsilon), \end{aligned} \quad (36)$$

where

$$C_n(\epsilon) = \frac{\sigma n \epsilon r_2 + \varrho_2 n \epsilon r_2 + (1/2)(n \epsilon r_2)^2 + \|h\|_{\varrho-\eta}}{\sigma n r_2 (\sigma n r_2 - \epsilon^2)},$$

for  $n = 1, 2$  and

$$C_3(\epsilon) = \frac{2(\varrho_2 - \eta_2)}{\sigma r_2 (\varrho_2 - \eta_2 - 2r_2 \epsilon)} + \frac{1}{\sigma r_2} + \frac{1}{\ell r_2}.$$

**Proof.** Our assumptions guarantee that there exists a canonical transformation  $U$  with a generating function  $\phi$  that solves the linear equation (32). More specifically, if  $b > 0$  has been chosen sufficiently small, then there exists  $\psi \in \mathcal{A}(\varrho - \eta)$  satisfying  $\|\psi\|_{\varrho-\eta} < \epsilon^3/\|\omega\|$  and  $g \in B^2$  of norm  $\|g\|_{\varrho-\eta-\delta'} < \sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta}$  that solves the equation (34).

Define the following one parameter ( $s \in \mathbb{C}$ ) family

$$\begin{aligned} F(s) &= -s(\|\omega\|\psi(x, y - sg) + y_2 \mathcal{D}_2 \psi(x, y - sg)) + \frac{1}{2}s^2(\mathcal{D}_2 \psi(x, y - sg))^2 \\ &\quad + h(x + s \nabla_{y-sg} \phi(x, y - sg), y - s \nabla_x \phi(x, y - sg)) \end{aligned}$$

passing through  $F(0) = H - H^0$  and  $F(1) = H \circ U - H^0$ , with  $F'(0) = \{H, \phi\}$ . Assuming that  $|s| \leq s_0 = \sigma\epsilon^{-3}n\delta_2'$ , where  $n \in \{1, 2\}$ , and using the propositions 5.1 and 5.2, we obtain the following bounds

$$\begin{aligned} \|sg\|_{\varrho-\eta-n\delta'} &\leq s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_2', \\ \|s \nabla_x \phi(x, y - sg)\|_{\varrho-\eta-n\delta'} &\leq s_0\sigma^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_2', \\ \|s \nabla_{y-sg} \phi(x, y - sg)\|_{\varrho-\eta-n\delta'} &\leq s_0[\kappa(\varrho_2 - \eta_2)]^{-1}\|\omega\|\|\psi\|_{\varrho-\eta} < n\delta_1'. \end{aligned} \quad (37)$$

These bounds, together with proposition 5.1, imply that  $F(s)$  belongs to  $\mathcal{A}(\varrho - \eta - n\delta')$ , whenever  $|s| \leq s_0$ . In fact, this is true on an open neighbourhood of the disc  $|s| \leq s_0$  as the inequalities (37) are strict due to  $A < A'$ . Now, we have

$$\begin{aligned} \|H \circ U - H - \{H, \phi\}\|_{\varrho - \eta - n\delta'} &= \|F(1) - F(0) - F'(0)\|_{\varrho - \eta - n\delta'} \\ &= \left\| \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s^2(s-1)} F(s) \right\|_{\varrho - \eta - n\delta'} \\ &\leq \frac{1}{s_0(s_0 - 1)} \left( s_0 \|\omega\| \|\psi\|_{\varrho - \eta} + s_0 \varrho_2 \sigma^{-1} \|\omega\| \|\psi\|_{\varrho - \eta} \right. \\ &\quad \left. + \frac{1}{2} s_0^2 \sigma^{-2} \|\omega\|^2 \|\psi\|_{\varrho - \eta}^2 + \|h\|_{\varrho - \eta} \right) \\ &\leq \frac{\sigma n \delta'_2 + \varrho_2 n \delta'_2 + (1/2)(n \delta'_2)^2 + \|h\|_{\varrho - \eta}}{\sigma \epsilon^{-3} n \delta'_2 (\sigma \epsilon^{-3} n \delta'_2 - 1)} = C_n(\epsilon) \epsilon^4. \end{aligned}$$

As  $\mathbb{I}^-(H \circ U) = \mathbb{I}^-(H \circ U - H - \{H, \phi\})$ , the first of the inequalities (36) immediately follows from this estimate for  $n = 2$ . From proposition 5.3 and the inequality

$$\|H \circ U - H\|_{\varrho - \eta - 2\delta'} \leq \|H \circ U - H - \{H, \phi\}\|_{\varrho - \eta - 2\delta'} + \|\{H, \phi\}\|_{\varrho - \eta - 2\delta'},$$

we find that  $\|H \circ U - H\|_{\varrho - \eta - 2\delta'} \leq (1 - A)^{-1} \|\mathbb{I}^- H\|_{\varrho - \eta} + C_2(\epsilon) \epsilon^4$ . This implies the second inequality in (36). The fact that the map  $U$  takes  $\mathcal{D}(\varrho - \eta - 2\delta')$  into  $\mathcal{D}(\varrho - \eta)$  follows from the bounds (37) and proposition 5.1.

From the definition of the operator  $L(h)$ , one can find that

$$\begin{aligned} \|L(H \circ U - H^0)\| &\leq \|L(H - H^0)\| + \left( \frac{2}{\kappa \delta'_1 (\varrho_2 - \eta_2 - 2\delta'_2)} + \frac{1}{\sigma \delta'_2} + \frac{1}{\|\omega\| \delta'_2} \right) \\ &\quad \times \|H \circ U - H\|_{\varrho - \eta - \delta'}. \end{aligned}$$

Using the inequality  $\|H \circ U - H\|_{\varrho - \eta - \delta'} \leq (1 + \epsilon C_1(\epsilon)) \epsilon^3$ , which can be obtained analogously to the second bound in (36), one can obtain the last desired inequality.  $\square$

**Theorem 5.6.** *Let  $\varrho > \varrho' > 0$ , componentwise,  $\varrho_2 < \sigma < 2\ell$  and  $0 < A' < 1$ . Let  $B$  be an open set of Hamiltonians  $H \in \mathcal{A}(\varrho)$ , for which  $\|H - H^0\|_{\varrho} < b$  and  $\|\mathbb{I}^- H\|_{\varrho} < (1 - A') \epsilon^3$ . If  $b > 0$  and  $\epsilon > 0$  are sufficiently small, then for every Hamiltonian  $H \in B$  there exists an analytic canonical transformation  $U_H : \mathcal{D}(\varrho') \rightarrow \mathcal{D}(\varrho)$  that solves the equation  $\mathbb{I}^- H \circ U_H = 0$ . The map  $H \mapsto H \circ U_H$  is analytic from  $B$  to  $\mathbb{I}^+ \mathcal{A}(\varrho')$  and*

$$\|H \circ U_H - H\|_{\varrho'} \leq \epsilon^3 + \mathcal{O}(\epsilon^4). \tag{38}$$

**Proof.** Let  $\varrho > \varrho - \eta > \varrho' > 0$  and  $r > 0$ , componentwise, with  $r_1 = \sigma r_2 / [\kappa(\varrho_2 - \eta_2)]$ . For sufficiently small  $b > 0$ , the norm of the operator  $L(H - H^0) : \mathbb{I}^- \mathcal{A}(\varrho - \eta) \rightarrow \mathcal{A}(\varrho - \eta)$  is bounded by a positive constant  $A < A' < 1$ .

By proposition 5.5, there exists a canonical transformation  $U : \mathcal{D}(\varrho - \eta - 2\epsilon r) \rightarrow \mathcal{D}(\varrho - \eta)$  such that  $\|\mathbb{I}^-(H \circ U)\|_{\varrho - \eta - 2\epsilon r} \leq C_2(\epsilon) \epsilon^4$ . We would like to iterate the map  $H \mapsto H \circ U$ , indefinitely. Introducing  $f(\epsilon) = [\epsilon C_2(\epsilon) / (1 - A')]^{1/3} \epsilon$ , we obtain  $\|\mathbb{I}^-(H \circ U)\|_{\varrho - \eta - 2\epsilon r} \leq (1 - A') f(\epsilon)^3$ . Let  $\epsilon_i = f^i(\epsilon)$ , for  $i \in \mathbb{N}_0$ . For sufficiently small  $\epsilon$ , the sum  $\sum_{i=0}^{\infty} 2\epsilon_i r$  converges to a limit  $\delta = \mathcal{O}(\epsilon)$ . The sums  $\sum_{i=0}^{\infty} \Delta b(\epsilon_i)$  and  $\sum_{i=0}^{\infty} \Delta A(\epsilon_i)$  converge to  $\Delta b = \mathcal{O}(\epsilon^3)$  and  $\Delta A = \mathcal{O}(\epsilon^2)$ , respectively. Thus, for sufficiently small  $\epsilon$  the map  $(H, \epsilon, \varrho - \eta) \mapsto (H \circ U, f(\epsilon), \varrho - \eta - 2\epsilon r)$  can be iterated indefinitely, and the iterations converge to a limit  $(H \circ U_H, 0, \varrho - \eta - \delta)$ . For sufficiently small  $\epsilon$ ,  $\varrho - \eta - \delta > \varrho'$ , componentwise.

As  $\epsilon_i$  are summable, the sequence of canonical transformations  $U$  generates a uniformly convergent sequence on  $\mathcal{D}(\varrho - \eta - \delta)$ . The analyticity of the map  $H \mapsto H \circ U_H$  follows from uniform convergence of our iteration scheme. The desired bound can be obtained from the second inequality in (36) and its iterations.  $\square$

Let  $\hat{H}_n$  be the Hamiltonian vector field generated by  $H_n$ , i.e.  $\hat{H}_n = (\mathbb{J}\nabla H_n) \cdot \nabla$ , where  $\mathbb{J}(q, p) = (p, -q)$  and  $\nabla = (\nabla_q, \nabla_p)$ .

The derivative of the map  $\mathcal{N}_{H_n} : H_n \mapsto H_n \circ U_{H_n}$  at a resonant Hamiltonian  $H_n^+$  is given by  $D\mathcal{N}_n(H_n^+) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{H}_n^+ (\mathbb{I}_n^- \hat{H}_n^+ \mathbb{I}_n^-)^{-1} \mathbb{I}_n^-$ .

Let  $K_n^0 = \omega_n \cdot p$  and let  $\mathbb{E}H_n = K_n^0 + f_n$ , where  $f_n = \frac{1}{2}(\Omega_n \cdot p)^2 + \mathbb{E}h_n$ . The derivative of the map  $\mathcal{N}_{H_n}$  at a  $q$ -independent Hamiltonian  $\mathbb{E}H_n$  is

$$D\mathcal{N}_n(\mathbb{E}H_n) = \mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^- (\mathbb{I}_n^- \hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^-)^{-1} \mathbb{I}_n^- \tag{39}$$

The norm of the operator  $\hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^- : \mathcal{A}_n(\varrho) \rightarrow \mathcal{A}_n(\varrho')$  satisfies the bound

$$\|\hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^-\| \leq \frac{\varrho_2'}{\sigma} + \frac{1}{\|\omega_n\|} \sum_{k \in \mathbb{N}_0^2} |(h_n)_{0,k}| \left( k_1 + k_2 \frac{\|\omega_n\|}{\sigma} \right) \varrho_2'^{\|k\|-1} \tag{40}$$

and thus

$$\|\hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^-\| \leq \frac{\varrho_2'}{\sigma} + \left( \frac{1}{\|\omega_n\|} + \frac{1}{\sigma} \right) \frac{\|\mathbb{E}h_n\|_{n,\varrho}}{\varrho_2(1 - \varrho_2'/\varrho_2)^2} \tag{41}$$

As  $\varrho_2 < \sigma$ , if  $\|\mathbb{E}h_n\|_{n,\varrho}$  is sufficiently small, then  $\|\hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^-\| < 1$  and thus the operator  $(\mathbb{I}_n^+ - \mathbb{I}_n^+ \hat{f}_n \hat{K}_n^0{}^{-1} \mathbb{I}_n^-)$  in (39) can be inverted by means of a Neumann series. If  $\varrho_2 < \sigma/2$  and  $\|\mathbb{E}h_n\|_{n,\varrho}$  is sufficiently small, then the operator norm  $\|D\mathcal{N}_n(\mathbb{E}H_n)\|$  can be bounded by 1.

### 6. Diophantine frequency vectors and some bounds

Recall that the  $n$ th-step renormalization operator is associated to the pair of vectors  $(\omega_n, \Omega_n)$  generated from a pair  $(\omega, \Omega)$ . We assume that  $\omega \in \mathbb{R}^2$  is of the form  $\omega = \ell(1, \alpha)^*$ , where  $\alpha > 1$  and  $\ell \in \mathbb{R}^+$ . For the vectors  $\omega = \ell(1, \alpha)^*$  with  $\alpha < 1$ , a canonical transformation of the phase space can be performed, generated by a matrix from  $GL(2, \mathbb{Z})$ , such that the ‘new’ vector  $\omega$  is of the desired form. Thus, without loss of generality we can assume that  $\alpha > 1$  and, consequently, for any  $n \in \mathbb{N}_0$ ,  $\omega_n = \ell(1, \alpha_n)^*$  with  $\alpha_n > 1$ .

The vectors  $(1, \alpha_n)^* = (-1)^n (p_{n-1} - \alpha q_{n-1})^{-1} P_{n-1}^* (1, \alpha)^*$  can be obtained using the convergent matrices (12). Therefore, we have

$$\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}} \tag{42}$$

The convergents  $p_n/q_n$  satisfy (theorem 171 in [15])

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \tag{43}$$

They are the best rational approximates of  $\alpha$ , in the sense that for any  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ , such that  $0 < q \leq q_n$  and  $p/q \neq p_n/q_n$ ,  $n \in \mathbb{N}$ , one has  $|p - \alpha q| < |p - \alpha q|$  (theorem 182 in [15]).

From the equality (42), for  $n \geq 0$ , we find that the fractional part of  $\alpha_n$  is

$$x_n = -\frac{p_n - \alpha q_n}{p_{n-1} - \alpha q_{n-1}} \tag{44}$$



Denoting the product of the first  $(n + 1)$  numbers  $x_i$  by  $\beta_n = \prod_{i=0}^n x_i = \prod_{i=0}^n \alpha_{i+1}^{-1}$ , we obtain that  $\beta_n = (-1)^{n+1}(p_n - \alpha q_n)$ . Notice that  $\det(P_n) = (-1)^{n+1}$ . Using the equality (42) again, one easily finds that  $\beta_n^{-1} = q_{n+1} + q_n x_{n+1}$ , and thus,  $q_{n+1} < \beta_n^{-1} < 2q_{n+1}$ .

Define  $\tilde{A}_n = \prod_{i=0}^n \alpha_i$ . As  $\beta_n = \alpha_0 \tilde{A}_{n+1}^{-1}$ , the previous bound implies

$$\alpha_0 q_n < \tilde{A}_n < 2\alpha_0 q_n \tag{45}$$

assuming that  $\alpha_0 > 0$ .

Emphasizing again that the values of  $q_n$  and  $p_n$  are associated to  $\alpha$  by writing explicitly  $q_n(\alpha)$  and  $p_n(\alpha)$ , notice that  $q_{n+1}(\alpha_0) = p_n(\alpha_1)$ . The double inequality (45) for  $(n + 1)$  instead of  $n$  can be written as

$$p_n(\alpha_1) < \frac{\tilde{A}_{n+1}}{\alpha_0} < 2p_n(\alpha_1) \tag{46}$$

implying the following bounds on  $p_n$ ,

$$p_n < \tilde{A}_n < 2p_n. \tag{47}$$

It is easy to show that  $\tilde{A}_n$  grows at least exponentially with  $n$ . If  $1 < \alpha_i \leq \gamma$ , for some  $i \in \{0, \dots, n - 1\}$ , then  $\alpha_i \alpha_{i+1} = \alpha_i / (\alpha_i - \alpha_i) \geq \gamma / (\gamma - 1) = \gamma^2$ , where  $\gamma = (1 + \sqrt{5})/2$  is the golden mean, the limit of the sequence of ratios  $F_{k+1}/F_k$  of successive Fibonacci numbers  $F_k$ , defined by  $F_{k+2} = F_{k+1} + F_k$ , for  $k \in \mathbb{N}$ , and  $F_1 = F_2 = 1$ . This implies that  $\tilde{A}_n / \tilde{A}_{j-1} \geq \gamma^{n-j}$ , for  $0 < j \leq n$ . If  $\alpha_0 > 1$ , then  $\tilde{A}_n \geq \gamma^n$  and the previous inequality is also valid for  $j = 0$ , with  $\tilde{A}_{-1} = 1$ .

This growth can be controlled if  $\alpha$  is a Diophantine number.

**Definition 6.1.** An irrational number  $\alpha$  will be called Diophantine of order  $\beta \geq 0$  if there exists a constant  $C > 0$ , such that for all  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ ,

$$\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2+\beta}}. \tag{48}$$

The set of all Diophantine numbers of order  $\beta$  will be denoted by  $D(\beta)$ . We will also call Diophantine a vector  $\omega \in \mathbb{R}^2$  with a Diophantine winding ratio. The Diophantine condition on  $\omega$  states that there exists  $C > 0$ , such that for all  $v \in \mathbb{Z}^2 \setminus \{0\}$ ,  $|\omega \cdot v| > C \|v\|^{-(1+\beta)}$ .

The Diophantine condition on  $\alpha$ , together with the inequality (43), imposes an upper bound on the growth rate of the denominators of its convergents. For  $\alpha \in D(\beta)$ , there exists a constant  $K > 0$  such that  $q_{n+1} < K q_n^{1+\beta}$ , for all  $n \in \mathbb{N}_0$ . Equivalently, using the inequalities (45), the Diophantine condition can be written as

$$\tilde{A}_{n+1} < \tilde{K} \tilde{A}_n^{1+\beta}, \tag{49}$$

where  $\tilde{K} > 0$  is a constant.

In particular, constant-type numbers, which have a bounded sequence of partial quotients, are Diophantine of order zero. Among them are quadratic irrationals, i.e. the roots of quadratic equations with integer coefficients, whose continued fraction expansions are eventually periodic. Constant-type numbers have zero Lebesgue measure in the real numbers. Diophantine numbers of any order  $\beta > 0$  are of full measure.

Though the construction of a one-step renormalization transformation is more general, the bound (49) plays an essential role in the proof of theorem 7.2 concerning the existence of a trivial attracting orbit of the sequence of renormalization operators which is associated to a Diophantine frequency vector  $\omega \in \mathbb{R}^2$ . To prove the theorem, we will also need the bounds obtained in the following proposition.

**Proposition 6.2.** *If  $\Omega_0 \in \mathbb{R}^2$  is suitable, so is every  $\Omega_n = T_{n-1}^{-1}\Omega_{n-1}/\|T_{n-1}^{-1}\Omega_{n-1}\|$ , for all  $n \in \mathbb{N}$ . For a suitable choice of  $\Omega_0$ , we have the following bounds,*

$$\frac{1 + \alpha_0}{4\alpha_0}(\tilde{A}_n + \tilde{A}_{n-1}) < \prod_{i=0}^n \|T_i^{-1}\Omega_i\| < (\tilde{A}_n + \tilde{A}_{n-1}). \tag{50}$$

**Proof.** The first part of the claim can be proved by induction. Notice that  $\|P_n^{*-1}\Omega_0\| = \|T_n^{-1} \dots T_0^{-1}\Omega_0\| = \prod_{i=0}^n \|T_i^{-1}\Omega_i\|$ , as the matrices  $T_i$  are symmetric. One easily finds that for a suitable  $\Omega_0 \in \mathbb{R}^2$ ,  $1/2(p_n + p_{n-1} + q_n + q_{n-1}) \leq \|P_n^{*-1}\Omega_0\| \leq p_n + p_{n-1}$ . The bounds (50) follow from the inequalities (45) and (47).  $\square$

### 7. Convergence of the renormalization scheme

The orbit of an integrable Hamiltonian  $H_0^0 = \omega \cdot p + 1/2(\Omega \cdot p)^2$  under the renormalization consists of Hamiltonians  $H_n^0 = \omega_n \cdot p + 1/2(\Omega_n \cdot p)^2$ , where  $n \in \mathbb{N}_0$ . The maps  $\omega_n \mapsto \omega_{n+1}$  and  $\Omega_n \mapsto \Omega_{n+1}$  are induced by the Gauss map of the inverse winding ratio of  $\omega$ .

The vectors  $\omega$  with the winding ratio  $\alpha = [a, a, \dots]$ , where  $a \in \mathbb{N}$ , are the fixed points of the first of these maps. A suitable vector  $\Omega$  is not, in general, a fixed point of the dynamics. However, in this case, it is possible to make a particular choice of  $\Omega$ , such that it is a fixed point of the map. The corresponding integrable Hamiltonian  $H_0^0$  is then a fixed point of the renormalization.

Similarly, if the winding ratio of a frequency vector  $\omega$  has a periodic continued fraction expansion one can make a particular choice of a suitable vector  $\Omega$ , such that the Hamiltonian  $H_0^0$  generates a periodic orbit of the renormalization. More generally, if the winding ratio of  $\omega$  is a quadratic irrational and a particular choice of a suitable vector  $\Omega$  is made, the dynamics of  $H_0^0$  eventually settles on a periodic orbit.

In the following, we show that if  $\omega_0$  is a Diophantine vector and  $\Omega_0$  is an arbitrary suitable vector, then the orbit of a resonant Hamiltonian  $H_0$ , sufficiently close to  $H_0^0$ , approaches the orbit of  $H_0^0$  exponentially fast under the action of the sequence of the renormalization operators  $\mathcal{R}_n$ ,  $n \in \mathbb{N}_0$ .

We first present a description of the eigenspaces of the derivative of the  $n$ th-step renormalization operator at  $H_n^0$ , which is given by its action on an arbitrary function  $f_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$ , where  $\rho' > 0$ , componentwise, as

$$D\mathcal{R}_n(H_n^0)f_n = \frac{\theta_n}{\mu_n} (\mathbb{I} - \mathbb{P}_{n+1}^{(0,0)} - \mathbb{P}_{n+1}^{(1,0)} - \mathbb{P}_{n+1}^{(0,1)} - \mathbb{P}_{n+1}^{(0,2)}) D\mathcal{N}_{n+1}(H_{n+1}^0)f_n \circ \mathcal{T}_n. \tag{51}$$

The space of  $q$ -independent Hamiltonians is an invariant subspace of the derivative operator. A constant Hamiltonian and the Hamiltonians  $(\hat{\omega}_n \cdot p)$ ,  $(\Omega_n \cdot p)$  and  $(\hat{\omega}_n \cdot p)^2$  are eigenvectors with eigenvalue zero. For  $k \in \mathbb{N}_0^2$  different from  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  and  $(0, 2)$ , the derivative operator maps the Hamiltonian  $(\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2}$  into

$$D\mathcal{R}_n(H_n^0)(\hat{\omega}_n \cdot p)^{k_1} (\Omega_n \cdot p)^{k_2} = \frac{\|\omega_{n+1}\|^{k_1}}{(\alpha_{n+1} \|T_n^{-1}\Omega_n\|)^{2(k_1-1)+k_2} \|\omega_n\|^{k_1}} (\hat{\omega}_{n+1} \cdot p)^{k_1} (\Omega_{n+1} \cdot p)^{k_2}.$$

Thus, the operator norm of  $D\mathcal{R}_n(H_n^0)$  acting on  $q$ -independent Hamiltonians can be bounded by

$$\|D\mathcal{R}_n(H_n^0)\mathbb{E}\| \leq \frac{\max\{1, \|\omega_{n+1}\|/\|\omega_n\|\}}{\alpha_{n+1} \|T_n^{-1}\Omega_n\|} \leq \frac{2}{3}. \tag{52}$$

The derivative of the  $n$ th-step renormalization operator  $\mathcal{R}_n$  at a  $q$ -independent Hamiltonian  $\mathbb{E}H_n$  is the linear operator  $\mathcal{L}_n = D\mathcal{R}_n(\mathbb{E}H_n) : \mathbb{I}_n^+ \mathcal{A}_n(\rho') \rightarrow \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho')$ , given by its action on an arbitrary  $f_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$ ,

$$\mathcal{L}_n f_n = \theta_n / \mu_n D\mathfrak{S}_{n+1}(\mathbb{E}\tilde{H}_{n+1}) D\mathcal{N}_{n+1}(\mathbb{E}H''_{n+1}) D\mathcal{V}_{n+1}(\mathbb{E}H'_{n+1}) f_n \circ \mathcal{T}_n. \tag{53}$$

The operators  $\mathfrak{S}_{n+1}$  and  $\mathcal{V}_{n+1}$  have been introduced in proposition 4.3 and proposition 4.2, respectively. The next proposition shows that there is a super-exponential shrinking of the  $q$ -dependent modes (which may also depend on the  $p$ -variables). In the case of the vector fields on a torus, where the  $p$ -variables are absent, a similar result has been obtained in [28].

Let  $H_0 = H_0^0 + h_0 \in \mathbb{I}_0^+ \mathcal{A}_0(\rho')$ , and let  $H_n = H_n^0 + h_n$ ,  $n \in \mathbb{N}$ , be the associated renormalization orbit of  $H_0$ .

**Proposition 7.1.** *Let  $\omega_0 \in D(\beta)$ , for some  $\beta \geq 0$ . There exist  $c_1, c_2 > 0$ , such that*

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq c_1^{n-j+1} \frac{\tilde{A}_{n+1}^2 \tilde{A}_n^2}{\tilde{A}_j^2 \tilde{A}_{j-1}^2} e^{-c_2 \Lambda_{j,n}}, \tag{54}$$

for  $n \geq 0$  and  $j = 0, \dots, n$  assuming that  $\|h_i\|_{i,\rho'} \leq \zeta_i/2$ , for  $i = j, \dots, n$ , where, as before,  $\zeta_i = C' / (\alpha_i \alpha_{i+1})^2$ ,  $C' > 0$ . Here,

$$\Lambda_{j,n}^{2+\beta} = \frac{\tilde{A}_{n+1}}{\max\{\sigma, \kappa\} \tilde{A}_{j-1}^{1+\beta}}. \tag{55}$$

**Proof.** The following properties of the linear operator  $\mathcal{L}_n$  will be useful to prove this proposition. First,  $\mathcal{L}_n = \mathbb{I}_{n+1}^+ \mathcal{L}_n$ . Second, when acting on a Fourier mode, this operator changes its index  $\nu$  into  $T_n^* \nu$ , as the derivatives of the elimination, translation and scaling maps do not change the value of  $\nu$ . We are interested in the action of the operator  $\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})$  on the modes indexed by  $\nu \neq 0$ . A mode with a particular value of  $k \in \mathbb{N}_0^2$  can in general produce modes with different  $k'$ -values in  $\mathbb{N}_0^2$ . We will denote by  $\|k'\|_{\min}$  the minimum of the norms of the vectors  $k'$  generated from a mode index  $k$ .

For every  $n \in \mathbb{N}_0$ , let

$$\begin{aligned} I_n^{1+} &= \{(\nu, k) \in I_n^+ : |\omega_n \cdot \nu| \leq \sigma |\Omega_n \cdot \nu|\}, \\ I_n^{2+} &= \{(\nu, k) \in I_n^+ : |\omega_n \cdot \nu| \leq \kappa \|k\|\} \end{aligned}$$

be the two subsets of  $I_n^+$ . We also define the following subset of  $I_j^{1+}$ , for  $j = 0, \dots, n$ ,

$$V_{j,n}^+ = \{(\nu, k) \in I_j^{1+} : (T_n^* \dots T_j^* \nu, k') \in I_{n+1}^{1+}\}.$$

For every  $(\nu, k) \in V_{j,n}^+$ ,  $\nu$  must satisfy the resonant condition

$$|\omega_{n+1} \cdot T_n^* \dots T_j^* \nu| \leq \sigma |\Omega_{n+1} \cdot T_n^* \dots T_j^* \nu|. \tag{56}$$

Notice that  $\omega_{n+1} \cdot T_n^* \dots T_j^* \nu = T_j \dots T_n \omega_{n+1} \cdot \nu$ . As  $T_n^{-1} \omega_n = \alpha_{n+1}^{-1} \omega_{n+1}$ , we obtain

$$T_j \dots T_n \omega_{n+1} = \omega_j \prod_{i=j+1}^{n+1} \alpha_i. \tag{57}$$

Similarly,  $\Omega_{n+1} \cdot T_n^* \dots T_j^* \nu = T_j \dots T_n \Omega_{n+1} \cdot \nu$ . From  $T_n^{-1} \Omega_n = \Omega_{n+1} \|T_n^{-1} \Omega_n\|$ , we find

$$T_j \dots T_n \Omega_{n+1} = \Omega_j \prod_{i=j}^n \frac{1}{\|T_i^{-1} \Omega_i\|}.$$

The condition (56) can then be written in the form

$$\left(\frac{1}{\sigma} \prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1} \Omega_{i-1}\| + \frac{1}{\|\omega_j\|}\right) |\omega_j \cdot v| \leq |\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|. \tag{58}$$

A lower bound on  $|\omega_j \cdot v|$  can be obtained by using the Diophantine property of  $\omega_0$ , i.e. that there exists  $C_0 > 0$ , such that  $|\omega_0 \cdot v| \geq C_0 \|v\|^{-(1+\beta)}$ , for any  $v \in \mathbb{Z}^2$ . This property implies that

$$|\omega_j \cdot v| = |\omega_0 \cdot T_0^{*-1} \dots T_{j-1}^{*-1} v| \prod_{i=1}^j \alpha_i \geq \frac{C_0 \prod_{i=1}^j \alpha_i}{\|T_0^{*-1} \dots T_{j-1}^{*-1} v\|^{1+\beta}},$$

for  $j \geq 1$ . As  $\|T_0^{*-1} \dots T_{j-1}^{*-1} v\| = \|P_{j-1}^{-1} v\| \leq (p_{j-1} + q_{j-1}) \|v\| \leq (1 + 1/\alpha_0) \tilde{A}_{j-1} \|v\|$  and  $\|v\| \leq 2(|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|)$ , there exists  $c > 1$ , such that

$$|\omega_j \cdot v| \geq \frac{C_0 \tilde{A}_j}{\alpha_0 (2c \tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|)^{1+\beta}}, \tag{59}$$

for  $j \geq 0$ . The bounds (58) and (59) imply that

$$|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v| \geq \frac{((1/\sigma) \prod_{i=j+1}^{n+1} \alpha_i \|T_{i-1}^{-1} \Omega_{i-1}\| + (1/\|\omega_j\|)) C_0 \tilde{A}_j}{\alpha_0 (2c \tilde{A}_{j-1})^{1+\beta} (|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|)^{1+\beta}}$$

and, thus,

$$(|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|)^{2+\beta} \geq \frac{C_0}{\alpha_0 (2c)^{1+\beta} \sigma} \frac{\tilde{A}_{n+1} \prod_{i=j+1}^{n+1} \|T_{i-1}^{-1} \Omega_{i-1}\|}{\tilde{A}_{j-1}^{1+\beta}}.$$

Using the bounds obtained in proposition 6.2, we find that there exists  $c' > 0$ , such that

$$(|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|) \geq c' \Lambda'_{j,n}, \tag{60}$$

where

$$\Lambda_{j,n}^{2+\beta} = \frac{\tilde{A}_{n+1} \tilde{A}_n}{\sigma \tilde{A}_{j-1}^{1+\beta} \tilde{A}_{j-1}}. \tag{61}$$

Now let us consider the modes indexed by  $(v, k) \in \mathbb{Z}^2 \times \mathbb{N}_0^2$  that belong to  $I_j^+ \setminus V_{j,n}^+$ . Let  $(T_n^* \dots T_j^* v, k') \in I_{n+1}^{2+}$  be the index of a mode generated by the action of the operator  $\mathcal{L}_n \circ \dots \circ \mathcal{L}_j (\mathbb{I} - \mathbb{E})$  on such a mode. The following condition must be satisfied,

$$|\omega_{n+1} \cdot T_n^* \dots T_j^* v| \leq \kappa \|k'\|_{\min} \leq \kappa \|k'\|. \tag{62}$$

From the inequality (62), using the identity (57) and the Diophantine bound (59), we find that

$$\|k'\| \geq \|k'\|_{\min} \geq \frac{C_0}{\kappa \alpha_0 (2c)^{1+\beta}} \frac{\tilde{A}_{n+1}}{\tilde{A}_{j-1}^{1+\beta} (|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|)^{1+\beta}}.$$

Define  $W_{j,n}^+ = \{(v, k) \in I_j^+ : (T_n^* \dots T_j^* v, k') \in I_{n+1}^{2+}, |\hat{\omega}_j \cdot v| + |\Omega_j \cdot v| \geq \|k'\|_{\min}\}$ . If  $(v, k) \in (I_j^+ \setminus V_{j,n}^+) \cap W_{j,n}^+$ , then

$$(|\hat{\omega}_j \cdot v| + |\Omega_j \cdot v|) \geq c'' \Lambda''_{j,n}, \tag{63}$$

where

$$\Lambda_{j,n}^{2+\beta} = \frac{\tilde{A}_{n+1}}{\kappa \tilde{A}_{j-1}^{1+\beta}} \tag{64}$$

and  $c'' > 0$ . If  $(T_n^* \dots T_j^* \nu, k') \in I_{n+1}^{2+}$  and  $(\nu, k) \in (I_j^+ \setminus V_{j,n}^+) \cap (I_j^+ \setminus W_{j,n}^+)$ , then

$$\|k'\|_{\min} \geq c'' \Lambda_{j,n}''. \tag{65}$$

Let  $\mathbb{V}_{j,n}^+ : \mathcal{A}_j(\rho') \rightarrow \mathcal{A}_j(\rho')$  be the projection operator on  $\mathcal{A}_j(\rho')$  over the indices in  $V_{j,n}^+$ , defined by the action

$$\mathbb{V}_{j,n}^+ f_j = \sum_{(\nu,k) \in V_{j,n}^+} (f_j)_{\nu,k} (\hat{\omega}_j \cdot p)^{k_1} (\Omega_j \cdot p)^{k_2} e^{iq \cdot \nu},$$

on an arbitrary  $f_j \in \mathcal{A}_j(\rho')$ . Let  $\bar{\mathbb{V}}_{j,n}^+ : \mathcal{A}_j(\rho') \rightarrow \mathcal{A}_j(\rho'')$  be the same projection followed by an analytic inclusion in the  $q$  variables, obtained by restricting the domain of  $f_j \in \mathcal{A}_j(\rho')$  to  $\mathcal{D}_j(\rho'')$ , where  $\rho'_1 > \rho''_1 > 0$  and  $\rho'_2 = \rho''_2$ . As

$$\|\bar{\mathbb{V}}_{j,n}^+ f_j\|_{j,\rho''} = \sum_{(\nu,k) \in V_{j,n}^+} |(f_j)_{\nu,k}| \rho_2^{\|k\|} e^{\rho'_1(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)} e^{-(\rho'_1 - \rho''_1)(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)},$$

we have  $\|\bar{\mathbb{V}}_{j,n}^+ f_j\|_{j,\rho''} \leq \|\bar{\mathbb{V}}_{j,n}^+\| \|f_j\|_{j,\rho'}$ , with  $\|\bar{\mathbb{V}}_{j,n}^+\| \leq e^{-(\rho'_1 - \rho''_1)c' \Lambda_{j,n}'}$ , as follows from the bound (60).

Similarly, let  $\mathbb{W}_{j,n}^+ : \mathcal{A}_j(\rho') \rightarrow \mathcal{A}_j(\rho')$  be the projection operator on  $\mathcal{A}_j(\rho')$  over the indexes in  $W_{j,n}^+$  and  $\bar{\mathbb{W}}_{j,n}^+ : \mathcal{A}_j(\rho') \rightarrow \mathcal{A}_j(\rho'')$  the same projection followed by an analytic inclusion in the  $q$  variables. Here, as before,  $\rho'_1 > \rho''_1 > 0$  and  $\rho'_2 = \rho''_2$ . As

$$\|\bar{\mathbb{W}}_{j,n}^+ f_j\|_{j,\rho''} = \sum_{(\nu,k) \in W_{j,n}^+} |(f_j)_{\nu,k}| \rho_2^{\|k\|} e^{\rho'_1(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)} e^{-(\rho'_1 - \rho''_1)(|\hat{\omega}_j \cdot \nu| + |\Omega_j \cdot \nu|)},$$

from the bound (63), we find that the operator norm of  $\bar{\mathbb{W}}_{j,n}^+$  satisfies  $\|\bar{\mathbb{W}}_{j,n}^+\| \leq e^{-(\rho'_1 - \rho''_1)c'' \Lambda_{j,n}'}$ .

Let  $\bar{\mathbb{I}}_n : \mathcal{A}_n(\rho'') \rightarrow \mathcal{A}_n(\rho')$  be the inclusion map obtained by restricting the domain of the functions  $f_n \in \mathcal{A}_n(\rho'')$  to  $\mathcal{D}_n(\rho')$ , where  $\rho''_2 > \rho'_2 > 0$  and  $\rho''_1 = \rho'_1$ . As

$$\|\bar{\mathbb{I}}_n f_n\|_{n,\rho'} = \sum_{(\nu,k') \in I_n} |(f_n)_{\nu,k'}| \rho_2^{\|k'\|} e^{\rho''_1(|\hat{\omega}_n \cdot \nu| + |\Omega_n \cdot \nu|)} e^{-(\ln \rho''_2 - \ln \rho'_2)\|k'\|} \leq \|\bar{\mathbb{I}}_n\| \|f_n\|_{n,\rho''},$$

the norm of the inclusion map  $\bar{\mathbb{I}}_n$  acting on functions that are composed only of modes with  $\|k'\| \geq \|k'\|_{\min}$  satisfying the inequality (65) can be bounded by  $\|\bar{\mathbb{I}}_n\| \leq e^{-(\ln \rho''_2 - \ln \rho'_2)c'' \Lambda_{j,n}'}$ .

To obtain the desired bound (54), we write the operator  $\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})$  as

$$\begin{aligned} \mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E}) &= \mathcal{L}_n \circ \dots \circ \mathcal{L}_{j+1}(\mathbb{I} - \mathbb{E}) \bar{\mathcal{L}}_j^{(1)} \bar{\mathbb{V}}_{j,n}^+ \\ &\quad + \mathcal{L}_n \circ \dots \circ \mathcal{L}_{j+1}(\mathbb{I} - \mathbb{E}) \bar{\mathcal{L}}_j^{(1)} \bar{\mathbb{W}}_{j,n}^+(\mathbb{I} - \mathbb{V}_{j,n}^+) \\ &\quad + \bar{\mathbb{I}}_{n+1} \bar{\mathcal{L}}_n^{(2)} \circ \mathcal{L}_{n-1} \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})(\mathbb{I} - \mathbb{W}_{j,n}^+)(\mathbb{I} - \mathbb{V}_{j,n}^+). \end{aligned} \tag{66}$$

Here,  $\bar{\mathcal{L}}_n^{(1)} : \mathbb{I}_n^+ \mathcal{A}_n(\rho'') \rightarrow \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho')$  and  $\bar{\mathcal{L}}_n^{(2)} : \mathbb{I}_n^+ \mathcal{A}_n(\rho') \rightarrow \mathbb{I}_{n+1}^+ \mathcal{A}_{n+1}(\rho'')$ ,  $n \in \mathbb{N}_0$ , are the derivatives of the  $n$ th-step renormalization operator at  $\mathbb{E}H_n$ . With superscripts (1) and (2), we explicitly emphasize that these operators actually improve analyticity in  $q$  and  $p$  variables, respectively. For some  $\rho'', \rho'' > 0$  satisfying  $0 < \rho''_1 < \rho'_1$ ,  $\rho''_2 = \rho'_2$ ,  $\rho''_1 = \rho'_1$  and  $\rho''_2 > \rho'_2$ , the construction of the analyticity improving operators is possible, as mentioned in remark 4.5.

The norm  $\|DR_i(H_i^0)\|$ ,  $i \in \mathbb{N}_0$ , can be bounded by a constant times  $\theta_i/\mu_i$ . By Cauchy's estimate, we have

$$\|DR_i(H_i) - DR_i(H_i^0)\| \leq \frac{2\|h_i\|_{i,\rho'}}{(\xi_i - \|h_i\|_{i,\rho'})^2}, \tag{67}$$

where  $DR_i(H_i)$  is the derivative of the  $i$ th-step renormalization operator  $\mathcal{R}_i$  at a Hamiltonian  $H_i = H_i^0 + h_i$ . In particular, the inequality (67) is satisfied by  $\mathcal{L}_i$ , the derivative of the  $i$ th-step

renormalization operator at  $\mathbb{E}H_i$ . As, by assumption,  $\|h_i\|_{i,\rho'} \leq \zeta_i/2$ , we have the bound  $\|\mathcal{L}_i\| \leq c_3\theta_i/\mu_i$ , with  $c_3 > 0$ .

Using the bounds on the operator norms  $\|\bar{\mathbb{V}}_{j,n}^+\|$ ,  $\|\bar{\mathbb{W}}_{j,n}^+\|$  and  $\|\bar{\mathbb{I}}_n\|$ , obtained above, we find that

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq c_3 e^{-c_2 \Lambda_{j,n}} \left( 2 \frac{\theta_j}{\mu_j} \|\mathcal{L}_n \circ \dots \circ \mathcal{L}_{j+1}(\mathbb{I} - \mathbb{E})\| + \frac{\theta_n}{\mu_n} \|\mathcal{L}_{n-1} \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \right),$$

where  $\Lambda_{j,n} \leq \min\{\Lambda'_{j,n}, \Lambda''_{j,n}\}$ , with  $\Lambda'_{j,n}$  and  $\Lambda''_{j,n}$  given by the expressions (61) and (64), respectively, and  $c_2 = \min\{(\rho'_1 - \rho''_1)c', (\rho'_1 - \rho''_1)c'', (\ln \varrho''_2 - \ln \rho'_2)c''\}$ .

Using the bounds on the norms of  $\mathcal{L}_i$ , for  $i = j, \dots, n$ , we obtain

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq 3c_3^{n-j+1} e^{-c_2 \Lambda_{j,n}} \prod_{i=j}^n \frac{\theta_i}{\mu_i}.$$

The bound (54) follows from this inequality and the bounds in proposition 6.2. □

Let  $h_0 \in \mathbb{I}_0^+ \mathcal{A}_0(\rho')$ ,  $\rho' > 0$ , componentwise. After the first  $(n + 1)$  renormalization steps the perturbation can be separated into two parts,

$$h_{n+1} = \mathbb{E}h_{n+1} + (\mathbb{I} - \mathbb{E})h_{n+1}. \tag{68}$$

The  $q$ -independent part of the perturbation can be determined by

$$\mathbb{E}h_{n+1} = D\mathcal{R}_n(H_n^0)\mathbb{E}h_n + \mathbb{E}\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2) \tag{69}$$

or after applying this equality recursively,

$$\begin{aligned} \mathbb{E}h_{n+1} &= D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0 \\ &+ \sum_{j=1}^n D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^0)\mathbb{E}\mathcal{O}_{j-1}^0(\|h_{j-1}\|_{j-1,\rho'}^2) + \mathbb{E}\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2). \end{aligned} \tag{70}$$

Here  $\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2)$  denotes the second-order remainder of the Taylor expansion of  $\mathcal{R}_n(H_n)$  about  $H_n^0$ . The norm of  $\mathcal{O}_n^0(\|h_n\|_{n,\rho'}^2)$  will be denoted by  $F_n^2$  and can be estimated by the bound (24).

In order to estimate the  $q$ -dependent part of the perturbation, we perform the Taylor expansion of  $\mathcal{R}_n(H_n)$  about  $\mathbb{E}H_n$ , the  $q$ -independent part of the Hamiltonian  $H_n$ . The  $q$ -dependent part of the perturbation is

$$(\mathbb{I} - \mathbb{E})h_{n+1} = \mathcal{L}_n(\mathbb{I} - \mathbb{E})h_n + (\mathbb{I} - \mathbb{E})\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2), \tag{71}$$

where  $\mathcal{L}_n = D\mathcal{R}_n(\mathbb{E}H_n)$  and  $\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2)$  denotes the second-order remainder of the Taylor expansion of  $\mathcal{R}_n(H_n)$  about  $\mathbb{E}H_n$ . The norm of this remainder is of the order of  $\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2$  and will be denoted by  $G_n^2$ .

After successive applications of the previous recursion relation, we obtain

$$\begin{aligned} (\mathbb{I} - \mathbb{E})h_{n+1} &= \mathcal{L}_n \dots \mathcal{L}_0(\mathbb{I} - \mathbb{E})h_0 + \sum_{j=1}^n \mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I} - \mathbb{E})\mathcal{O}_{j-1}(\|(\mathbb{I} - \mathbb{E})h_{j-1}\|_{j-1,\rho'}^2) \\ &+ (\mathbb{I} - \mathbb{E})\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2). \end{aligned} \tag{72}$$

We will use this identity to estimate the decrease of the norm of the  $q$ -dependent part of the perturbation in the following theorem.

**Theorem 7.2.** *Let  $\omega_0 \in D(\beta)$ ,  $0 \leq \beta < \sqrt{2} - 1$ , and let  $\rho'_1 > 0$ . There exist  $\tau > 2$  and  $\rho'_2, \sigma, \kappa, C > 0$ , such that if  $H_0 = H_0^0 + h_0 \in \mathbb{I}_0^+ \mathcal{A}_0(\rho')$  and  $\|h_0\|_{0,\rho'} \leq C^2 < 1$ , then  $\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'} \leq C \tilde{A}_n^{-\tau} < \zeta_n/2$ ,  $n \geq 0$ , where  $\zeta_n = C'/(\alpha_n \alpha_{n+1})^2$ ,  $C' > 0$ . Furthermore, if  $\beta < (\sqrt{161} - 11)/10$ , then  $\|h_n\|_{n,\rho'} \leq C \tilde{A}_{n-1}^{-2} < \zeta_n/2$ .*

**Proof.** We find first, using the Diophantine bound (49), that if  $\beta \leq \sqrt{2} - 1$  and  $\tau > 2$ , there exists  $C > 0$ , such that

$$\frac{\zeta_n}{2} = \frac{C'}{2\alpha_n^2 \alpha_{n+1}^2} \geq \frac{C'}{2\tilde{K}^4 \tilde{A}_{n-1}^{2\beta} \tilde{A}_n^{2\beta}} \geq \frac{C'}{2\tilde{K}^{4+2\beta} \tilde{A}_{n-1}^{2\beta} \tilde{A}_{n-1}^{2\beta(1+\beta)}} > \frac{C}{\tilde{A}_{n-1}^2} > \frac{C}{\tilde{A}_n^\tau}.$$

Let  $\tau' = \ln c_1 / \ln \gamma$ , where  $c_1 > 1$  is the constant from proposition 7.1. As  $\tilde{A}_n / \tilde{A}_{j-1} \geq \gamma^{n-j}$ ,  $0 \leq j \leq n$ , the inequality (54) implies

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq \frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2 \Lambda_{j,n}}. \tag{73}$$

Using the inequality  $e^{-t} \leq (s/t)^s$ , valid for any  $t > 0$  and  $s > 0$ , we find that, for any  $\tau'' > 0$ ,

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2 \Lambda_{j,n}} \leq (\max\{\sigma, \kappa\})^{\tau''} \left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)} \frac{\tilde{A}_{j-1}^{\tau''(1+\beta)-(4+\tau')}}{\tilde{A}_{n+1}^{\tau''-(4+\tau')}}.$$

Let  $\tau'' > 0$  be given, such that  $\tau = \tau'' - \tau' - 4 > 2$ . There exist  $\sigma, \kappa > 0$ , such that

$$\frac{\tilde{A}_{n+1}^{4+\tau'}}{\tilde{A}_{j-1}^{4+\tau'}} e^{-c_2 \Lambda_{j,n}} \leq (\max\{\sigma, \kappa\})^{\tau''} \left(\frac{\tau''(2+\beta)}{c_2}\right)^{\tau''(2+\beta)} \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{\tilde{A}_{n+1}^\tau} \leq \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^\tau}$$

and thus,

$$\|\mathcal{L}_n \circ \dots \circ \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \leq \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau')\beta}}{6\tilde{A}_{n+1}^\tau}. \tag{74}$$

We will prove the theorem by induction. There exists  $0 < C < 1$ , such that

$$\|(\mathbb{I} - \mathbb{E})h_0\|_{0,\rho'} \leq \|h_0\|_{0,\rho'} \leq C^2 < C \tilde{A}_0^{-\tau} < C < \zeta_0/2. \tag{75}$$

Therefore, for  $n = 0$ , the claim is true. Assume that the claim holds for  $0 < j \leq n$ . Thus, there exists  $C > 0$ , such that  $\|(\mathbb{I} - \mathbb{E})h_j\|_{j,\rho'} \leq C \tilde{A}_j^{-\tau} < \zeta_j/2$  and  $\|h_j\|_{j,\rho'} \leq C \tilde{A}_{j-1}^{-2} < \zeta_j/2$ . We will show that the claim is true for  $j = n + 1$ .

Using the identity (72), we obtain

$$\|(\mathbb{I} - \mathbb{E})h_{n+1}\|_{n+1,\rho'} \leq \|\mathcal{L}_n \dots \mathcal{L}_0(\mathbb{I} - \mathbb{E})h_0\|_{n+1,\rho'} + \sum_{j=1}^n \|\mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \cdot G_{j-1}^2 + G_n^2. \tag{76}$$

We will estimate the size of the terms on the right-hand side of the inequality (76).

As  $\|h_0\|_{0,\rho'} \leq C^2 < 1$ , the inequality (74), for  $j = 0$ , implies a bound on the first term

$$\|\mathcal{L}_n \dots \mathcal{L}_0(\mathbb{I} - \mathbb{E})h_0\|_{n+1,\rho'} \leq \frac{C}{6\tilde{A}_{n+1}^\tau}. \tag{77}$$

Using Cauchy's formula, one can obtain an estimate on the norm of the second-order remainder  $\mathcal{O}_n(\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2)$ , analogous to the bound (24),

$$G_n^2 \leq \frac{\|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}^2}{(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'})(\zeta_n - \|\mathbb{E}h_n\|_{n,\rho'} - \|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'})}.$$

Applying the inductive hypothesis and the Diophantine condition in the form (49), or equivalently,  $\alpha_{n+1} \leq \tilde{K}\tilde{A}_n^\beta$ , we further obtain that, for some  $C > 0$ ,

$$G_n^2 \leq \frac{4C^2}{\zeta_n^2 \tilde{A}_n^{2\tau}} \leq \frac{4C^2 \alpha_n^4 \alpha_{n+1}^4}{C^2 \tilde{A}_n^{2\tau}} \leq \frac{4C^2 \tilde{K}^{2\tau/(1+\beta)+8} \tilde{A}_n^{4\beta} \tilde{A}_{n-1}^{4\beta}}{C^2 \tilde{A}_{n+1}^{2\tau/(1+\beta)}} < \frac{C}{6\tilde{A}_{n+1}^\tau}, \tag{78}$$

if  $0 \leq \beta < 1$  and  $\tau \geq 8\beta(1 + \beta)/(1 - \beta)$ . Using the bounds (74) and (78), we can estimate the sum

$$\begin{aligned} \sum_{j=1}^n \|\mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \cdot G_{j-1}^2 &\leq \frac{2C^2}{3\tilde{A}_{n+1}^\tau} \sum_{j=1}^n \frac{\tilde{A}_{j-1}^{\tau(1+\beta)+(4+\tau)\beta}}{\zeta_{j-1}^2 \tilde{A}_{j-1}^{2\tau}} \\ &\leq \frac{2C^2 \tilde{K}^8}{3C^2 \tilde{A}_{n+1}^\tau} \sum_{j=1}^n \frac{1}{\tilde{A}_{j-1}^{\tau(1-\beta)-(12+\tau)\beta}}. \end{aligned}$$

Since for  $\alpha_0 > 1$ , we have  $\tilde{A}_{j-1} \geq \gamma^{j-1}$ , the previous sum can be bounded by some positive constant, if  $0 \leq \beta < 1$  and  $\tau > \beta(\tau' + 12)/(1 - \beta)$ . Thus, there exists  $C > 0$ , such that

$$\sum_{j=1}^n \|\mathcal{L}_n \dots \mathcal{L}_j(\mathbb{I} - \mathbb{E})\| \cdot G_{j-1}^2 \leq \frac{C}{6\tilde{A}_{n+1}^\tau}. \tag{79}$$

Finally, if  $0 \leq \beta < 1$  is given, and constants  $\tau > 2$  and  $\sigma, \kappa, C > 0$  are chosen according to the various conditions stated above, using the bounds (77), (78) and (79), we obtain

$$\|(\mathbb{I} - \mathbb{E})h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{6\tilde{A}_{n+1}^\tau} + \frac{C}{6\tilde{A}_{n+1}^\tau} + \frac{C}{6\tilde{A}_{n+1}^\tau} = \frac{C}{2\tilde{A}_{n+1}^\tau} < \frac{C}{\tilde{A}_{n+1}^\tau} < \frac{\zeta_{n+1}}{2}. \tag{80}$$

Concerning the  $q$ -independent part of the perturbation, from the identity (70), we find that

$$\begin{aligned} \|\mathbb{E}h_{n+1}\|_{n+1,\rho'} &\leq \|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n+1,\rho'} \\ &+ \sum_{j=1}^n \|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \cdot F_{j-1}^2 + F_n^2. \end{aligned} \tag{81}$$

Using the bound (52) on the derivative of a one-step renormalization operator acting on  $q$ -independent Hamiltonians and the first of the inequalities (50), one obtains that there exists  $C > 0$ , such that

$$\|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_0(H_0^0)\mathbb{E}h_0\|_{n,\rho'} \leq \frac{4\alpha_0^2}{(1 + \alpha_0)\tilde{A}_n^2} \|h_0\|_{0,\rho'} \leq \frac{4\alpha_0^2}{(1 + \alpha_0)\tilde{A}_n^2} C^2 < \frac{C}{6\tilde{A}_n^2}. \tag{82}$$

From the estimate (24) obtained in theorem 4.4, we have the following bound on the norm of the second-order remainder,  $F_n^2 \leq \zeta_n^{-1}(\zeta_n - \|h_n\|_{n,\rho'})^{-1} \|h_n\|_{n,\rho'}^2$ . Using the inductive hypothesis and the Diophantine property of  $\omega_0$ , we find that there exists  $C > 0$ , such that

$$F_j^2 \leq \frac{2C^2}{\zeta_j^2 \tilde{A}_{j-1}^4} \leq \frac{2C^2 \tilde{K}^{4/(1+\beta)} \alpha_j^4 \alpha_{j+1}^4}{C^2 \tilde{A}_j^{4/(1+\beta)}} \leq \frac{2C^2 \tilde{K}^{4/(1+\beta)+8} \tilde{A}_{j-1}^{4\beta} \tilde{A}_j^{4\beta}}{C^2 \tilde{A}_j^{4/(1+\beta)}} < \frac{C}{6\tilde{A}_j^2}, \tag{83}$$

for  $0 \leq \beta < (\sqrt{41} - 5)/8$ . Since, from the bound (52) and the inequalities (50), one has

$$\|D\mathcal{R}_n(H_n^0) \dots D\mathcal{R}_j(H_j^0)\mathbb{E}\| \leq \frac{8\alpha_0 \tilde{A}_j \tilde{A}_{j-1}}{(1 + \alpha_0)\tilde{A}_n^2},$$



the sum on the right-hand side of the inequality (81) can be estimated by

$$\begin{aligned} \sum_{j=1}^n \|DR_n(H_n^0) \dots DR_j(H_j^0)\mathbb{E}\| \cdot F_{j-1}^2 &\leq \frac{16\alpha_0 C^2 \tilde{K}^8}{(1 + \alpha_0)C^2 \tilde{A}_n^2} \sum_{j=1}^n \frac{\tilde{A}_j \tilde{A}_{j-1}^{1+4\beta} \tilde{A}_{j-2}^{4\beta}}{\tilde{A}_{j-2}^4} \\ &\leq \frac{16\alpha_0 C^2 \tilde{K}^{11+5\beta}}{(1 + \alpha_0)C^2 \tilde{A}_n^2} \sum_{j=1}^n \frac{1}{\tilde{A}_{j-2}^{4-(2+5\beta)(1+\beta)-4\beta}}. \end{aligned}$$

If  $2 - 11\beta - 5\beta^2 > 0$ , the sum on the right side can be bounded by a positive constant, as  $\tilde{A}_j$  grows at least exponentially with  $j$ . Thus, there exists  $C > 0$ , such that

$$\sum_{j=1}^n \|DR_n(H_n^0) \dots DR_j(H_j^0)\mathbb{E}\| \cdot F_{j-1}^2 \leq \frac{C}{6\tilde{A}_n^2}. \tag{84}$$

Using the bounds (82), (83) and (84), the inequality (81) implies

$$\|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} + \frac{C}{6\tilde{A}_n^2} = \frac{C}{2\tilde{A}_n^2}. \tag{85}$$

Finally, taking into account the estimates (80) and (85), we find that

$$\|h_{n+1}\|_{n+1,\rho'} = \|(\mathbb{I} - \mathbb{E})h_{n+1}\|_{n+1,\rho'} + \|\mathbb{E}h_{n+1}\|_{n+1,\rho'} \leq \frac{C}{2\tilde{A}_{n+1}^2} + \frac{C}{2\tilde{A}_n^2} \leq \frac{C}{\tilde{A}_n^2} < \frac{\zeta_{n+1}}{2}.$$

This completes the proof of the claim. □

**Corollary 7.3.** *Let  $\omega_0 \in D(\beta)$ , with  $0 \leq \beta < (\sqrt{161} - 11)/10$ . Given  $\rho'_1 > 0$ , there exist  $\rho'_2, \sigma, \kappa, \tilde{C} > 0$  and a non-empty open neighbourhood  $B_{0,\rho'}^+ \subset \mathbb{I}_0^+ \mathcal{A}_0(\rho')$  of  $H_0^0$ , such that for all Hamiltonians  $H_0 \in B_{0,\rho'}^+$  and  $n \in \mathbb{N}_0$ ,  $\|H_n - H_n^0\|_{n,\rho'} \leq \tilde{C}\gamma^{-2n}$ .*

**Proof.** The proof of this corollary follows directly from theorem 7.2 and the fact that  $\tilde{A}_n$  grows at least exponentially with  $n \geq 0$ , i.e.  $\tilde{A}_n \geq \gamma^n$ , for  $\alpha_0 > 1$ . □

**Remark 7.4.** In the context of remark 4.6, the set of Hamiltonians in  $\mathbb{I}_0^+ \mathcal{A}_0(\rho')$  that satisfies the assumptions of theorem 7.2 and the neighbourhood  $B_{0,\rho'}^+$  in corollary 7.3 could be replaced with an open ball  $B_{0,\rho'} \subset \mathcal{A}_0(\rho')$ .

### 8. Definition of invariant tori and formal identities

In the remaining part of this paper, we apply the result about the convergence of the constructed renormalization scheme for  $\omega_0 \in D(\beta)$  to prove a KAM theorem for near-integrable Hamiltonians. We construct the invariant tori with Diophantine frequency vectors for Hamiltonians approaching the trivial limit set under the renormalization. We start by providing some definitions and some formal identities that will be used in the construction.

**Definition 8.1.** *Given  $\delta_1 > 0$ , we define*

$$\mathcal{D}_{n,0}(\delta_1) = \{q \in \mathbb{C}^2 : |\text{Im } \omega'_n \cdot q| < \delta_1, |\text{Im } \Omega'_n \cdot q| = 0\} \times \{0\}. \tag{86}$$

**Definition 8.2.** *Given  $r > 0$  and  $\delta_1 > 0$ , let  $\mathcal{A}_{n,0}(\delta_1)$  be the space of complex valued functions  $f$ , analytic on  $\mathcal{D}_{n,0}(\delta_1)$ ,  $2\pi$ -periodic in both  $q$ -variables, with the finite norm*

$$\|f\|_{n,\delta_1} = \sum_{v \in \mathbb{Z}^2} |f_v| (1 + |\Omega_n \cdot v|)^r e^{\delta_1 |\hat{\omega}_n \cdot v|}, \tag{87}$$

where  $f_v$  are the Fourier coefficients of  $f$ .

**Definition 8.3.** We say that  $H_n$  has an invariant torus with frequency vector  $\omega_n$  if there exists a continuous map  $\Gamma_n : \mathcal{D}_{n,0}(\delta_1) \rightarrow \mathcal{D}_n(\rho)$ , with  $\rho > 0$ , componentwise, and a continuous function  $t : \mathbb{R} \rightarrow \mathbb{R}$ , such that for all  $s \in \mathbb{R}$ ,

$$\Phi_n^{t(s)} \circ \Gamma_n = \Gamma_n \circ \Psi_n^s. \tag{88}$$

Here  $\Phi_n$  is the flow for the Hamiltonian  $H_n$ , and  $\Psi_n$  is the flow for the Hamiltonian  $K_n = \omega_n \cdot p$ , i.e.  $\Psi_n^s(q, 0) = (q + \omega_n s, 0)$ .

Notice that an invariant torus of a Hamiltonian is defined as the conjugacy between the Hamiltonian flow and a linear flow of an integrable Hamiltonian.

The flow  $\Phi_n$  for the Hamiltonian  $H_n$  and the flow  $\Phi_{n+1}$  for the renormalized Hamiltonian  $H_{n+1} = \mathcal{R}_n(H_n)$  can be related by

$$\Lambda_n \circ \Phi_{n+1}^t = \Phi_n^{\theta'_n t} \circ \Lambda_n. \tag{89}$$

This identity is valid for  $t \in \mathbb{C}$  on any domain where the compositions are well defined. The identity formally follows from

$$\frac{d}{dt} f \circ \Phi_n^{\theta'_n t} \circ \Lambda_n|_{t=0} = \theta'_n \{f, H_n\} \circ \Lambda_n = \frac{\theta'_n}{\mu'_n} \{f \circ \Lambda_n, H_n \circ \Lambda_n\} = \{f \circ \Lambda_n, \mathcal{R}_n(H_n)\},$$

where we have used the identity  $\mu'_n{}^{-1} \{f \circ \Lambda_n, g \circ \Lambda_n\} = \{f, g\} \circ \Lambda_n$ , for complex valued functions  $f$  and  $g$  defined on a neighbourhood of  $\mathbb{T}^2 \times \mathbb{R}^2$ . Here  $\{f, g\} = \nabla_q f \cdot \nabla_p g - \nabla_q g \cdot \nabla_p f$  is the Poisson bracket of the functions  $f$  and  $g$ .

Now, we have the identities

$$\begin{aligned} \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} \circ \Psi_n^s &= \Lambda_n \circ \Gamma_{n+1} \circ \Psi_{n+1}^{\alpha_{n+1}^{-1} s} \circ \mathcal{T}_n^{-1} \\ &= \Lambda_n \circ \Phi_{n+1}^{t_{n+1}(\alpha_{n+1}^{-1} s)} \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} = \Phi_n^{\theta'_n t_{n+1}(\alpha_{n+1}^{-1} s)} \circ \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}. \end{aligned} \tag{90}$$

Thus, formally, if  $\Gamma_{n+1}$  is an invariant torus with frequency vector  $\omega_{n+1}$  of  $H_{n+1}$ , then

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1} = \mathcal{M}_n(\Gamma_{n+1}) \tag{91}$$

is the invariant torus with frequency vector  $\omega_n$  of  $H_n$ . The flow parameters  $t_n$  and  $t_{n+1}$  are related by  $t_n(s) = \theta'_n t_{n+1}(\alpha_{n+1}^{-1} s)$ . One solution of this functional equation is  $t_n(s) = \xi s / \prod_{i=1}^n \tau_i$ , where  $\xi$  is a constant.

### 9. Construction of invariant tori

The formal relationship (91) between invariant tori of a Hamiltonian and its renormalization image motivates the construction of an invariant torus. In this section we study the properties of the maps  $\mathcal{M}_n$ , defined by (91), and use them to construct invariant tori for near-integrable Hamiltonians in a space of functions of low regularity. In the following, assume that  $\omega_0 \in D(\beta)$ , with  $0 \leq \beta < (\sqrt{161} - 11)/10$ , and that  $H_n \in \mathbb{I}_n^+ \mathcal{A}_n(\rho')$ ,  $n \in \mathbb{N}_0$ , is the renormalization orbit of a given Hamiltonian  $H_0 \in B_{0,\rho'}^+(C^2)$  in the domain of attraction of the integrable limit set.

Introduce the coordinates  $x_1 = \omega'_n \cdot q$ ,  $x_2 = \Omega'_n \cdot q$ ,  $y_1 = \hat{\omega}_n \cdot p$  and  $y_2 = \Omega_n \cdot p$ , for a given  $n \in \mathbb{N}_0$ . Denote the partial derivative with respect to  $x_1, x_2, y_1$  and  $y_2$  as  $\partial_i$ , for  $i = 1, 2, 3$  and 4, respectively. We can define the components of a vector valued function  $f = (f^q, f^p)$  on  $\mathcal{D}_{n,0}(\delta_1)$ ,  $\delta_1 > 0$ , as  $f^1 = \omega'_n \cdot f^q$ ,  $f^2 = \Omega'_n \cdot f^q$ ,  $f^3 = \hat{\omega}_n \cdot f^p$ ,  $f^4 = \Omega_n \cdot f^p$ .

**Definition 9.1.** Given  $r > 0$  and  $\delta_1 > 0$ , let  $\mathcal{B}_{n,0}(\delta_1)$  be the Banach space of vector-valued functions  $f = (f^q, f^p)$  on  $\mathcal{D}_{n,0}(\delta_1)$ , whose components  $f^i$ ,  $i = 1, 2, 3, 4$ , belong to  $\mathcal{A}_{n,0}(\delta_1)$ , with the norm

$$\|f\|_{n,\delta_1} = \max\{c_n \|f^1\|_{n,\delta_1}, \|f^2\|_{n,\delta_1}, \|f^3\|_{n,\delta_1}, \|f^4\|_{n,\delta_1}\}, \tag{92}$$

where  $c_n = \tilde{A}_n \prod_{i=0}^{n-1} \|T_i^{-1} \Omega_i\|$ .

The maps  $\mathcal{M}_n$  are defined formally by the action on a vector valued function  $F = I + f$ , where  $I$  is the identity map and  $f \in \mathcal{B}_{n+1,0}(\delta_1)$ ,

$$\mathcal{M}_n(F) = \Lambda_n \circ F \circ \mathcal{T}_n^{-1}. \tag{93}$$

The weight factor  $c_n$  has been introduced in the norm (92) in order to make the map  $\mathcal{M}_n$  a contraction; if the norm were defined without that factor, the scaling map in  $\Lambda_n$  would actually expand the first component of  $F$ .

Similarly, for a vector valued function  $g = (g^q, g^p)$  on  $\mathcal{D}_n(\rho)$  with components  $g^1 = \omega'_n \cdot g^q, g^2 = \Omega'_n \cdot g^q, g^3 = \hat{\omega}_n \cdot g^p$  and  $g^4 = \Omega_n \cdot g^p$  in  $\mathcal{A}_n(\rho)$ , define the norms

$$\|g\|_{n,\rho} = \max\{c_n \|g^1\|_{n,\rho}, \|g^2\|_{n,\rho}, \|g^3\|_{n,\rho}, \|g^4\|_{n,\rho}\} \tag{94}$$

and

$$\|g\|'_{n,\rho} = \max \left\{ \|\partial_1 g^1\|_{n,\rho} + c_n \sum_{i=2}^4 \|\partial_i g^1\|_{n,\rho}, c_n^{-1} \|\partial_1 g^2\|_{n,\rho} + \sum_{i=2}^4 \|\partial_i g^2\|_{n,\rho}, c_n^{-1} \|\partial_1 g^3\|_{n,\rho} + \sum_{i=2}^4 \|\partial_i g^3\|_{n,\rho}, c_n^{-1} \|\partial_1 g^4\|_{n,\rho} + \sum_{i=2}^4 \|\partial_i g^4\|_{n,\rho} \right\}. \tag{95}$$

Before proving that in the above norm the maps  $\mathcal{M}_n$  are contractions, let us state the following proposition.

**Proposition 9.2.** *Let  $\delta_1 > 0, \delta_1^< = \delta_1 \|\omega_{n+1}\| / (\alpha_{n+1} \|\omega_n\|) < 2\delta_1/3$  and  $\rho > 0$ , componentwise, with  $\rho_1 > \delta_1$ . Let also  $f_n, g_n \in \mathcal{A}_{n,0}(\delta_1), h_n \in \mathcal{A}_n(\rho)$  and  $X_n, Y_n \in \mathcal{A}_{n,0}^2(\delta_1)$ , such that  $\delta_1 + \|\omega'_n \cdot X_n\|_{n,\delta_1} < \rho_1, \|\Omega'_n \cdot X_n\|_{n,\delta_1} < \rho_1, \|\hat{\omega}_n \cdot Y_n\|_{n,\delta_1} < \rho_2$  and  $\|\Omega_n \cdot Y_n\|_{n,\delta_1} < \rho_2$ . If  $U(q, 0) = (q + X_n(q, 0), Y_n(q, 0))$  is a given change of variables, then*

- (i)  $|f_n(q, 0)| \leq \|f_n\|_{n,\delta_1}, \forall (q, 0) \in \mathcal{D}_{n,0}(\delta_1),$
- (ii)  $f_n g_n \in \mathcal{A}_{n,0}(\delta_1)$  and  $\|f_n g_n\|_{n,\delta_1} \leq \|f_n\|_{n,\delta_1} \|g_n\|_{n,\delta_1},$
- (iii)  $\|f_{n+1} \circ \mathcal{T}_n^{-1}\|_{n,\delta_1} \leq \|\mathcal{T}_n^{-1} \Omega_n\|^r \|f_{n+1}\|_{n+1,\delta_1^<},$
- (iv)  $\|h_n \circ U\|_{n,\delta_1} \leq c_{\rho_1} (\|\Omega'_n \cdot X_n\|_{n,\delta_1}) \|h_n\|_{n,\rho},$

where  $c_{\rho_1}(s) = \sup_{t \geq 0} (1+t)^r e^{-(\rho_1-s)t}$ , for  $|s| < \rho_1$ .

The proof of this proposition is straightforward and will be omitted. Denote by  $B_n(b)$  an open ball of radius  $b > 0$  in the affine space  $I + \mathcal{B}_{n,0}(\delta_1)$ , centred at the identity.

**Lemma 9.3.** *Let  $r < 1$  and let  $0 < \delta_1 < \rho'_1$ . For sufficiently small  $b, C > 0$  and all Hamiltonians  $H_n, n \in \mathbb{N}_0$ , of the renormalization orbit of  $H_0 \in \mathcal{B}_{0,\rho'}^+(C^2) \subset \mathbb{I}_0^+ \mathcal{A}_0(\rho')$ , the map  $\mathcal{M}_n$  is a contraction with the contraction rate  $a < 1$  (independent of  $H_0$  and  $n$ ) from  $B_{n+1}(b)$  into  $B_n(b)$ .*

**Proof.** Let  $F_{n+1} \in B_{n+1}(b)$  and let  $f_{n+1} = F_{n+1} - I$ . Define the map  $\mathcal{N}_n : B_{n+1}(b) \rightarrow \mathcal{B}_{n,0}(\delta_1)$  by

$$\mathcal{N}_n(f_{n+1}) = \mathcal{M}_n(F_{n+1}) - I = \Lambda_n \circ (I + f_{n+1}) \circ \mathcal{T}_n^{-1} - I.$$

The fact that the maps  $V_{H'_{n+1}}, U_{H''_{n+1}}$  and  $S_{\tilde{H}_{n+1}}$ , included in  $\Lambda_n$ , are close to the identity motivates us to write  $\Lambda_n = \mathcal{T}_n \circ (I + g_{n+1})$ . Therefore,

$$\mathcal{N}_n(f_{n+1}) = \mathcal{T}_n \circ (f_{n+1} + g_{n+1} \circ (I + f_{n+1})) \circ \mathcal{T}_n^{-1}.$$

As the maps  $V_{H'_{n+1}}, U_{H''_{n+1}}$  and  $S_{\tilde{H}_{n+1}}$  depend only on the  $q$ -dependent part of the Hamiltonian,  $\|g_{n+1}^i\|_{n+1,\rho'} \leq C_1 \|(\mathbb{I} - \mathbb{E})h_n\|_{n,\rho'}$ , for  $i = 1, 2, 3, 4$ , i.e.  $\|g_{n+1}^i\|_{n+1,\rho'} \leq CC_1 \tilde{A}_n^{-\tau}$ , where  $C_1 > 0$  is an  $n$ -independent constant. Here, we assume that  $C > 0$  is sufficiently small such that the

estimates of theorem 7.2 are valid. Using the Diophantine bound (49) and the inequalities (50), we find that, if  $\tau \geq 2 + \beta$ , then

$$\|g_{n+1}\|_{n+1,\rho'} \leq \tilde{A}_{n+1} \prod_{i=0}^n \|T_i^{-1}\Omega_i\| \frac{CC_1}{\tilde{A}_n^\tau} \leq \frac{CC_2}{\tilde{A}_n^{\tau-2-\beta}} \leq CC_2,$$

where  $C_2 > 0$  is a constant. These inequalities show that the growth of the weight factor with  $n$  is slower than the decrease of the norm of the  $q$ -dependent part of Hamiltonian.

The derivative of the operator  $\mathcal{N}_n$  at  $f_{n+1}$  is given by its action on an arbitrary  $\tilde{f} \in B_{n+1}(b)$ ,

$$D\mathcal{N}_n(f_{n+1})\tilde{f} = T_n \circ (\tilde{f} + Dg_{n+1} \circ (I + f_{n+1})\tilde{f}) \circ T_n^{-1}.$$

Using the proposition 9.2, we find that for  $(\delta_1, 0) < \rho'' < \rho'$ , componentwise,

$$\begin{aligned} \|D\mathcal{N}_n(f_{n+1})\tilde{f}\|_{n,\delta_1} &\leq \|T_n^{-1}\Omega_n\|^{r-1} \|\tilde{f}\|_{n+1,\delta_1^c} \\ &\quad + \|T_n^{-1}\Omega_n\|^{r-1} c_{\rho_1''} (\|f_{n+1}^2\|_{n+1,\delta_1^c}) \|g_{n+1}\|'_{n+1,\rho''} \|\tilde{f}\|_{n+1,\delta_1^c}, \end{aligned}$$

providing that  $\delta_1^c + \|f_{n+1}^1\|_{n+1,\delta_1^c} < \rho_1''$ ,  $\|f_{n+1}^2\|_{n+1,\delta_1^c} < \rho_1''$ ,  $\|f_{n+1}^3\|_{n+1,\delta_1^c} < \rho_2''$  and  $\|f_{n+1}^4\|_{n+1,\delta_1^c} < \rho_2''$ . These conditions are satisfied for sufficiently small  $b > 0$ . Using Cauchy estimates on the norms of the derivatives  $\|\partial_i g_{n+1}^j\|_{n+1,\rho''}$ , for  $i, j = 1, 2, 3, 4$ , we find that  $\|g_{n+1}\|'_{n+1,\rho''} \leq C_3 \|g_{n+1}\|_{n+1,\rho'}$ , with  $C_3 > 0$ . Thus, as  $r < 1$  and  $\|T_n^{-1}\Omega_n\| \geq 3/2$ , for sufficiently small  $C > 0$ , the operator norm  $\|D\mathcal{N}_n(f_{n+1})\|$  can be bounded by a positive constant  $a < 1$ .

As  $\|\mathcal{N}_n(0)\|_{n,\delta_1} \leq \|T_n^{-1}\Omega_n\|^{r-1} c_{\rho_1''}(0) \|g_{n+1}\|_{n+1,\rho'}$ , for sufficiently small  $C > 0$ ,  $\|\mathcal{N}_n(0)\|_{n,\delta_1} < (1-a)b$ . This shows that  $\mathcal{N}_n$  maps  $B_{n+1}(b)$  into  $B_n(b)$  and contracts distances at least by a factor  $a < 1$ . □

Before we prove that every Hamiltonian  $H_n$ , of the renormalization orbit of  $H_0 \in B_{0,\rho'}^+(C^2)$ , with sufficiently small  $C > 0$ , has an invariant torus of Diophantine frequency vector  $\omega_n$ , we will prove the following proposition.

**Proposition 9.4.** *Let  $0 < \delta_1 < \rho_1'$  and  $\rho_2' > 0$ . For sufficiently small  $b > 0$ , the following holds. If  $H_0 \in B_{0,\rho'}^+(C^2)$ ,  $|t_m - s_m| \leq C_4 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'}$  and  $|t_m| < C_5$ , for some  $C_4, C_5 > 0$ ,  $t_m, s_m \in \mathbb{R}$  and for all  $m \in \mathbb{N}_0$  and if  $C > 0$  is sufficiently small, then*

$$\|(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - I)\|_{m,\delta_1} \leq b, \tag{96}$$

where  $\Phi_{m,0}$  is the flow for the Hamiltonian  $H_m^0$  restricted to  $\mathcal{D}_{m,0}(\delta_1)$ .

**Proof.** Let  $s \in \mathbb{R}$  be given and let  $(q, 0) \in \mathcal{D}_{m,0}(\delta_1)$ . The flow  $\Phi_m$  for the Hamiltonian  $H_m$  satisfies the equation,

$$\frac{d}{dt}(\Phi_m^t \circ \Phi_{m,0}^{-s} - I)(q, 0) = (\mathbb{J} \cdot \nabla H_m \circ \Phi_m^t \circ \Phi_{m,0}^{-s})(q, 0), \tag{97}$$

where  $\mathbb{J}(q, p) = (p, -q)$  and  $\nabla = (\nabla_q, \nabla_p)$ . Introducing the function

$$\Upsilon_s(q, t) = (\Phi_m^t \circ \Phi_{m,0}^{-s} - I)(q, 0),$$

$2\pi$ -periodic in both  $q$ -variables, we can integrate the equation (97) to obtain the integral equation

$$\Upsilon_s(q, t) = -(s\omega_m, 0) + \int_0^t dt' \{ \mathbb{J} \cdot \nabla H_m \circ [(q, 0) + \Upsilon_s(q, t')] \}. \tag{98}$$

This equation can be viewed as the fixed point equation of the operator  $\Xi_s$ , defined by the action

$$\Xi_s(\Upsilon_s(q, t), q) = -(s\omega_m, 0) + \int_0^t dt' \{ \mathbb{J} \cdot \nabla H_m \circ [(q, 0) + \Upsilon_s(q, t')] \} \tag{99}$$

on a space of functions  $\Upsilon_s : \mathcal{D}_{m,0}(\delta_1) \times J \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$ ,  $J = [0, t] \subset \mathbb{R}$ , with the norm

$$|||\Upsilon_s||| = \sup_{t' \in J} \max\{c_m \|\Upsilon_s^1(\cdot, t')\|_{m,\delta_1}, \|\Upsilon_s^2(\cdot, t')\|_{m,\delta_1}, c_m \|\Upsilon_s^3(\cdot, t')\|_{m,\delta_1}, c_m \|\Upsilon_s^4(\cdot, t')\|_{m,\delta_1}\},$$

where  $\Upsilon_s^1 = \omega'_m \cdot \Upsilon_s^q$ ,  $\Upsilon_s^2 = \Omega'_m \cdot \Upsilon_s^q$ ,  $\Upsilon_s^3 = \hat{\omega}_m \cdot \Upsilon_s^p$ ,  $\Upsilon_s^4 = \Omega_m \cdot \Upsilon_s^p$  and  $\Upsilon_s = (\Upsilon_s^q, \Upsilon_s^p)$ .

We consider an open ball of functions  $\Upsilon_s$  in that space that satisfy  $|||\Upsilon_s||| < b$ , with  $0 < b < \min\{\rho'_1 - \delta_1, \rho_2\}$ . This justifies the formal use of equations (98) and (99), as for any given  $t' \in J$  one has  $\|\Upsilon_s(\cdot, t')\|_{m,\delta_1} \leq |||\Upsilon_s|||$ .

Now, if  $\Upsilon_s^{(1)}$  and  $\Upsilon_s^{(2)}$  are two functions from that ball, we have

$$\Xi_s(\Upsilon_s^{(2)}(q, t), q) - \Xi_s(\Upsilon_s^{(1)}(q, t), q) = \int_0^t dt' \int_{\Upsilon_s^{(1)}(q,t')}^{\Upsilon_s^{(2)}(q,t')} d\Upsilon \cdot \nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q, 0) + \Upsilon].$$

Thus,

$$|||\Xi_s(\Upsilon_s^{(2)}, \cdot) - \Xi_s(\Upsilon_s^{(1)}, \cdot)||| \leq |t| \cdot \|\Upsilon_s^{(2)} - \Upsilon_s^{(1)}\| \cdot \|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [(q, 0) + \Upsilon]\|, \quad (100)$$

where

$$\begin{aligned} \|\nabla(\mathbb{J} \cdot \nabla H_m) \circ [\Upsilon + (q, 0)]\| \leq & \max \left\{ c_m \|\partial_2 \partial_3 H_m\|_{m,\rho''} + \sum_{i=1, i \neq 2}^4 \|\partial_i \partial_3 H_m\|_{m,\rho''}, \right. \\ & \|\partial_2 \partial_4 H_m\|_{m,\rho''} + c_m^{-1} \sum_{i=1, i \neq 2}^4 \|\partial_i \partial_4 H_m\|_{m,\rho''}, c_m \|\partial_2 \partial_1 H_m\|_{m,\rho''} \\ & \left. + \sum_{i=1, i \neq 2}^4 \|\partial_i \partial_1 H_m\|_{m,\rho''}, c_m \|\partial_2 \partial_2 H_m\|_{m,\rho''} + \sum_{i=1, i \neq 2}^4 \|\partial_i \partial_2 H_m\|_{m,\rho''} \right\}, \end{aligned}$$

with  $b + \delta_1 < \rho''_1 < \rho'_1$  and  $b < \rho''_2 < \rho'_2$ .

Using Cauchy estimates on the norms of the derivatives and the fact that  $\|\partial_2 \partial_i H_m\|_{m,\rho''} \leq C_6 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'}$ , for  $i = 1, 2, 3$ , where  $C_6 > 0$ , this norm can be bounded by an  $m$ -independent constant. Here, we again use the fact that  $\|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'}$  drops faster with  $m$  than  $c_m$  increases. The inequality (100) then implies that, if  $|t| = |t_m| < C_5$ ,  $H_0 \in B_{0,\rho'}^+(C^2)$  and  $C > 0$  is sufficiently small,  $\Xi_s$  is a contraction with a contraction rate  $\bar{a} < 1$ .

Notice that

$$\Xi_s(0, q) = ((t - s)\omega_m, 0) + \mathbb{J} \cdot \nabla h_m(q, 0)t,$$

and that  $\nabla \mathbb{E}h_m(q, 0) = 0$ . Therefore, since  $|t_m - s_m| \leq C_4 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'}$ , where  $C_4 > 0$  and  $|t_m| < C_5$ , if  $t = t_m$  and  $s = s_m$ , then, for sufficiently small  $C > 0$ , we have  $|||\Xi_{s_m}(0, q)||| < (1 - \bar{a})b$ . This shows that  $\Xi_{s_m}(\cdot, q)$  is a contraction on a ball of radius  $b > 0$  and has a unique fixed point  $\Upsilon_{s_m}$ , of norm  $|||\Upsilon_{s_m}||| < b$ , that satisfies the equation (98). The claim follows from the fact that for  $|t_m| < C_5$  one has  $\|\Upsilon_{s_m}(\cdot, t_m)\|_{m,\delta_1} \leq |||\Upsilon_{s_m}|||$ .  $\square$

Let, in the following,  $b > 0$  and  $C > 0$  be chosen sufficiently small such that the assumptions of lemma 9.3 and proposition 9.4 are satisfied. Let  $F_m$ ,  $m \in \mathbb{N}_0$ , be arbitrary maps in  $B_m(b)$ . Define

$$\Gamma_{n,m} = (\mathcal{M}_n \circ \dots \circ \mathcal{M}_{m-1})(F_m), \quad (101)$$

for  $0 \leq n < m$ , or, taking into account the definition (93) of the operators  $\mathcal{M}_n$ ,

$$\Gamma_{n,m} = \Lambda_n \circ \dots \circ \Lambda_{m-1} \circ F_m \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_n^{-1}. \quad (102)$$

Here, the maps  $\Lambda_n$  and the operators  $\mathcal{M}_n$  are associated to the sequence of Hamiltonians  $H_n$ ,  $n \in \mathbb{N}_0$ .

**Theorem 9.5.** *Let  $0 < \delta_1 < \rho'_1$ . If  $H_0 \in B_{0,\rho'}^+(C^2)$  and  $b, C > 0$  are sufficiently small, the limits  $\Gamma_n = \lim_{m \rightarrow \infty} \Gamma_{n,m}$ ,  $n \in \mathbb{N}_0$ , exist in  $B_n(b)$  and satisfy*

$$\|\Gamma_n - I\|_{n,\delta_1} \leq b. \tag{103}$$

$\Gamma_n$  is an invariant torus of  $H_n$  with frequency vector  $\omega_n$ . The invariant tori satisfy

$$\Gamma_n = \Lambda_n \circ \Gamma_{n+1} \circ \mathcal{T}_n^{-1}. \tag{104}$$

**Proof.** By lemma 9.3, if  $n < m < i$ , then  $\|\Gamma_{n,i} - \Gamma_{n,m}\|_{n,\delta_1} \leq 2ba^{m-n}$ , where  $a < 1$  is the constant from the same lemma. Thus, the sequence  $m \mapsto \Gamma_{n,m}$  is Cauchy and converges in this norm to a limit  $\Gamma_n$ . Moreover, this limit is independent of  $F_m$ . The relationship (104) follows from the continuity of  $\mathcal{M}_n$ .

Now, we want to show that, in the above norm,  $\Gamma_n \rightarrow I$ , when  $n \rightarrow \infty$ . Consider a one-parameter family of Hamiltonians  $H_n(s) = H_n^0 + s(H_n - H_n^0)$ ,  $s \in \mathbb{C}$ , and let  $\Lambda_n(s)$  and  $\Gamma_n(s)$  be the corresponding one parameter families of maps. The map  $s \mapsto \Lambda_n(s)$  is analytic in the domain  $|s| < \zeta_n \tilde{A}_{n-1}^2 / C$  containing the unit disc. By uniform convergence, the map  $s \mapsto \Gamma_n(s)$  is analytic from the same domain into  $B_n(b)$ . As  $\Gamma_n(0) = I$ , by Schwartz's lemma,  $\|\Gamma_n(1) - I\|_{n,\delta_1} \leq 2bC / (\zeta_n \tilde{A}_{n-1}^2)$ . The inequality (103) then follows from theorem 7.2.

It remains to be proved that  $\Gamma_n$  is an invariant torus of  $H_n$  with frequency vector  $\omega_n$ . First, we formally have the identities

$$\begin{aligned} \Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} &= \Phi_n^{t_n} \circ \Lambda_n \circ \dots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_n^{-1} \circ \Psi_n^{-s_n} \\ &= \Lambda_n \circ \Phi_{n+1}^{\theta_n^{t_n}} \circ \dots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \Psi_{n+1}^{-\alpha_{n+1}^{-1} s_n} \circ \mathcal{T}_n^{-1} \\ &= \Lambda_n \circ \dots \circ \Lambda_{m-1} \circ \Phi_m^{t_m} \circ \Psi_m^{-s_m} \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_n^{-1} \end{aligned} \tag{105}$$

on any domain where the compositions are well defined, where  $t_m = \theta_{m-1}^{t_{m-1}} \dots \theta_0^{t_0}$  and  $s_m = \alpha_m^{-1} \dots \alpha_1^{-1} s_0$ ,  $m \in \mathbb{N}$ . As the map  $\Lambda_n$  maps  $\mathcal{D}_n(\rho')$  into  $\mathcal{D}_n(\rho'')$ , with  $\rho'' < \rho'$ , componentwise, for  $t_0$  in an open interval containing zero and  $H_0$  sufficiently close to  $H_0^0$ , the equality  $\Phi_n^{t_n} \circ \Lambda_n = \Lambda_n \circ \Phi_{n+1}^{\theta_n^{t_n}}$ ,  $n \in \mathbb{N}_0$ , is an identity between maps on  $\mathcal{D}_n(\rho')$ , with a range contained in  $\mathcal{D}_n(\rho')$ .

As both sides of (105) are the maps on  $\mathcal{B}_{n,0}(\delta_1)$ , we have the identity

$$\Phi_n^{t_n} \circ \Gamma_{n,m} \circ \Psi_n^{-s_n} = (\mathcal{M}_n \circ \dots \circ \mathcal{M}_{m-1})(\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}).$$

Here, we have also used the fact that  $I$  is the invariant torus of frequency  $\omega_m$  of  $H_m^0$ . It suffices to show that the map  $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$  belongs to the domain of  $\mathcal{M}_{m-1}$ , for sufficiently large  $m$ .

Let  $t_0 = \xi s_0$ , where  $\xi = \prod_{j=1}^{\infty} \tau_j = \lim_{k \rightarrow \infty} \prod_{j=1}^k \tau_j$  and  $\tau_j = \tau_{\tilde{H}_j}$ . The existence of this limit follows from the convergence of  $\sum_{j=0}^{\infty} |\tau_{j+1} - 1|$  and the mean value theorem. The convergence of the sum itself follows from the fact that  $|\tau_{j+1} - 1| \leq C_7 \|(\mathbb{I} - \mathbb{E})h_j\|_{j,\rho'}$ , where  $C_7 > 0$ , and theorem 7.2. We have,

$$\frac{\tilde{A}_m}{\alpha_0}(t_m - s_m) = t_0 \prod_{i=1}^m \tau_i^{-1} - s_0 = \left( \prod_{i=m+1}^{\infty} \tau_i - 1 \right) s_0 \leq s_0 C_8 \|(\mathbb{I} - \mathbb{E})H_m\|_{m,\rho'},$$

with  $C_8 > 0$ .

Thus, for sufficiently small  $b, C > 0$ , the assumptions of proposition 9.4 are satisfied. The same proposition then implies that  $\|\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m} - I\|_{m,\delta_1} < b$  and that the map  $\Phi_m^{t_m} \circ \Phi_{m,0}^{-s_m}$  belongs to the domain of  $\mathcal{M}_{m-1}$ . The right-hand side of (105) converges in  $\mathcal{B}_{n,0}(\delta_1)$  to  $\Gamma_n$  when  $m \rightarrow \infty$ . As the convergence implies pointwise convergence and since the maps  $\Phi_n^{t_n}$  and  $\Psi_n^{s_n}$  are both continuous and invertible, we conclude that  $\Phi_n^{t_n} \circ \Gamma_n \circ \Psi_n^{-s_n} = \Gamma_n$ . Thus,  $\Gamma_n$  is an invariant torus of frequency  $\omega_n$  associated to the Hamiltonian  $H_n$ .  $\square$

## 10. Analyticity of invariant tori

Theorem 9.5 shows that every Hamiltonian  $H_0$  sufficiently close to  $H_0^0$  has an invariant torus  $\Gamma_0$  with a Diophantine frequency vector  $\omega_0$ . In this section, we show that the so-constructed invariant tori can be extended to analytic functions. We use the fact that  $\Gamma_0$  depends analytically on  $H_0$ .

Define the translation  $J_v : (q, p) \mapsto (q + v, p)$ , for  $v \in \mathbb{R}^2$ , a one parameter family of Hamiltonians  $H_n(v) = H_n \circ J_v$  and the map  $\mathcal{J}_v : H_n \mapsto H_n \circ J_v^{-1}$ . Let  $\Lambda_n(v)$  be the one parameter family of maps introduced in definition 4.1 associated to  $H_n(v)$ . Let  $0 < \rho' < \varrho' < \varrho < \rho < 3\rho'/2$ , componentwise, be chosen as in definition 4.1.

**Proposition 10.1.** *Let  $v \in \mathbb{R}^2$ . For sufficiently small  $c > 0$  and for all Hamiltonians  $H_n$  in the ball  $B_{n,\varrho}(c) \subset \mathcal{A}_n(\varrho)$ , of radius  $c$ , centred at  $H_n^0$ , the identity  $U_H \circ J_v = J_v \circ U_{H \circ J_v}$ , is valid on  $\mathcal{D}_n(\varrho')$ ,  $n \in \mathbb{N}$ .*

**Proof.** The proof of this proposition follows from the construction of the transformation  $U_{H_n}$  that satisfies the equation  $\mathbb{I}^- H_n \circ U_{H_n} = 0$ . In section 5, this transformation is constructed as the composition of a sequence of canonical transformations  $U_\phi$  generated by functions  $\phi$ . As the translation  $J_v$  commutes with  $\mathbb{I}^\pm$  and differentiation, we have  $\phi_{H_n \circ J_v} = \phi_{H_n} \circ J_v$ , where  $\phi_{H_n}$  is the generating function associated to  $H_n$ . Then,  $U_{\phi_{H_n}} \circ J_v = J_v \circ U_{\phi_{H_n \circ J_v}}$ . The claim follows by taking the limit of the product of the above mentioned sequence.  $\square$

**Proposition 10.2.** *Let  $v \in \mathbb{R}^2$ . For sufficiently small  $C > 0$  and for all Hamiltonians  $H_0 \in B_{0,\rho'}^+(C^2) \subset \mathbb{I}_0^+ \mathcal{A}_n(\rho')$ , if the sequence  $H_n$ ,  $n \in \mathbb{N}_0$ , is the renormalization orbit of  $H_0$ , then the identity  $\Lambda_n \circ J_v = J_{T_n v} \circ \Lambda_n(T_n v)$  is valid on  $\mathcal{D}_n(\rho')$ ,  $n \in \mathbb{N}$ .*

**Proof.** Notice first that  $\mathcal{T}_n \circ J_v = J_{T_n v} \circ \mathcal{T}_n$  and that  $V_{H_{n+1}'}$  and  $S_{\tilde{H}_{n+1}}$  commute with  $J_v$ . Further, notice that  $V_{H_{n+1}' \circ J_v} = V_{H_{n+1}'}$  and  $S_{\tilde{H}_{n+1} \circ J_v} = S_{\tilde{H}_{n+1}}$ . Recall that  $\Lambda_n = \mathcal{T}_n \circ V_{H_{n+1}'}$  and  $\tilde{H}_{n+1} = H_{n+1}'' \circ U_{H_{n+1}''}$ . Let  $H_{n+1}'(v) = (\theta_n/\mu_n)H_n(v) \circ \mathcal{T}_n$ ,  $H_{n+1}''(v) = H_{n+1}'(v) \circ V_{H_{n+1}'(v)}$  and  $\tilde{H}_{n+1}(v) = H_{n+1}''(v) \circ U_{H_{n+1}''(v)}$ . Using the previous relations and the proposition 10.1, we find that  $\Lambda_n \circ J_v = J_{T_n v} \circ \mathcal{T}_n \circ V_{H_{n+1}'(T_n v)} \circ U_{H_{n+1}''(T_n v)} \circ S_{\tilde{H}_{n+1}(T_n v)}$ , and the claim immediately follows.  $\square$

**Proposition 10.3.** *Let  $v \in \mathbb{R}^2$ . For sufficiently small  $C > 0$  and for all Hamiltonians  $H_0 \in B_{0,\rho'}^+(C^2) \subset \mathbb{I}_0^+ \mathcal{A}_n(\rho')$ , we have  $J_v \circ \Gamma_0(H_0 \circ J_v) = \Gamma_0(H_0) \circ J_v$ . Here, by writing  $\Gamma_0(H_0)$ , we have emphasized that the invariant torus  $\Gamma_0$  is associated to a Hamiltonian  $H_0$ .*

**Proof.** Let  $v \in \mathbb{R}^2$  be fixed. Since  $\mathcal{J}_v^{-1}$  is an isometry on  $\mathcal{A}_n(\rho)$ , for sufficiently small  $C > 0$  and for all Hamiltonians  $H_0 \in B_{0,\rho'}^+(C^2)$ ,  $\mathcal{J}_v^{-1}(H_0)$  belongs to  $B_{0,\rho}^+(C^2)$  and the orbit  $\mathcal{R}_{n-1} \circ \dots \circ \mathcal{R}_0(\mathcal{J}_v^{-1}(H_0))$ ,  $n \in \mathbb{N}$ , satisfies the bounds analogous to those obtained in theorem 7.2. Thus, we can construct the maps

$$\Gamma_{0,m}(H_0 \circ J_v) = \Lambda_0(v) \circ \dots \circ \Lambda_{m-1}(T_{m-2}^{-1} \dots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_0^{-1},$$

for  $m \in \mathbb{N}$ . By theorem 9.5, the limit  $\Gamma_0(H_0 \circ J_v) = \lim_{m \rightarrow \infty} \Gamma_{0,m}(H_0 \circ J_v)$  is an invariant torus of  $H_0 \circ J_v$  with frequency vector  $\omega_0$ . Using proposition 10.2, we find

$$\begin{aligned} \Gamma_{0,m} \circ J_v &= \Lambda_0 \circ \dots \circ \Lambda_{m-1} \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_0^{-1} \circ J_v \\ &= J_v \circ \Lambda_0(v) \circ \dots \circ \Lambda_{m-1}(T_{m-2}^{-1} \dots T_0^{-1} v) \circ \mathcal{T}_{m-1}^{-1} \circ \dots \circ \mathcal{T}_0^{-1} \\ &= J_v \circ \Gamma_{0,m}(H_0 \circ J_v). \end{aligned}$$

Further, by taking the limit  $m \rightarrow \infty$  of this identity, we find that  $J_v \circ \Gamma_0(H_0 \circ J_v) = \Gamma_0(H_0) \circ J_v$ . This holds for all Hamiltonians  $H_0$  in a ball  $B_{0,\rho'}^+(C^2)$  independent of  $v \in \mathbb{R}^2$ .  $\square$

The relationship between  $\Gamma_0(H_0 \circ J_v)$  and  $\Gamma_0(H_0)$ , valid *a priori* for  $v \in \mathbb{R}^2$ , can be used to analytically continue  $\Gamma_0(H_0)$  away from  $\mathbb{R}^2 \times \{0\}$ .

In the following, given any two sets  $X$  and  $Y$ , if  $x \in X$ , then  $\mathcal{E}_x$  denotes the evaluation functional defined by  $\mathcal{E}_x f = f(x)$ , for all functions  $f \in Y^X$ .

**Theorem 10.4.** *Let  $\rho_1 - \rho'_1 > \delta_1 > \delta'_1 > 0$  and  $\rho_2 - \rho'_2 > 0$ . There exists an open neighbourhood  $B$  of  $H_0^0$  in  $\mathcal{A}_0(\rho)$ , such that for every Hamiltonian  $H_0 \in B$ , the map*

$$\mathcal{G}_{H_0}(q, p) = \mathcal{E}_0 J_q \circ \Gamma_0(H_0 \circ J_q), \quad q \in \mathcal{D}_{0,1}(\delta'_1), \quad p \in \mathbb{C}^2 \quad (106)$$

*defines an analytic function  $\mathcal{G}_{H_0}$  on  $\mathcal{D}_{0,1}(\delta'_1) \times \mathbb{C}^2$  whose restriction to  $\mathbb{R}^2 \times \{0\}$  coincides with  $\Gamma_0(H_0)$ .*

**Proof.** Let  $\delta = (\delta_1, 0)$ . The map  $(q, H) \mapsto H \circ J_q$  is analytic from  $\mathcal{D}_{0,1}(\delta_1) \times \mathcal{A}_0(\rho)$  into  $\mathcal{A}_0(\rho')$ . This follows from the fact that the map  $\mathcal{J}_q^{-1}$  is bounded with norm 1 from  $\mathcal{A}_0(\rho' + \delta)$  to  $\mathcal{A}_0(\rho')$ , for every  $q \in \mathcal{D}_{0,1}(\delta_1)$ , and the differentiation is bounded from  $\mathcal{A}_0(\rho)$  to  $\mathcal{A}_0(\rho' + \delta)$ .

Define the map  $G : H_0 \mapsto G(H_0)$ , by setting  $G(H_0)(q, p) = \Gamma_0(H_0)(q, 0) - S_0(q, p)$ , for  $H_0$  in the domain of  $\Gamma_0$ . Here  $S_0(q, p) = (q, 0)$ . As  $\mathcal{J}_q^{-1}$  is bounded from  $\mathcal{A}_0(\rho)$  to  $\mathcal{A}_0(\rho')$ , there exists an open ball  $B$  in  $\mathcal{A}_0(\rho)$  containing  $H_0^0$ , such that for every  $q \in \mathcal{D}_{0,1}(\delta_1)$ , both  $H_0$  and  $\mathcal{J}_q^{-1}H_0$  belong to the domain of  $\Gamma_0$ . Since  $\Gamma_0$  depends analytically on  $H_0$ , so does  $G$ , and as,  $\mathcal{E}_0$  is bounded, the map  $(q, H_0) \mapsto \mathcal{E}_0 G(\mathcal{J}_q^{-1}H_0)$  is analytic on  $\mathcal{D}_{0,1}(\delta_1) \times B$ . The analyticity of  $\mathcal{G}_{H_0}$  for  $H_0 \in B$  follows from the identity  $(\mathcal{G}_{H_0} - S_0)(q, p) = \mathcal{E}_0 G(\mathcal{J}_q^{-1}H_0)$ .

Using proposition 10.3, we find

$$\mathcal{E}_{(q,0)}\Gamma_0(H_0) = \mathcal{E}_0(\Gamma_0(H_0) \circ J_q) = \mathcal{E}_0 J_q \circ \Gamma_0(H_0 \circ J_q) = \mathcal{G}_{H_0}(q, 0),$$

for all  $q \in \mathbb{R}^2$  and  $H_0 \in B$ . This shows that  $\Gamma_0(H_0)$  and  $\mathcal{G}_{H_0}$  agree on  $\mathbb{R}^2 \times \{0\}$  as claimed.  $\square$

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## References

- [1] Arnold V I 1963 Proof of A N Kolmogorov's theorem on the preservation of quasiperiodic motions under small perturbations of the Hamiltonian *Usp. Mat. Nauk. SSSR* **18** 13–40
- [2] Bricmont J, Gawedzki K and Kupiainen A 1999 KAM theorem and quantum field theory *Commun. Math. Phys.* **201** 699–727
- [3] Chandre C and Moussa P 2001 Scaling law for the critical function of an approximate renormalization *Nonlinearity* **14** 803–16
- [4] Chandre C and Jauslin H R 2002 Renormalization-group analysis for the transition to chaos in Hamiltonian systems *Phys. Rep.* **365** 1–64
- [5] Couillet P and Tresser C 1978 Iteration d'endomorphismes et groupe de renormalisation *J. Physique Coll.* **39** C5–25
- [6] Escande DF and Doveil F 1981 Renormalization method for computing the threshold of the large-scale stochastic instability in two degree-of-freedom Hamiltonian systems *J. Stat. Phys.* **26** 257–84
- [7] Feigenbaum M J 1978 Quantitative universality for a class of nonlinear transformations *J. Stat. Phys.* **19** 25–52
- [8] Feigenbaum M J 1979 The universal metric properties of nonlinear transformations *J. Stat. Phys.* **21** 669



- [9] Feigenbaum M J, Kadanoff L P and Shenker S J 1982 Quasiperiodicity in dissipative systems: a renormalization analysis *Physica D* **5** 370–86
- [10] Gaidashev D and Koch H 2004 Renormalization and shearless invariant tori: numerical results *Nonlinearity* **17** 1713–22
- [11] Gaidashev D 2005 Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori *Discrete Continuous Dyn. Syst.* **13** 63–102
- [12] Gallavotti G, Gentile G and Mastropietro V 1995 Field theory and KAM tori *Math. Phys. Electron. J.* **1** paper 5
- [13] Gallavotti G and Gentile G 2002 Hyperbolic low-dimensional invariant tori and summations of divergent series *Commun. Math. Phys.* **227** 421–60
- [14] Govin M, Chandre C and Jauslin H R 1997 Kolmogorov–Arnold–Moser-renormalization group analysis of stability of Hamiltonian flows *Phys. Rev. Lett.* **79** 3881–4
- [15] Hardy G H and Wright E M 1990 *An Introduction to the Theory of Numbers* (Oxford: Oxford Science Publications)
- [16] Haydn N T A 1990 On invariant curves under renormalization *Nonlinearity* **3** 887–912
- [17] Kadanoff L P 1966 Scaling laws for Ising models near  $T_c$  *Physics* **2** 263
- [18] Kadanoff L P 1981 Scaling for a critical Kolmogorov–Arnold–Moser trajectory *Phys. Rev. Lett.* **47** 1641–3
- [19] Khanin K M and Sinai Ya G 1986 The renormalization group method and Kolmogorov–Arnold–Moser theory *Nonlinear Phenomena in Plasma Physics and Hydrodynamics* ed R Z Sagdeev (Moscow: Mir) pp 93–118
- [20] Koch H 1999 A renormalization group for Hamiltonians, with applications to KAM tori *Ergod. Theory Dynam. Syst.* **19** 475–521
- [21] Koch H 2002 On the renormalization of Hamiltonian flows, and critical invariant tori *Discrete Continuous Dynam. Syst.* **8** 633–46
- [22] Koch H 2002 A renormalization group fixed point associated with the breakup of golden invariant tori *Preprint mp-arc 02-175*
- [23] Koch H 2004 Existence of critical invariant tori *Preprint mp-arc 04-210*
- [24] Kolmogorov A N 1954 On conservation of quasiperiodic motions under small perturbations of the Hamiltonian *Dokl. Akad. Nauk SSSR* **98** 527–30
- [25] Lanford O E III 1988 Renormalization group methods for circle mappings *Nonlinear Evolution and Chaotic Phenomena* (New York: Plenum) pp 25–36
- [26] de la Llave R 2001 A tutorial in KAM theory *Smooth Ergodic Theory and its Applications (Proc. Symp. in Pure Mathematics, vol 69, Seattle 1999)* ed A Katok *et al* (Providence, RI: AMS) pp 175–292
- [27] Lopes-Dias J 2002 Renormalization of flows on the multidimensional torus close to a KT frequency vector *Nonlinearity* **15** 647–64
- [28] Lopes-Dias J 2002 Renormalization scheme for vector fields on  $\mathbb{T}^2$  with a diophantine frequency *Nonlinearity* **15** 665–79
- [29] Lopes-Dias J 2004 Brjuno condition and renormalization for Poincaré flows *Preprint*
- [30] MacKay R S 1982 Renormalization in area-preserving maps *PhD Thesis* Princeton University
- [31] MacKay R S 1983 A renormalization approach to invariant circles in area-preserving maps *Physica D* **7** 283–300
- [32] MacKay R S 1988 Exact results for an approximate renormalization scheme and some predictions for the break-up of invariant tori *Physica D* **33** 240–65
- [33] MacKay R S, Meiss J D and Stark J 1994 An approximate renormalization for the break-up of invariant tori with three frequencies *Phys. Lett. A* **190** 417–24
- [34] MacKay R S 1995 Three topics in Hamiltonian dynamics *Dynamical Systems and Chaos* vol 2, ed Y Aizawa *et al* (Singapore: World Scientific) pp 34–43
- [35] Moser J 1962 On invariant curves of area-preserving mappings of an annulus *Nachr. Akad. Wiss. Gött. II. Math. Phys. Kl. IIa* 1–20
- [36] Ostlund S, Rand D, Sethna J and Siggia E 1983 Universal properties of the transition from quasi-periodicity to chaos in dissipative systems *Physica D* **8** 303–42
- [37] Shenker S J and Kadanoff L P 1982 Critical behavior of a KAM surface: empirical results *J. Stat. Phys.* **27** 631–56
- [38] Stirmemann A 1993 Renormalization for golden circles *Commun. Math. Phys.* **152** 369–431
- [39] Stueckelberg E C G and Petermann A 1953 La normalisation des constantes dans la théorie des quanta *Helv. Phys. Acta* **26** 499–520