Generic rigidity for circle diffeomorphisms with breaks

Saša Kocić *
Department of Mathematics, The University of Mississippi
Hume Hall 305, University, MS 38677-1848

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Abstract

We prove that C^r-smooth ($r > 2$) circle diffeomorphisms with a break, i.e., circle diffeomorphisms with a single singular point where the derivative has a jump discontinuity, are generically, i.e., for almost all irrational rotation numbers, not $C^{1+\varepsilon}$-rigid, for any $\varepsilon > 0$. This result complements our recent proof, joint with K. Khanin [12], that such maps are generically C^1-rigid. It stands in remarkable contrast to the result of J.-C. Yoccoz [22] that C^r-smooth circle diffeomorphisms are generically $C^{r-1-\varkappa}$-rigid, for any $\varkappa > 0$.

1 Introduction

Rigidity theory of circle diffeomorphisms is a classic topic in dynamical systems. It concerns smooth conjugacy of circle diffeomorphisms within the same topological conjugacy class. The first rigidity result is probably a local result due to Arnol'd [1] who proved, using the methods of Kolmogorov-Arnol'd-Moser theory, that any analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rigid rotation $R_\rho : x \mapsto x + \rho \mod 1$, is analytically conjugate to R_ρ. Arnol'd conjectured that the closeness to the rigid rotation is not necessary for this claim to hold true. That conjecture, which is a global version of his result, was proved by Herman [8] almost two decades later. Herman proved that any C^∞- smooth (C^ω) circle diffeomorphism with a Diophantine rotation number ρ is C^∞- smoothly (C^ω) conjugated to the rotation R_ρ. The theory was further developed by Yoccoz [22] and implies that any two sufficiently smooth circle diffeomorphisms with the same Diophantine rotation number are smoothly conjugate to each other. The smoothness of the conjugacy depends on the Diophantine

*Email: skocic@olemiss.edu

properties of the rotation number. In fact, a result of Yoccoz [22], improved by Katznelson and Ornstein [9], shows that for C^r-smooth ($r \geq 3$) circle diffeomorphisms with a Diophantine rotation number of class $D(\delta)$, the conjugacy is $C^{r-1-\delta-\epsilon}$-smooth, for any $\epsilon > 0$. Here, a map is said to be C^r-smooth if it has a Hölder continuous $[r]$-th derivative ($[r] := \max\{k \in \mathbb{Z} : k \leq r\}$) with Hölder exponent $\{r\} := r - [r]$. A number ρ is said to be Diophantine of class $D(\delta)$, for some $\delta \geq 0$, if there is a constant $\mathcal{C} > 0$ such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$. Since for $\delta > 0$, Diophantine numbers of class $D(\delta)$ form a set of full Lebesgue measure, the result of Yoccoz implies that, for any $\varkappa > 0$, and almost all irrational $\rho \in (0, 1)$, C^r-smooth circle diffeomorphisms with rotation number ρ are $C^{r-1-\varkappa}$-rigid. On the other hand, smooth conjugacy is not guaranteed for all irrational rotation numbers. Arnol'd even constructed examples of analytic circle diffeomorphisms with some Liouville (non-Diophantine) irrational rotation numbers for which the conjugacy to the rotation is essentially singular. These results are at the core of rigidity theory of circle diffeomorphisms.

A natural question to ask is what are the rigidity properties of circle maps that fail to be diffeomorphisms only at a single point. Over the last two decades great effort has been put into understanding rigidity of smooth circle diffeomorphisms with a single singular point where the derivative vanishes (critical circle maps) or has a jump discontinuity (circle diffeomorphisms/maps with a break). The main technical tool in the proofs of the rigidity results has been renormalization. Renormalizations of a circle homeomorphism T form a sequence of appropriately rescaled increasing powers T^{q_n} of the map T, restricted to a small interval around a (singular) point. Rigidity theory of analytic critical circle maps is rather complete. The first result on convergence of renormalizations for critical circle maps is due to de Faria and de Melo [7]. They proved that, for a set of zero Lebesgue measure irrational numbers ρ, the renormalizations of any two analytic critical circle maps with the same irrational rotation number and the same order of the critical point approach each other exponentially fast. Here, the critical point x_c is said to be of order $\beta > 1$ if the derivative of the map at a point x near x_c is of the order $|x - x_c|^{\beta-1}$. This result was later extended to all irrational rotation numbers by Yampolsky [21]. De Faria and de Melo [6] also proved that, for almost all irrational rotation numbers, the conjugacy is $C^{1+\varepsilon}$-smooth, for some $\varepsilon > 0$. Finally, for any $\varepsilon > 0$, they constructed examples of C^∞-smooth critical circle maps with the same irrational rotation number and the same order of the critical point that are not $C^{1+\varepsilon}$-smoothly conjugate to each other. Khmelev and Yampolsky proved that for analytic critical circle maps with the same irrational rotation number and the same order of the critical point the conjugacy is always $C^{1+\varepsilon}$-smooth at the critical point, for some $\varepsilon > 0$, suggesting that the analytic case may be different. Avila [2], however, showed that, for any $\varepsilon > 0$, not even analytic critical circle maps are $C^{1+\varepsilon}$-rigid, for all irrational rotation numbers. Nevertheless, analytic critical circle maps are C^1-rigid, for all irrational rotation numbers, as was shown by Khanin and Teplinsky [14]. The rigidity results of [6] and [14] would also hold true in the case of C^r-smooth ($r \geq 3$)

critical circle maps, provided one had the exponential convergence of renormalizations in that case. The proof of the exponential convergence of renormalizations for non-analytic critical circle maps is at present, however, still an open problem.

For C^r-smooth ($r > 2$) circle maps with a break, the first result on the exponential convergence of renormalizations, for a countable set of rotation numbers, is due to Khanin and Khmelev [10]. The result was extended to a larger set of zero measure in [15]. The full proof of the exponential convergence of renormalizations, for all irrational rotation numbers, was obtained only recently [12]. In a joint work with Khanin [12], we proved that renormalizations f_n and $\widetilde{f}_n$ of any two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle maps with breaks T_ρ and $\widetilde{T}_\rho$, with the same irrational rotation number ρ and the same size of the break $c \in \mathbb{R}_+ \setminus \{1\}$ (i.e., the square root of the ratio of the left and right derivatives at the break point) approach each other (at least) exponentially fast (in the C^2-topology), i.e., there exist $\lambda \in (0,1)$ and $C > 0$ such that $\|f_n - \widetilde{f}_n\|_{C^2} \leq C\lambda^n$. Together with our earlier result [13], this implies that C^r-smooth ($r > 2$) circle maps with a break are generically C^1-rigid: for almost all irrational ρ, every two C^r-smooth circle maps with a break, with the same rotation number ρ and the same size of the break $c \in \mathbb{R}_+ \setminus \{1\}$, are C^1-smoothly conjugate to each other. We also proved that this result cannot be extended to all irrational rotation numbers. In a joint work with Khanin [11], we constructed examples of analytic circle maps with breaks of the same size and the same irrational rotation number, for which no conjugacy is Lipschitz continuous. These results are analogous to those in the case circle diffeomorphisms, although the set of rotation numbers for which C^1-rigidity holds is not Diophantine.

The main result of this paper stands in striking contrast to the case of circle diffeomorphisms and analytic critical circle maps. We prove that circle maps with breaks are generically not $C^{1+\varepsilon}$-rigid, for any $\varepsilon > 0$. The main result of this paper can be summarized in the following theorem, which can be considered an extension of Herman's theory on the linearization of circle diffeomorphisms. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Theorem 1.1 *For any $c \in \mathbb{R}_+ \setminus \{1\}$ and $r > 2$ there exists a set $S \subset (0,1) \setminus \mathbb{Q}$ of Lebesgue measure 1 such that, for any $\rho \in S$: (i) for any two C^r-smooth circle diffeomorphisms with a break T_ρ and $\widetilde{T}_\rho$, with the same rotation number ρ and the same size of the break c, there is a C^1-smooth diffeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ T_\rho \circ h^{-1} = \widetilde{T}_\rho$, and (ii) for any $\varepsilon > 0$, there is a pair of C^r-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same rotation number ρ and the same size of the break c, such that the conjugacy h is not $C^{1+\varepsilon}$-smooth.*

Remark 1 The only conjugacy between circle maps with breaks T and $\widetilde{T}$, which can be C^1-smooth, is the conjugacy that maps the break point of T into the break point of $\widetilde{T}$. The conjugacy h mentioned in Theorem 1.1 is this particular conjugacy.

Remark 2 The set S is the intersection of two sets, S_{rig} for which C^1-rigidity holds and S_{non} for which $C^{1+\varepsilon}$-rigidity does not hold, for any $\varepsilon > 0$. The results of [12, 13] imply that, for any $c \in \mathbb{R}_+ \backslash \{1\}$ and $r > 2$, there is a set $S_{rig} \subset (0,1)\backslash\mathbb{Q}$ of Lebesgue measure 1 such that, for any $\rho \in S_{rig}$ and any two C^r-smooth circle diffeomorphisms with a break, with the same rotation number ρ and the same size of the break c, the conjugacy h is a C^1-smooth diffeomorphism. The set S_{rig} of rotation numbers $\rho \in (0,1)\backslash\mathbb{Q}$ with the continued fraction expansion $\rho = [k_1, k_2, \dots]$ can be characterized as follows. For a given $\lambda_1 \in (\lambda, 1)$, we define the set $S_{rig} = S_{rig}(\lambda_1)$ to be the set of all such ρ for which there exists a constant $\mathcal{C}_1 > 0$ such that $k_n \le \mathcal{C}_1 \lambda_1^{-n}$, for all $n \in 2\mathbb{N} - 1$, if $c < 1$, or for all $n \in 2\mathbb{N}$, if $c > 1$. Notice that even the set $\cap_{\lambda_1 \in (0,1)} S_{rig}(\lambda_1)$ has Lebesgue measure 1. The difference between n odd and n even comes from the difference in the behavior of the corresponding subsequences of renormalizations. A set $S_{non} \subset (0,1)\backslash\mathbb{Q}$ for which, for any $\rho \in S_{non}$ and any $\varepsilon > 0$, one can find two C^r-smooth circle diffeomorphisms with a break, with the same rotation number ρ and the same size of the break, which are not $C^{1+\varepsilon}$-smoothly conjugate to each other, can be chosen as follows. We define S_{non} to be the set of all such ρ with the property that, for every $A \in \mathbb{N}$, there exists an (infinite) subsequence $\sigma_n \in 2\mathbb{N} - 1$, if $c < 1$ or $\sigma_n \in 2\mathbb{N}$, if $c > 1$, such that $k_{\sigma_n} \ge A\sigma_n$. We show in Section 4 that this set has a full Lebesgue measure.

Remark 3 We actually prove a stronger statement than (ii). We prove that for any $\rho \in S_{non}$ and for any $\varepsilon > 0$, there is a pair of C^r-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same rotation number ρ and the same size of the break $c \in \mathbb{R}_+ \backslash \{1\}$, such that the conjugacy h is not $C^{1+\varepsilon}$-smooth at the break point.

We say that the conjugacy $h : \mathbb{S}^1 \to \mathbb{S}^1$ is $C^{1+\varepsilon}$-smooth at x_0 if, for any $x \in \mathbb{S}^1$,

$$h'(x) - h'(x_0) = O(|x - x_0|^{\varepsilon}). \tag{1.1}$$

Remark 4 Examples of circle diffeomorphisms with breaks which are C^1- but not $C^{1+\varepsilon}$-smoothly conjugate to each other have previously been constructed in [5, 11, 20], using the parabolic renormalization approach of Avila for critical circle maps [2]. Those examples were constructed using the features of circle maps with a break which are similar to those of critical circle maps (a parabolic “almost tangency” of a subsequence of renormalizations). Those examples are *exceptional* and that approach cannot be used to prove Theorem 1.1. The technical tools that we develop and apply in this paper exploit the features of circle maps with breaks not shared by the critical circle maps. In particular, it is the *strongly unbounded geometry* of circle maps with breaks which makes these maps generically non-$C^{1+\varepsilon}$-rigid, for any $\varepsilon > 0$.

We say that the geometry of a circle map T is *bounded* if all neighboring intervals of all dynamical partitions of the circle $\mathcal{P}_n$ (defined in the next section) are of comparable length, i.e., if the ratios of lengths of these intervals are uniformly bounded (by positive constants). If this is not the case, we say that the geometry is unbounded. The geometry of dynamical partitions plays a major role in the rigidity properties of circle maps. Critical circle maps have bounded geometry. Both circle diffeomorphisms and circle maps with breaks are characterized by unbounded geometry. It is this unbounded geometry which is the reason for the absence of robust rigidity, i.e., C^1-rigidity of these maps for all irrational rotation numbers. However, while, in the case of circle diffeomorphisms, the maximal ratio of lengths of neighboring intervals of the dynamical partition $\mathcal{P}_n$ can increase at most linearly with the corresponding partial quotient k_{n+1} of the rotation number ρ, it can grow exponentially in the case of circle maps with a break. It is this difference that ultimately causes the difference in the $C^{1+\varepsilon}$-smoothness of conjugations: smooth circle diffeomorphisms are generically $C^{1+\varepsilon}$-rigid while, as we prove in this paper, circle maps with a break are generically not such, for any $\varepsilon > 0$.

Remark 5 The proof of part (ii) of Theorem 1.1 is based on the construction of two C^r-smooth circle maps with a break T and $\widetilde{T}$, with the same irrational rotation number ρ and the same size of the break, for which there is a subsequence $(\pi_n)_{n\in\mathbb{N}}$ such that the corresponding renormalizations f_{π_n} and $\widetilde{f}_{\pi_n}$ have left derivatives at the break point which are at least exponentially different in n. This is the statement of Lemma 5.6 below, which implies that there exist $\mathscr{C} > 0$ and $\mu \in (0,1)$ such that $\|f_{\pi_n} - \widetilde{f}_{\pi_n}\|_{C^2} \geq \mathscr{C}\mu^{\pi_n}$, for a sequence $\{\pi_n\}_{n\in\mathbb{N}}$, and, thus, in general, one cannot obtain faster than exponential convergence of renormalizatons for such maps. In that sense, the result of [12] on the exponential convergence of renormalizations for circle maps with breaks is optimal.

Remark 6 The above construction, used in the proof of part (ii) of Theorem 1.1, is based on the following general result on the geometry of dynamical partitions for circle maps with breaks. We prove that the length of the longest element of the dynamical partition $\mathcal{P}_n$ of a circle map with a break decreases at most exponentially fast with n. This complements the result of Sinai and Khanin [19] that the length of the longest element of the dynamical partition $\mathcal{P}_n$ decreases at least exponentially with n.

Remark 7 An almost the same construction gives examples of circle maps with breaks, with the same irrational rotation number ρ and the same size of the break, that are C^1 but not $C^{1+\varepsilon}$-smoothly conjugate to each other, for any $\varepsilon > 0$.

At the end of this introduction let us mention that the results for circle maps with breaks are also relevant for understanding the properties of generalized interval exchange transformations, recently introduced by Marmi, Moussa and Yoccoz [18]. These transformations are obtained by replacing continuous linear branches of an interval exchange transformation by smooth homeomorphisms. Like an interval exchange transformation of two intervals can be seen as a rigid rotation on a circle, a generalized interval exchange transformation of two intervals is a circle map with two break points. As these two points lie on the same orbit of the map, the map can be piecewise smoothly conjugated to a circle map with one point of break. Marmi, Moussa and Yoccoz considered the linearizable case of more intervals [18], when there are no breaks of the derivatives. The special case of cyclic permutations, which corresponds to circle maps with more points of break, but with product of the sizes of breaks equal to 1, was considered by Cunha and Smania [3, 4]. In this case, renormalizations approach piecewise linear maps. In our situation, the renormalizations are essentially non-linear. Theorem 1.1 summarizes the only generic rigidity results in this case.

This paper is organized as follows. In the next section, we define dynamical partitions of the circle and the renormalization operator. In Section 3, we prove some general estimates concerning the geometry of the dynamical partitions, that should be useful in other problems as well. In particular, we prove that the length of the longest element of the dynamical partition $\mathcal{P}_n$ does not decrease faster than exponentially with n. In Section 4, we construct the set S of irrational numbers of full Lebesgue measure for which the claim of Theorem 1.1 holds. In Section 5, we prove the main lemmas and our main result.

2 Preliminaries

For every orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$ of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ there exists a (unique up to an additive integer constant) continuous and strictly increasing function $\mathcal{T} : \mathbb{R} \to \mathbb{R}$, called a lift of T, that satisfies $\mathcal{T}(x+1) = \mathcal{T}(x)+1$, for every $x \in \mathbb{R}$. Poincaré showed that for every orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$ there is a unique rotation number ρ, given by the limit $\rho := \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is any lift of T. Renormalizations of an orientation-preserving homeomorphism of a circle T, with a rotation number $\rho \in (0, 1)$ are defined using the continued fraction expansion

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}, \tag{2.1}$$

that we also write as $\rho = [k_1, k_2, k_3, \dots]$. The sequence of integers $(k_n)_{n\in\mathbb{N}}$, called *partial quotients*, is infinite if and only if ρ is irrational. Every irrational ρ defines uniquely the sequence of partial quotients. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$. It is well-known that this sequence forms the sequence of best rational approximates of an irrational ρ, i.e., there are no rational numbers with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$. The convergents satisfy $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We define $\rho_n := [k_n, k_{n+1}, k_{n+2}, \dots]$. It is easy to show by induction that, for $n \in \mathbb{N}$,

$$\rho = \frac{p_{n-1}\rho_n^{-1} + p_{n-2}}{q_{n-1}\rho_n^{-1} + q_{n-2}}. \tag{2.2}$$

To define the renormalizations of an orientation-preserving homeomorphism of a circle T, with an irrational rotation number ρ, we start with a *marked point* $x_0 \in \mathbb{S}^1$, and consider the marked trajectory $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}}$ indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, will be called the sequence of *dynamical convergents*. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $(x_{q_n})_{n\in\mathbb{N}}$ for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left and the subsequence with n even approaches x_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$ (or $\Delta_0^{(n)}(T)$, if we need to emphasize which map T it is associated to) and called the n-th renormalization segments associated to x_0. The n-th renormalization segment associated to x_i will be denoted by $\Delta_i^{(n)}$ (or $\Delta_i^{(n)}(T)$). We also have the following important property: the only points of the orbit $\{x_i : 0 < i \le q_{n+1}\}$ that belong to $\Delta_0^{(n-1)}$ are $\{x_{q_{n-1}+iq_n} : 0 \le i \le k_{n+1}\}$.

Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterates of a map T with rotation number ρ, cover the whole circle without overlapping beyond the end points and form the n-th *dynamical partition* of the circle

$$\mathcal{P}_n := \{T^i\Delta_0^{(n-1)} : 0 \le i < q_n\} \cup \{T^i\Delta_0^{(n)} : 0 \le i < q_{n-1}\}. \tag{2.3}$$

The set of end points of the intervals of partition $\mathcal{P}_n$ will be denoted by Ξ_n. The intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ will be called the *fundamental intervals* of $\mathcal{P}_n$. When necessary to emphasize that $\mathcal{P}_n$ and Ξ_n are associated to a map T, they will be denoted by $\mathcal{P}_n(T)$ and $\Xi_n(T)$, respectively. We note that these sequences are finite if the rotation number of the map is rational.

The n-th *renormalization* of an orientation-preserving homeomorphism $T : \mathbb{S}^1 \to \mathbb{S}^1$, with rotation number ρ, with respect to the marked point $x_0 \in \mathbb{S}^1$, is a function f_n (or $f_n(T)$) : $[-1, 0] \to \mathbb{R}$ obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.4}$$

Definition (2.4) is valid for all $n \in \mathbb{N}_0$ if and only if ρ is irrational; if ρ is rational, n must be less than or equal to the length of the continued fraction expansion of ρ. If we identify

x_0 with zero, then τ_n is exactly the multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{S}^1$.

This paper concerns circle maps with a break, i.e., homeomorphisms of a circle for which there exists a point $x_{br} \in \mathbb{S}^1$, such that $T \in C^r(\mathbb{S}^1 \backslash \{x_{br}\})$, $T'(x)$ is bounded from below by a positive constant and the one-sided derivatives at x_{br} are such that the *size of the break*

$$c := \sqrt{\frac{T'_-(x_{br})}{T'_+(x_{br})}} \neq 1.$$

In the following, we only consider renormalizations with marked point $x_0 = x_{br}$.

It was proved in [16] that the renormalizations of circle maps with a break approach a particular family of fractional linear transformations. The sequence of renormalizations $(f_n)_{n\in\mathbb{N}_0}$ of a C^r-smooth ($r > 2$) circle map T with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ converges to a sequence of fractional linear transformations $F_n := F_{a_n,b_n,M_n,c_n} : [-1, 0] \to \mathbb{R}$,

$$F_n(z) := \frac{a_n + (a_n + b_n M_n)z}{1 - (M_n - 1)z}, \tag{2.5}$$

with

$$a_n := \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad b_n := \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad M_n = \exp\left(\sum_{i=0}^{q_n-1} \int_{x_{q_{n-1}+i}}^{x_i} \frac{T''(x)}{2T'(x)}\, dx\right). \tag{2.6}$$

In fact, this convergence is uniform within certain families of maps that will be described in the next section. The precise statements will be given below.

3 General estimates

In this section, we prove some general estimates on the geometry of dynamical partitions.

We start with a circle map with a break T_0, with a fixed point $T_0(0) = 0$ and a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at the break point $x_{br} = 0$. We consider a one-parameter family $T_\Omega := T_0 + \Omega$, with $\Omega \in [0, 1)$. For a set $B \subset [0, 1)$, the set of parameters Ω such that the rotation number $\rho(T_\Omega) \in B$ is denoted by $\rho^{-1}(B)$. It is well-known that the rotation number ρ of a map in this family is a continuous and non-decreasing function of the parameter Ω. Moreover, for every irrational ρ, there is a unique parameter value Ω_ρ such that the map T_{Ω_ρ} has the rotation number ρ. For every rational rotation number $p/q \in \mathbb{Q}$, there is a mode-locking interval $[\Omega_{p/q}^{(L)}, \Omega_{p/q}^{(R)}]$ of parameter values corresponding to p/q. If p/q has a sufficiently long continued fraction expansion, then the following holds. When the parameter value Ω is equal to $\Omega^{(L)}_{p/q}$, in the case $c > 1$, or $\Omega^{(R)}_{p/q}$, in the case $0 < c < 1$, the map T_Ω has a single periodic orbit of the type (p, q) and the break point belongs to this periodic orbit, i.e., a lift $\mathcal{T}_\Omega : \mathbb{R} \to \mathbb{R}$ of T_Ω satisfies $\mathcal{T}^q_\Omega(0) = p$. Let us denote that unique value of the parameter Ω by $\Omega_{p/q}$. When the parameter value Ω equals the other end point ($\Omega^{(R)}_{p/q}$, in the case $c > 1$; $\Omega^{(L)}_{p/q}$, in the case $0 < c < 1$), the map T_Ω has a single periodic orbit of the type (p, q), which is neutral. Obviously, the break point x_{br} does not belong to it. For all other values of the parameter inside the mode-locking interval, the map has two periodic orbits of type (p, q), one stable and one unstable [16].

The following monotonicity statement clearly holds for any family of orientation-preserving circle homeomorphisms T_Ω. To be specific, let $\mathcal{T}_\Omega$ be the lift of T_Ω that satisfies $\mathcal{T}_\Omega(0) \in [0, 1)$.

Proposition 3.1 *If $\rho_1 > \rho_2$, then $\mathcal{T}^i_{\Omega_{\rho_1}}(0) > \mathcal{T}^i_{\Omega_{\rho_2}}(0)$, for every $i \in \mathbb{N}$.*

Proof. The claim follows by induction. If $\rho_1 > \rho_2$, then $\Omega_{\rho_1} > \Omega_{\rho_2}$ and, thus, $\mathcal{T}_{\Omega_{\rho_1}}(0) > \mathcal{T}_{\Omega_{\rho_2}}(0)$. Assume that, for some $i \in \mathbb{N}$, we have $\mathcal{T}^i_{\Omega_{\rho_1}}(0) > \mathcal{T}^i_{\Omega_{\rho_2}}(0)$. Then, since $\mathcal{T}_0$ is strictly increasing, we have $\mathcal{T}^{i+1}_{\Omega_{\rho_1}}(0) > \mathcal{T}_{\Omega_{\rho_2}}(\mathcal{T}^i_{\Omega_{\rho_1}}(0)) > \mathcal{T}^{i+1}_{\Omega_{\rho_2}}(0)$. **QED**

Now, let us fix an irrational number $\rho = [k_1, k_2, \ldots] \in (0, 1)$ and consider the sequence $p_n/q_n = [k_1, \ldots, k_n]$, $n \in \mathbb{N}$, of its convergents. The open interval with end points $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$ will be denoted by B_n. Clearly, $\rho = \cap_{n=1}^{\infty} B_n$. To simplify the notation, let $\Omega_n := \Omega_{p_n/q_n}$. We also define $x_i(\Omega_\rho) := T^i_{\Omega_\rho}(0)$, $x_i(\Omega_n) := T^i_{\Omega_n}(0)$. Let also $I_i(\Omega_n) := [x_i(\Omega_n), x_i(\Omega_\rho)]$, if $\rho > p_n/q_n$ and $I_i(\Omega_n) := [x_i(\Omega_\rho), x_i(\Omega_n)]$, if $\rho < p_n/q_n$. Finally, let $\Delta^{(n)}_0(\Omega) := \Delta^{(n)}_0(T_\Omega)$, $\mathcal{P}_n(\Omega) := \mathcal{P}_n(T_\Omega)$ and $\Xi_n(\Omega) := \Xi_n(T_\Omega)$.

Proposition 3.2, Proposition 3.3 and Proposition 3.4 lead to estimates that relate the geometry of the dynamical partitions of T_{Ω_ρ} and T_{Ω_n} (Proposition 3.5 and Corollary 3.6), that will be used later on.

Proposition 3.2 *$[\mathcal{T}^i_{\Omega_\rho}(0)] = [\mathcal{T}^i_{\Omega_n}(0)]$, for $i = 1, \ldots, q_n - 1$.*

Proof. To be specific, let us consider the case $\rho > p_n/q_n$. Assume that for some $q < q_n$, $[\mathcal{T}^q_{\Omega_\rho}(0)] \neq [\mathcal{T}^q_{\Omega_n}(0)]$. Then, it follows from Proposition 3.1 that there exists $\Omega \in [\Omega_n, \Omega_\rho]$, such that $\mathcal{T}^q_\Omega(0) = p$. Therefore, T_Ω has a periodic orbit of period q and, consequently, a rational rotation number $p/q \in (p_n/q_n, \rho)$. This contradicts the assumption that p_n/q_n is a rational convergent for ρ, due to the properties of the continued fractions. **QED**

As a corollary of Proposition 3.1 and Proposition 3.2, we obtain following statement.

Proposition 3.3 *If $\rho > p_n/q_n$, then $T^i_{\Omega_\rho}(0) > T^i_{\Omega_n}(0)$, for $i = 1, \ldots, q_n - 1$. If $\rho < p_n/q_n$, then $T^i_{\Omega_\rho}(0) < T^i_{\Omega_n}(0)$, for $i = 1, \ldots, q_n - 1$.*

We can now prove the following claim.

Proposition 3.4 *$I_i(\Omega_n) \cap I_j(\Omega_n) = \emptyset$, for $i, j = 1, \dots, q_n$ such that $i \neq j$.*

Proof. Since $I_{q_n}(\Omega_n) = \Delta_0^{(n)}(\Omega_\rho) \subset \Delta_{q_n}^{(n-1)}(\Omega_\rho)$, the intervals $T_{\Omega_\rho}^{-i}(I_{q_n}(\Omega_n)) \subset \Delta_{q_n-i}^{(n-1)}(\Omega_\rho)$, for $i = 0, \dots, q_n - 1$, are proper subsets of different intervals of partition $\mathcal{P}_n(T_{\Omega_\rho})$, and they do not overlap, even at the end points. Due to Proposition 3.3, the same is true for the intervals $I_{q_n-i}(\Omega_n) \subset T_{\Omega_\rho}^{-i}(I_{q_n}(\Omega_n))$. **QED**

Since for circle maps with a break T_Ω, the variation of the logarithm of the derivative $V := \mathrm{Var}_{\mathbb{S}^1} \ln T'_\Omega < \infty$, all maps T_Ω with rotation numbers in the closed interval with end points ρ and p_n/q_n, satisfy $|\ln(T_\Omega^{q_n})'(x)| \leq V$, for all $x \in \mathbb{S}^1$. We will refer to this statement as the Denjoy lemma [19]. Therefore, we have a uniform bound on the logarithm of the derivative of the renormalizations $f_n(\Omega) := f_n(T_\Omega)$,

(A) $\quad |\ln (f_n(\Omega))'(x)| \leq V$,

for all $x \in [-1, 0]$.

Proposition 3.5 *$I_i(\Omega_n) \subset \Delta_i^{(n-1)}(\Omega_\rho)$ and there exists $C_1 \geq 1$ such that $|I_i(\Omega_n)| \leq C_1 a_n |\Delta_i^{(n-1)}(\Omega_\rho)|$, for all $i = 1, \dots, q_n$.*

Proof. We first show that the estimate holds for $i = q_n$, then extend it to other intervals. Clearly $I_{q_n}(\Omega_n) = \Delta_0^{(n)}(\Omega_\rho) \subset \Delta_{q_n}^{(n-1)}(\Omega_\rho)$. Since it follows from (A) that $|\Delta_{q_n}^{(n-1)}(\Omega_\rho)| \geq e^{-V}|\Delta_0^{(n-1)}(\Omega_\rho)|$, we have $|I_{q_n}(\Omega_n)| \leq e^V a_n |\Delta_{q_n}^{(n-1)}(\Omega_\rho)|$. It follows from Proposition 3.3 that, for $j = 1, \dots, q_n - 1$, $I_i(\Omega_n) \subset \Delta_i^{(n-1)}(\Omega_n)$, and for some $\zeta_j, \xi_j \in \Delta_j^{(n-1)}(\Omega_\rho)$, we have

$$\frac{|I_j(\Omega_n)|}{|\Delta_j^{(n-1)}(\Omega_\rho)|} \leq \frac{|T_{\Omega_\rho}^{-1}(I_{j+1}(\Omega_n))|}{|T_{\Omega_\rho}^{-1}(\Delta_{j+1}^{(n-1)}(\Omega_\rho))|} = \frac{|I_{j+1}(\Omega_n)|}{|\Delta_{j+1}^{(n-1)}(\Omega_\rho)|} \frac{T'_{\Omega_\rho}(\zeta_j)}{T'_{\Omega_\rho}(\xi_j)}. \tag{3.1}$$

Therefore, we obtain the estimate

$$\frac{|I_j(\Omega_n)|}{|\Delta_j^{(n-1)}(\Omega_\rho)|} \leq \frac{|I_{j+1}(\Omega_n)|}{|\Delta_{j+1}^{(n-1)}(\Omega_\rho)|} \left(1 + \frac{\max_{x\in[0,1]} |T''_{\Omega_\rho}(x)|}{\min_{x\in[0,1]} T'_{\Omega_\rho}(x)} |\Delta_j^{(n-1)}(\Omega_\rho)|\right). \tag{3.2}$$

By iterating the latter inequality, we find

$$\frac{|I_j(\Omega_n)|}{|\Delta_j^{(n-1)}(\Omega_\rho)|} \leq \frac{|I_{q_n}(\Omega_n)|}{|\Delta_{q_n}^{(n-1)}(\Omega_\rho)|} \prod_{i=j}^{q_n-1} \left(1 + \frac{\max_{x\in[0,1]} |T''_{\Omega_\rho}(x)|}{\min_{x\in[0,1]} T'_{\Omega_\rho}(x)} |\Delta_i^{(n-1)}(\Omega_\rho)|\right). \tag{3.3}$$

Since, $\sum_{i=1}^{q_n-1} |\Delta_i^{(n-1)}(\Omega_\rho)| \leq 1$, we find

$$|I_j(\Omega_n)| \leq \exp\left(V + \frac{\max_{x\in[0,1]} |T''_{\Omega_\rho}(x)|}{\min_{x\in[0,1]} T'_{\Omega_\rho}(x)}\right) a_n |\Delta_j^{(n-1)}(\Omega_\rho)|. \tag{3.4}$$

QED

Since $\sum_{j=0}^{q_n-1}|\Delta_j^{(n-1)}(\Omega_n)|\leq 1$, from Proposition 3.5, we immediately obtain

Corollary 3.6 $\sum_{i=1}^{q_n}|I_i(\Omega_n)|\leq C_1 a_n$.

Proof. $\sum_{i=1}^{q_n}|I_i(\Omega_n)|\leq\sum_{i=1}^{q_n-1}C_1 a_n|\Delta_i^{(n-1)}(\Omega_\rho)|+a_n|\Delta_0^{(n-1)}(\Omega_\rho)|\leq C_1 a_n$. **QED**

It was shown in [19] that the length of the longest element of the dynamical partition $\mathcal{P}_n$ of an orientation preserving circle homeomorphism T, satisfying the Denjoy lemma, decreases at least exponentially with n. Here, we prove that, for circle maps with a break, this result is optimal, i.e., that the length of the longest element of the dynamical partition $\mathcal{P}_n$ decreases at most exponentially fast with n (see Corollary 3.15 below). We denote the longest of the intervals $\Delta_i^{(n-1)}$, for $i=0,\ldots,q_n-1$, of the dynamical partition $\mathcal{P}_n$ (defined with the marked point $x_0=x_{br}=0$) by $\Delta_{\max}^{(n-1)}$. It follows from [19] that $|\Delta_{\max}^{(n-1)}|\leq\text{const }\lambda^n$, where $\lambda:=(1+e^{-V})^{-1/2}$. To prove that $|\Delta_{\max}^{(n-1)}|$ decreases at most exponentially fast with n, we will need the following result of Khanin and Vul [16].

Proposition 3.7 *For every $C^{2+\alpha}$-smooth circle map T_0 with a break of size $c\in\mathbb{R}_+\backslash\{1\}$ at the fixed point, there is $C>0$ such that, for all irrational $\rho\in(0,1)$, all $n\in\mathbb{N}$ and all $\Omega\in\rho^{-1}(B_n)$*

(B) $\|f_n(\Omega)-F_n(\Omega)\|_{C^2}\leq C\lambda^{\alpha n}$,

(C) $|a_n+b_nM_n-c_n|\leq Ca_n\lambda^{\alpha n}$,

(D) $|M_{n+1}-c_{n+1}(1+a_{n+1}a_n(M_n-1))|\leq Ca_{n+1}a_n\lambda^{\alpha n}$.

where $c_n:=c^{(-1)^n}$. Here, we denote by $F_n(\Omega)$ the map F_n with parameters a_n, b_n and M_n associated to T_Ω.

The renormalization estimates of the following six claims will be used in the proofs of Proposition 3.15 and Corollary 3.15, which are the main objectives of the remaining part of this section. It is easy to see that the following holds.

Proposition 3.8 $\exp\left(-\frac{\max_{x\in[0,1]}|T''(x)|}{2\min_{x\in[0,1]}T'(x)}\right)\leq M_n\leq\exp\left(\frac{\max_{x\in[0,1]}|T''(x)|}{2\min_{x\in[0,1]}T'(x)}\right)$.

Proof. The claim follows directly from the definition of M_n (see (2.6)). **QED**

Proposition 3.9 *There exists $\delta>0$ such that $a_na_{n+1}<1-\delta$, for sufficiently large $n\in\mathbb{N}$.*

Proof. The estimate (C) implies that $a_n\leq c_n-b_nM_n+Ca_n\lambda^{\alpha n}$. If $a_na_{n+1}\geq 1-\bar{\delta}$, for some $\bar{\delta}>0$ and some n, then $a_n\leq c_n-(1-\delta)M_n+Ca_n\lambda^{\alpha n}$. It follows from Proposition 3.8 that there exists $\delta>0$ such that $a_{n+1}a_n<1-\delta$, uniformly in n, if $\bar{\delta}$ has been chosen small enough and n is sufficiently large. **QED**

Proposition 3.10 *There exists $\delta' > 0$ such that $\frac{M_n-1}{c_n-1} > \delta'$, for sufficiently large $n \in \mathbb{N}$.*

Proof. The estimate (D) leads to the following recursion relation

$$\frac{M_{n+2}-1}{c_{n+2}-1} = 1 - a_{n+2}a_{n+1} + a_{n+2}a_{n+1}a_{n+1}a_n\frac{M_n-1}{c_n-1} + O(\lambda^{\alpha n}). \tag{3.5}$$

Taking into account Proposition 3.9, this recursion relation implies that, for sufficiently large n, the left hand side of (3.5) is bounded from below by a positive constant. **QED**

Estimate (B), Proposition 3.8 and Proposition 3.10 imply the following.

Proposition 3.11 *For all irrational $\rho \in (0,1)$ and sufficiently large n, $(f_n(\Omega))''(z)$ is bounded, bounded away from zero (uniformly in n and $\Omega \in \rho^{-1}(B_n)$) on $[-1,0]$ and negative if $c_n < 1$ and positive if $c_n > 1$.*

Proof. It follows from (B) that, for sufficiently large n, $(f_n(\Omega))''(z)$ is close to $(F_n(\Omega))''(z) = 2(M_n-1)\frac{(a_n+b_n)M_n}{(1+(1-M_n)z)^3}$. Due to Proposition 3.8, the fraction $\frac{(a_n+b_n)M_n}{(1+(1-M_n)z)^3}$ is bounded both from below and above, by positive constants, uniformly in $z \in [-1,0]$. The claim now follows from Proposition 3.10. **QED**

Proposition 3.12 *If $c_n > 1$, there exists $\epsilon > 0$ such that $a_n > \epsilon$, for sufficiently large n.*

Proof. We will prove the claim by contraposition. If a_n is close to zero, then $(f_n(\Omega))'(0)$ is close to $(F_n(\Omega))'(0) = a_n + b_nM_n + (M_n-1)a_n$ which is close to $c_n > 1$. Since, for sufficiently large n, the second derivative $(f_n(\Omega))''(z)$ is bounded, by Proposition 3.11, there is a point $z^\star \in (-1,0)$, such that $f_n(\Omega)(z^\star) = z^\star$. This contradicts the fact that the rotation number of T_Ω is not p_n/q_n. The claim follows. **QED**

Let

$$\bar{F}_n(\Omega)(z) := \frac{a_n + c_nz}{1 - v_nz}, \qquad v_n := \frac{c_n - a_n - b_n}{b_n}. \tag{3.6}$$

Proposition 3.7 and Proposition 3.12 immediately imply the following.

Corollary 3.13 *Under the assumptions of Proposition 3.7, if $c_n < 1$, we have*

$$\|f_n(\Omega) - \bar{F}_n(\Omega)\|_{C^2} \leq \bar{C}\lambda^{\alpha n}. \tag{3.7}$$

Proof. It follows from property (C) of Proposition 3.7 that

$$|b_n(M_n - v_n - 1))| \leq Ca_n\lambda^{\alpha n}. \tag{3.8}$$

Since $b_n \geq a_na_{n+1}$ and since $a_{n+1} > \epsilon$ by Proposition 3.12, the claim follows. **QED**

The next proposition shows that the length of the longest elements of dynamical partitions $\mathcal{P}_n(\Omega)$ decreases at most geometrically, with a uniform rate.

Proposition 3.14 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at the fixed point, there exist $\gamma > 0$ and $N \in \mathbb{N}$ such that, for all irrational $\rho \in (0,1)$, all $n \geq N$ and all $\Omega \in \rho^{-1}(B_n)$, $|\Delta^{(n)}_{\max}(\Omega)| > \gamma|\Delta^{(n-1)}_{\max}(\Omega)|$.*

Proof. We will prove first that there exist constants $C_2 > 0$ and $N \in \mathbb{N}$ such that, for $n \geq N$, there exists $0 \leq j < k_{n+1}$ and an element $\Delta^{(n)}_{q_{n-1}+jq_n}(\Omega) \subset \Delta^{(n-1)}_0(\Omega)$ of partition $\mathcal{P}_{n+1}(\Omega)$ satisfying $|\Delta^{(n)}_{q_{n-1}+jq_n}(\Omega)|/|\Delta^{(n-1)}_0(\Omega)| > C_2$. Then, we will use a bounded distortion argument to find an interval of partition $\mathcal{P}_{n+1}(\Omega)$ whose length is at least a constant fraction of the length of the longest element of partition $\mathcal{P}_n(\Omega)$. If $a_n > \epsilon$, then, since $\Delta^{(n)}_{q_{n-1}}(\Omega) = T^{q_{n-1}}_\Omega(\Delta^{(n)}_0(\Omega))$, this inequality holds for $j = 0$ with $C_2 = e^{-V}\epsilon$. If $a_n \leq \epsilon$, due to Proposition 3.12, it suffices to consider the case $c_n < 1$ only. Then $b_n \geq 1 - \epsilon e^V$ and $c_n - \epsilon < v_n + 1 \leq c_n/(1 - \epsilon e^V)$. We note that parameters a_n, b_n and v_n in this proof are associated to the map T_Ω. If $\epsilon > 0$ has been chosen such that $\epsilon e^V < 1/2$, then $c_n - 1 - \epsilon < v_n \leq c_n - 1 + 2c_n\epsilon e^V$. These estimates show that, if $\epsilon > 0$ has been chosen small enough, $(\bar{F}_n(\Omega))'(-1) = \frac{c_n+a_nv_n}{(1+v_n)^2}$ and $(\bar{F}_n(\Omega))'(0) = c_n + a_nv_n$ are uniformly (in both n and Ω) close to c_n^{-1} and c_n, respectively, and that $(\bar{F}_n(\Omega))''(z) = 2v_n\frac{c_n+a_nv_n}{(1-v_nz)^3}$ is uniformly bounded on $[-1, 0]$ and negative. Proposition 3.7 then implies that there exist $C_3 > 0$, $C_4 > 1$ and $N \in \mathbb{N}$ such that $|(f_n(\Omega))'(-1) - c_n^{-1}| < C_3\epsilon$, $|(f_n(\Omega))'(0) - c_n| < C_3\epsilon$ and $-C_4 < (f_n(\Omega))''(z) < -C_4^{-1}$, for $n \geq N$. Using the above estimates on $(f_n(\Omega))'(-1)$ and $(f_n(\Omega))''(z)$, we find that there is $\epsilon > 0$ such that $(f_n(\Omega))'(z) > c_n^{-1} - C_3\epsilon - 3C_4\epsilon > 1$ for $z \in [-1, -1 + 3\epsilon]$. This implies that $f_n(\Omega)(-1 + \epsilon) > -1 + c_n^{-1}\epsilon - C_3\epsilon^2 - 3C_4\epsilon^2$, and thus $f_n(\Omega)(-1 + \epsilon) + 1 - \epsilon > (c_n^{-1} - 1)\epsilon - C_3\epsilon^2 - 3C_4\epsilon^2 > \frac{1}{2}(c_n^{-1} - 1)\epsilon$, if $\epsilon > 0$ is small enough. If there is $0 \leq j < k_{n+1}$ such that $(f_n(\Omega))^j(-1) \in [-1 + \epsilon, -1 + 2\epsilon]$ and $(f_n(\Omega))^{j+1}(-1) \in [-1 + \epsilon, -1 + 3\epsilon]$, then $|\Delta^{(n)}_{q_{n-1}+jq_n}(\Omega)|/|\Delta^{(n-1)}_0(\Omega)| > \frac{1}{2}(c_n^{-1} - 1)\epsilon$. If this is not the case, then either there is not j satisfying $0 \leq j < k_{n+1}$, such that $(f_n(\Omega))^j(-1) \in [-1+\epsilon, -1+2\epsilon]$ or there is such j but $(f_n(\Omega))^{j+1}(-1) \in [-1+\epsilon, -1+3\epsilon]$. In the first case, let j be the maximal of all $0 \leq i < k_{n+1}$, such that $(f_n(\Omega))^i(-1) < -1+\epsilon$. Since $(f_n(\Omega))^{j+1}(-1) > -1 + 2\epsilon$, $|\Delta^{(n)}_{q_{n-1}+jq_n}(\Omega)|/|\Delta^{(n-1)}_0(\Omega)| > \epsilon$. In the second case, the same inequality holds since $(f_n(\Omega))^{j+1}(-1) > -1 + 3\epsilon$. This proves the claim from the beginning of this proof.

The claim of the proposition now follows from the fact that the distortion of the ratio of lengths $\mathcal{D}(\mathcal{J}_1, \mathcal{J}_2; T^k_\Omega) := \frac{|T^k_\Omega(\mathcal{J}_2)|}{|T^k_\Omega(\mathcal{J}_1)|}\frac{|\mathcal{J}_1|}{|\mathcal{J}_2|}$ of any two intervals $\mathcal{J}_1, \mathcal{J}_2 \subset \Delta^{(n-1)}_0(\Omega)$ under the action of T^k_Ω, for $k = 1, \ldots, q_n - 1$, is bounded by

$$\mathcal{D}(\mathcal{J}_1, \mathcal{J}_2; T^k_\Omega) \leq \exp\left(\frac{\max_{x\in[0,1]} T''(x)}{\min_{x\in[0,1]} T'(x)} \sum_{i=0}^{k-1} |T^i_\Omega(\Delta^{(n-1)}_0(\Omega))|\right). \tag{3.9}$$

Since the intervals $\Delta^{(n-1)}_i(\Omega) = T^i_\Omega(\Delta^{(n-1)}_0(\Omega))$, for $i = 0, 1, \ldots, q_n - 1$, belong to the same partition $\mathcal{P}_n(\Omega)$, this distortion is uniformly bounded. It follows that $\Delta^{(n-1)}_{\max}(\Omega)$, for

$n \geq N$, contains, as a subset, an element of partition $\mathcal{P}_{n+1}(\Omega)$ whose length is bounded from below by $C_2|\Delta_{\max}^{(n-1)}(\Omega)|/\exp(\frac{\max_{x\in[0,1]} T''(x)}{\min_{x\in[0,1]} T'(x)})$. **QED**

Corollary 3.15 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at a fixed point, there exist $\gamma > 0$ and, for all irrational $\rho \in (0,1)$, a constant $K_0 > 0$ such that, for all $n \in \mathbb{N}$, $|\Delta_{\max}^{(n-1)}(\Omega_\rho)| > K_0\gamma^{n-1}$.*

4 The set of rotation numbers

In this section, we define a set $S \subset (0,1)\backslash\mathbb{Q}$ of Lebesgue measure 1, for which Theorem 1.1 holds.

Proposition 4.1 *For any $A \in \mathbb{R}^+$, there are sets $S_A^{(o)}, S_A^{(e)} \subset (0,1)\backslash\mathbb{Q}$ of Lebesgue measure 1, such that the following holds. For every $\rho \in S_A^{(o)}$, there is a subsequence $\sigma_n^{(o)} \in 2\mathbb{N}-1$, $n \in \mathbb{N}$, such that $k_{\sigma_n^{(o)}} \geq A\sigma_n^{(o)}$. For every $\rho \in S_A^{(e)}$, there is a subsequence $\sigma_n^{(e)} \in 2\mathbb{N}$, $n \in \mathbb{N}$, such that $k_{\sigma_n^{(e)}} \geq A\sigma_n^{(e)}$.*

Proof. We will prove the second claim. The proof of the first claim is analogous.

Let $J_n = J_n(\ell_1,\ldots,\ell_n) := \{\rho \in (0,1) : k_1 = \ell_1,\ldots,k_n = \ell_n\}$. It is not difficult to see from (2.2) that J_n is an interval of length $|J_n| = \frac{1}{q_n^2(1+\chi_n)}$, where $\chi_n := q_{n-1}/q_n$. For $N \in \mathbb{N}$, let $J_{n+1,N}$ be a subset of J_n defined by an additional requirement that $k_{n+1} < N$, i.e., $J_{n+1,N} = J_{n+1,N}(\ell_1,\ldots,\ell_n) := \{\rho \in (0,1) : k_1 = \ell_1,\ldots,k_n = \ell_n; k_{n+1} < N\}$. Since, $\rho_{n+1}^{-1} = k_{n+1} + \rho_{n+2}$ and $\rho_{n+2} \in (0,1)$, the length of $J_{n,N}$ is given by

$$|J_{n,N}| = \left|\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_nN + p_{n-1}}{q_nN + q_{n-1}}\right| = \frac{1}{q_n^2(1+\chi_n)}\frac{N-1}{N+\chi_n} < \left(1 - \frac{1}{N}\right)|J_n|. \qquad (4.1)$$

We will show that the Lebesgue measure of ${S_A^{(e)}}^C$, the complement of $S_A^{(e)}$, is zero. Notice that ${S_A^{(e)}}^C$ is the set of all $\rho \in (0,1)$ for which there exists $n_0 \in 2\mathbb{N}$ such that, for all $n \in 2\mathbb{N}$, satisfying $n \geq n_0$, $k_n < An$. In other words, ${S_A^{(e)}}^C = \cup_{n_0\in 2\mathbb{N}} B_{n_0}$, where B_{n_0} is the set of all $\rho \in (0,1)$ for which, for all $n \in 2\mathbb{N}$, satisfying $n \geq n_0$, $k_n < An$. Since the Lebesgue measure $\ell({S_A^{(e)}}^C) \leq \sum_{n_0\in 2\mathbb{N}} \ell(B_{n_0})$, it suffices to show that, for any fixed $n_0 \in 2\mathbb{N}$, $\ell(B_{n_0}) = 0$. For $n_0, m \in 2\mathbb{N}$, let $B_{n_0,m}$ be the set of all $\rho \in (0,1)$ such that, for all $n \in 2\mathbb{N}$, satisfying $n_0 \leq n \leq m$, $k_n < An$. It follows from (4.1) that the the Lebesgue measure

$$\ell(B_{n_0,m}) \leq \prod_{n\in 2\mathbb{N}, n=n_0}^{m} \left(1 - \frac{1}{An}\right) < \exp\left(-\sum_{n\in 2\mathbb{N}, n=n_0}^{m} \frac{1}{An}\right). \qquad (4.2)$$

Here, we have used that $1-x<e^{-x}$, for $x>0$. Since $\ell(B_{n_0,m})\to 0$, as $m\to\infty$, and $B_{n_0}\subset B_{n_0,m}$, for any $m\in 2\mathbb{N}$, we have $\ell(B_{n_0})=0$. The claim follows. **QED**

Proposition 4.2 *There are sets $S^{(o)}, S^{(e)}\subset(0,1)\backslash\mathbb{Q}$ of Lebesgue measure 1, such that the following holds. For every $\rho\in S^{(o)}$ and every $A\in\mathbb{N}$, there exists a subsequence $\sigma_n^{(o)}\in 2\mathbb{N}-1$, $n\in\mathbb{N}$, such that $k_{\sigma_n^{(o)}}\geq A\sigma_n^{(o)}$. For every $\rho\in S^{(e)}$ and every $A\in\mathbb{N}$, there exists a subsequence $\sigma_n^{(e)}\in 2\mathbb{N}$, $n\in\mathbb{N}$, such that $k_{\sigma_n^{(e)}}\geq A\sigma_n^{(e)}$.*

Proof. We will prove the second claim. The proof of the first claim is analogous. Let $S^{(e)}=\bigcap_{A\in\mathbb{N}}S_A^{(e)}$. For every $\rho\in S^{(e)}$ and every $A\in\mathbb{N}$, by Proposition 4.1, there exists a subsequence $\sigma_n^{(e)}\in 2\mathbb{N}$, $n\in\mathbb{N}$, such that $k_{\sigma_n^{(e)}}\geq A\sigma_n^{(e)}$. Moreover, the Lebesgue measure ℓ of the complement $\ell({S^{(e)}}^C)=\ell(\bigcup_{A\in\mathbb{N}}{S_A^{(e)}}^C)\leq\sum_{A\in\mathbb{N}}\ell({S_A^{(e)}}^C)=0$ and, thus, $\ell(S^{(e)})=1-\ell({S^{(e)}}^C)=1$. **QED**

We define the set S_{non} as follows. If $c<1$, then $S_{non}:=S^{(o)}$; if $c>1$, then $S_{non}:=S^{(e)}$. The set S, for which Theorem 1.1 holds, is defined as $S:=S_{rig}\cap S_{non}$, where S_{rig} is the set introduced in Remark 2. Since both S_{rig} and S_{non} have full Lebesgue measure in $(0,1)$, so does S.

5 Proof of the main theorem

In the following, for a given $\rho\in S_{non}$ and $A\in\mathbb{N}$, σ_n is the sequence $\sigma_n^{(o)}$, if $c<1$, or the sequence $\sigma_n^{(e)}$, if $c>1$, whose existence is guaranteed by Proposition 4.2.

We first show that the ratio of lengths of fundamental intervals of dynamical partition $\mathcal{P}_n$ is exponentially small with k_{n+1}.

Proposition 5.1 *For every $\epsilon_0>0$, there exist $N_0, A_0\in\mathbb{N}$ and $C_5>0$ such that if $n\geq N_0$, $k_{n+1}\geq A_0$ and $c_n<1$ then $a_n\leq C_5(c_n+\epsilon_0)^{\frac{1}{2}k_{n+1}}$.*

Proof. Let $\epsilon_0>0$. If $k_{n+1}\geq A_0$ then, since $f_n^j(-1)\in[-1,0)$, for $0\leq j\leq k_{n+1}$, there must be a point $z^\star\in[-1,0]$ such that $f_n(z^\star)-z^\star<A_0^{-1}$. Proposition 3.11 then implies that $\min\{1-b_n,a_n\}=\min\{f_n(-1)+1,f_n(0)\}=\min_{z\in[-1,0]}f_n(z)-z<A_0^{-1}$. Since, $1-b_n\geq a_ne^{-V}$, it follows that $a_n<A_0^{-1}e^V$. As in the proof of Proposition 3.15, one can show that, for every $\epsilon_1>0$, there exist $N_0, A_0\in\mathbb{N}$ such that $|(f_n(\Omega))'(-1)-c_n^{-1}|<\epsilon_1$ and $|(f_n(\Omega))'(0)-c_n|<\epsilon_1$, for $n\geq N_0$. Due to Proposition 3.11, there exists $\epsilon_2>0$ such that $|(f_n(\Omega))'(z)-c_n^{-1}|<\epsilon_0$ for $z\in[-1,-1+\epsilon_2]$ and $|(f_n(\Omega))'(z)-c_n|<\epsilon_0$ for $z\in[-\epsilon_2,0]$. Furthermore, the total number of elements of the set $\{f_n^j(-1):1\leq j\leq k_{n+1}\}$ outside of these two intervals is bounded. Since, by the Denjoy estimate (A), the lengths of the intervals $\Delta_{q_{n-1}}^{(n)}$ and $\Delta_{q_{n+1}}^{(n)}$ are of the same order, we obtain $a_n\leq C_5(c_n+\epsilon_0)^{\frac{1}{2}k_{n+1}}$, for some $C_5>0$. **QED**

The next proposition shows that, for every $\rho \in S_{non}$ and sufficiently large $A \in \mathbb{N}$, there are two neighboring sufficiently long (at least the size of an exponentially small quantity in σ_n) intervals of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$ which contain no end points of the previous partitions $\Xi_{\sigma_m-1}(\Omega_{\sigma_m-1})$, for $m = 1, \ldots, n-1$. These "long" intervals will be used in Proposition 5.3, Lemma 5.4 and Lemma 5.6 to create a perturbed circle map $\widetilde{T}_0$ of a map T_0, with the same size of the break c, such that the one-sided derivatives at zero of $(\sigma_n - 1)$-th renormalizations (i.e., q_{σ_n-1}-th powers) of maps $T_0 + \Omega_\rho$ and $\widetilde{T}_0 + \widetilde{\Omega}_\rho$, with the same irrational rotation number $\rho \in S_{non}$, are sufficiently different (by at least an exponentially small quantity in σ_n). In Lemma 5.4, we show that for such maps T_0 and $\widetilde{T}_0$ the one-sided derivatives at the break point of the q_{σ_n-1}-th powers of the maps $T_0 + \Omega_{\sigma_n-1}$ and $\widetilde{T}_0 + \widetilde{\Omega}_{\sigma_n-1}$, with rational rotation numbers, are sufficiently different (by at least an exponentially small quantity in σ_n). Lemma 5.5 relates the derivatives of these maps with an irrational rotation number and those of its rational convergents.

Proposition 5.2 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at the fixed point, there exist $\gamma_1 > 0$, $N_1, A_1 \in \mathbb{N}$ and, for every $\rho \in S_{non}$, a constant $K_1 > 0$ such that, for every $A \geq A_1$ and every $n \geq N_1$, the following holds. There exists $0 < j_n < q_{\sigma_n-1}$,*

$$|\Delta^{(\sigma_n-2)}_{j_n-q_{\sigma_n-2}}(\Omega_{\sigma_n-1})|, |\Delta^{(\sigma_n-2)}_{j_n}(\Omega_{\sigma_n-1})| > K_1\gamma_1^{\sigma_n} \tag{5.1}$$

and

$$\bigcup_{m=1}^{n-1} \Xi_{\sigma_m-1}(\Omega_{\sigma_m-1}) \bigcap \left(\Delta^{(\sigma_n-2)}_{j_n-q_{\sigma_n-2}}(\Omega_{\sigma_n-1}) \bigcup \Delta^{(\sigma_n-2)}_{j_n}(\Omega_{\sigma_n-1})\right) = \emptyset. \tag{5.2}$$

Proof. We will first show that, for $\rho \in S_{non}$, there are "long" intervals of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$; we will then show that there are such neighboring "long" intervals that do not contain the endpoints of the previous partitions $\Xi_{\sigma_m-1}(\Omega_{\sigma_m-1})$, for $m = 1, \ldots, n-1$.

By Corollary 3.15, for every $\rho \in (0,1)\backslash\mathbb{Q}$ there exists an interval $\Delta^{(\sigma_n-2)}_{\max}(\Omega_\rho)$ of partition $\mathcal{P}_{\sigma_n-1}(\Omega_\rho)$ and a constant $K_0 > 0$ such that $|\Delta^{(\sigma_n-2)}_{\max}(\Omega_\rho)| > K_0\gamma^{\sigma_n-2}$. It follows from Proposition 3.5, that $|\Delta^{(\sigma_n-2)}_{\max}(\Omega_{\sigma_n-1})| > (1 - 2C_1a_{\sigma_n-1})K_0\gamma^{\sigma_n-2}$. Proposition 5.1 then implies that for some $N_1, A_1 \in \mathbb{N}$, $n \geq N_1$ and $k_{\sigma_n} \geq A\sigma_n \geq A_1\sigma_n$, $a_{\sigma_n-1} < (4C_1)^{-1}$.

The interval $\Delta^{(\sigma_n-2)}_{\max}(\Omega_{\sigma_n-1})$ belongs to an interval $\Delta^{(\sigma_{n-1}-2)}_i(\Omega_{\sigma_n-1})$ of the partition $\mathcal{P}_{\sigma_{n-1}-1}(\Omega_{\sigma_n-1})$, for some $0 \leq i < q_{\sigma_{n-1}-1}$, and, for all $m < n-1$, an interval of partition $\mathcal{P}_{\sigma_m-1}(\Omega_{\sigma_n-1})$. It follows from Proposition 3.5 and Proposition 5.1 that, if N_1 and A_1 have been chosen large enough, for all $m < n$, at most one of the points in $\Xi_{\sigma_m-1}(\Omega_{\sigma_m-1})$ belongs to $\Delta^{(\sigma_{n-1}-2)}_i(\Omega_{\sigma_n-1})$. Since the number of intervals of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$ inside $\Delta^{(\sigma_{n-1}-2)}_i(\Omega_{\sigma_n-1})$ is larger than or equal to $k_{\sigma_n} \geq A\sigma_n$ and since the neighboring intervals of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$ are mapped one into another by the map $T^{q_{\sigma_n-2}}_{\Omega_{\sigma_n-1}}$, it follows from the Denjoy estimate (A) that, if N_1 and A_1 have been chosen large enough, there exists $0 < j_n < q_{\sigma_n - 1}$ and neighboring elements $\Delta^{(\sigma_n-2)}_{j_n - q_{\sigma_n-2}}(\Omega_{\sigma_n-1})$ and $\Delta^{(\sigma_n-2)}_{j_n}(\Omega_{\sigma_n-1})$, of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$, of length larger than $\frac{1}{2}K_0\gamma^{\sigma_n-2}e^{-(2n-1)V}$, that contain no points in $\Xi_{\sigma_m-1}(\Omega_{\sigma_m-1})$, for any $m = 1, \dots, n-1$. Since $\sigma_n \geq 2n-1$, the claim follows with $\gamma_1 = \gamma e^{-V}$ and $K_1 = 2^{-1}K_0\gamma^{-2}$. **QED**

In the proposition that follows, we use the "long" intervals that do not contain points of the previous partitions to construct a sequence of C^r-smooth functions of rapidly decreasing norm that vanish and have prescribed derivatives at some points. They will be used in Lemma 5.4 to perturb a circle map T_0 in a way that produces controlled changes in the derivatives of the $(\sigma_n - 1)$-th renormalizations of $T_0 + \Omega_{\sigma_n-1}$, while preserving some periodic orbits.

In the following, we identify the points on $\mathbb{S}^1$ with the corresponding points in the interval $[0, 1]$.

Proposition 5.3 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at the fixed point, there exist $\gamma_1 > 0$, $N_1, A_1 \in \mathbb{N}$ and, for every $\rho \in S$, a constant $K_2 > 0$ such that, for every $A \geq A_1$, the following holds. There is a sequence of C^r-smooth functions $\phi_n : [0,1] \to \mathbb{R}$ satisfying the following conditions for $n \geq N_1$:*

(i) $\phi_n(x_{i_m}(\Omega_{\sigma_m-1})) = 0$, *for* $m = N_1, \dots, n+1$ *and* $0 \leq i_m < q_{\sigma_m-1}$,

(ii) $\phi'_n(x_{i_m}(\Omega_{\sigma_m-1})) = 0$, *for* $m = N_1, \dots, n+1$ *and* $0 \leq i_m < q_{\sigma_m-1}$ *such that* $i_{n+1} \neq j_{n+1}$,

(iii) $\phi'_n(x_{j_{n+1}}(\Omega_{\sigma_{n+1}-1})) = 1$,

(iv) $\|\phi_n\|_{C^r} \leq K_2\gamma_1^{-(r-1)\sigma_{n+1}}$.

Here $j_{n+1} < q_{\sigma_{n+1}}$ is a natural number whose existence is guaranteed by Proposition 5.2.

Proof. To fix the orientation, we will prove the claim in the case $c < 1$ only. If $c > 1$, the proof is similar. Proposition 5.2 guarantees that there exist $\gamma_1 > 0$, $N_1, A_1 \in \mathbb{N}$ and, for every $\rho \in S_{non}$, a constant $K_1 > 0$ such that, for every $A \geq A_1$ and $n \geq N_1$, there is $0 < j_n < q_{\sigma_n}$ and elements $\Delta^{(\sigma_n-2)}_{j_n-q_{\sigma_n-2}}(\Omega_{\sigma_n-1})$, $\Delta^{(\sigma_n-2)}_{j_n}(\Omega_{\sigma_n-1})$ of partition $\mathcal{P}_{\sigma_n-1}(\Omega_{\sigma_n-1})$ satisfying (5.1) and (5.2). We define ϕ_n piecewise as follows. For $x \in \Delta_- \cup \Delta_+$, where $\Delta_- := \Delta^{(\sigma_{n+1}-2)}_{j_{n+1}-q_{\sigma_{n+1}-2}}(\Omega_{\sigma_{n+1}-1})$ and $\Delta_+ := \Delta^{(\sigma_{n+1}-2)}_{j_{n+1}}(\Omega_{\sigma_{n+1}-1})$, we define

$$\phi_n(x) := \frac{1}{|\Delta_-|^{r+1}|\Delta_+|^{r+1}}(x-\chi)|x-\chi_-|^{r+1}|x-\chi_+|^{r+1}, \tag{5.3}$$

where $\chi := x_{j_{n+1}}(\Omega_{\sigma_{n+1}-1})$, $\chi_- := x_{j_{n+1}-q_{\sigma_{n+1}-2}}(\Omega_{\sigma_{n+1}-1})$, $\chi_+ := x_{j_{n+1}+q_{\sigma_{n+1}-2}}(\Omega_{\sigma_{n+1}-1})$. For $x \in [0,1]\backslash(\Delta_- \cup \Delta_+)$, we define $\phi_n(x) := 0$. It is easy to see that ϕ_n is C^r-smooth on $[0, 1]$. It follows from the definition of ϕ_n and the condition (5.2) that the conditions (i), (ii) and (iii) are satisfied. Condition (5.1), together with the fact that the lengths of the intervals Δ_- and Δ_+ are of the same order, due to the Denjoy estimate (A), implies that the condition (iv) is satisfied as well, for some $K_2 > 0$ (depending on r and ρ). **QED**

Lemma 5.4 *For every C^r-smooth $(r>2)$ circle map T_0 with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$ at the fixed point, there exist $\mu \in (0,1)$ and $N_1, A_1 \in \mathbb{N}$ such that the following holds. For every $\rho \in S_{non}$, there exists a constant $K>0$ and for every $A \geq A_1$, there exists a C^r-smooth circle map $\widetilde{T}_0$ with a break of size c at the fixed point such that, for every $m \geq N_1$,*

$$\left| \left(T^{q_{\sigma_m -1}}_{\Omega_{\sigma_m -1}}\right)'_+ (0) - \left(\widetilde{T}^{q_{\sigma_m -1}}_{\tilde{\Omega}_{\sigma_m -1}}\right)'_+ (0) \right| \geq K \mu^{\sigma_m} . \tag{5.4}$$

Proof. Proposition 5.3 guarantees that there exist $\gamma_1 > 0$, $N_1, A_1 \in \mathbb{N}$ and, for every $\rho \in S_{non}$, a constant $K_2 > 0$ such that, for every $A \geq A_1$ there is a sequence σ_n and a sequence of C^r-smooth functions $\phi_n : [0,1] \to \mathbb{R}$, $n \geq N_1$, satisfying conditions (i), (ii), (iii) and (iv).

Let T_0 be given and let $T_0^{[N_1]} := T_0$. We will construct the map $\widetilde{T}_0$ as the limit of a sequence of C^r-smooth maps $T_0^{[n]}$, with a break of size c at the fixed point, that satisfy the condition (5.4), for $N_1 \leq m \leq n$.

In order to simplify the notation, the parameter values Ω_m, associated to the map $T_0^{[n]}$, will be denoted by $\Omega_m^{[n]}$. We also define $T_m^{[n]} := T^{[n]}_{\Omega_m^{[n]}}$ and the corresponding orbit $x_i^{[n]}(\Omega_m^{[n]}) := (T_m^{[n]})^i(0)$. We choose $T_0^{[n+1]} := T_0^{[n]} + \epsilon_n \phi_n$, where $\epsilon_n = 0$ if the map $\widetilde{T}_0 = T_0^{[n]}$ satisfies condition (5.4) for $N_1 \leq m \leq n+1$; otherwise $\epsilon_n = \epsilon_0 \mu^{\sigma_{n+1}}$, where $\epsilon_0 > 0$ and $\mu \in (0,1)$ will be chosen below.

Condition (i) guarantees that $T_0^{[n]}$ defines a circle map with a fixed point at 0 and condition (ii) guarantees that the size of the break at 0 is c. If $\mu > 0$ is chosen such that $\mu < \gamma_1^{r-1}$, it follows from condition (iv) that $\|\epsilon_n \phi_n\|_{C^r} \leq \epsilon_0 K_2 (\mu \gamma_1^{-(r-1)})^{\sigma_{n+1}}$ and, consequently, $\| \sum_{k=m}^n \epsilon_k \phi_k \|_{C^r} \leq \epsilon_0 K_2 (1 - \mu \gamma_1^{-(r-1)})^{-1}$. If $\epsilon_0 > 0$ has been chosen small enough, then for all $n \geq N_1$, $\|T_0^{[n]}\|_{C^r}$ is uniformly (order of ϵ_0) close to $\|T_0\|_{C^r}$. The sequence of functions $T_0^{[n]}$ converges uniformly to a C^r- smooth function $\widetilde{T}_0$ with the same property.

By construction (condition (i)), $\Omega^{[n+1]}_{\sigma_m -1} = \Omega^{[n]}_{\sigma_m -1}$ and $x_i^{[n+1]}(\Omega^{[n+1]}_{\sigma_m -1}) = x_i^{[n]}(\Omega^{[n]}_{\sigma_m -1})$, for $N_1 \leq m \leq n+1$. Therefore, we have

$$\left(\left(T^{[n+1]}_{\sigma_m -1} \right)^{q_{\sigma_m -1}} \right)'_+ (0) \quad = \quad \prod_{i=0}^{q_{\sigma_m -1}-1} (T_0^{[n]} + \epsilon_n \phi_n)'_+ (x_i^{[n]}(\Omega^{[n]}_{\sigma_m -1})). \tag{5.5}$$

It follows from (5.5) and condition (ii) that $\left(\left(T^{[n+1]}_{\sigma_m -1} \right)^{q_{\sigma_m -1}} \right)'_+ (0) = \left(\left(T^{[n]}_{\sigma_m -1} \right)^{q_{\sigma_m -1}} \right)'_+ (0)$, for $N_1 \leq m \leq n$. Therefore, if $\widetilde{T}_0 = T_0^{[n]}$ satisfies the condition (5.4), for $N_1 \leq m \leq n$, so does $\widetilde{T}_0 = T_0^{[n+1]}$. Conditions (ii) and (iii), together with (5.5), imply that

$$\left(\left(T^{[n+1]}_{\sigma_{n+1} -1} \right)^{q_{\sigma_{n+1} -1}} \right)'_+ (0) = \quad \left(\left(T^{[n]}_{\sigma_{n+1} -1} \right)^{q_{\sigma_{n+1} -1}} \right)'_+ (0) \left(1 + \frac{\epsilon_n}{\left(T_0^{[n]}\right)'(x^{[n]}_{j_{n+1}}(\Omega^{[n]}_{\sigma_{n+1}-1}))} \right) \tag{5.6}$$

Since the derivative of $T_0^{[n]}$ is uniformly (order of ϵ_0) close to the derivative of T_0 (both pointwise and in n), which is bounded away from zero, $(T_0^{[n]})'(x)$ is uniformly (both in x and in n) bounded away from zero. Furthermore, $V_n := \mathrm{Var}_{\mathbb{S}^1} \ln(T_0^{[n]})'$ is uniformly (in n) close to V, and by the Denjoy lemma (A), $|\ln(T_{\sigma_k-1}^{[n]})^{q_{\sigma_k-1}}| \le V_n$. The equality (5.6), therefore, implies that the map $\widetilde{T}_0 = T_0^{[n+1]}$ satisfies condition (5.4) for $m = n+1$ as well, if K is small enough. The limiting map $\widetilde{T}_0 := \lim_{n\to\infty} T_0^{[n]}$, therefore, satisfies (5.4), for all $n \ge N_1$. **QED**

We now prove a general estimate that relates the derivatives of the renormalizations of circle maps with a break in a family $T_0 + \Omega$ with irrational rotation numbers and those with their rational convergents.

Lemma 5.5 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at the fixed point, there exists $C_6 > 0$, such that, for all irrational $\rho \in (0,1)$, and all $n \in \mathbb{N}$,*

$$|(f_n(\Omega_\rho))'_-(0) - (f_n(\Omega_n))'_-(0)| \le C_6 a_n. \tag{5.7}$$

Proof. It follows from the definition of the renormalizations (2.4) that

$$(f_n(\Omega_\rho))'_-(0) - (f_n(\Omega_n))'_-(0) = \prod_{i=0}^{q_n-1} (T_0)'_-(x_i(\Omega_\rho)) - \prod_{i=0}^{q_n-1} (T_0)'_-(x_i(\Omega_n)). \tag{5.8}$$

It follows from the mean value theorem that, for $1 \le i < q_n$,

$$|T_0'(x_i(\Omega_\rho)) - T_0'(x_i(\Omega_n)| = |T_0''(\xi_i)||I_i(\Omega_n)| \le \max_{x\in[0,1]} |T_0''(x)||I_i(\Omega_n)|, \tag{5.9}$$

where $\xi_i \in I_i(\Omega_n)$. Therefore, we have

$$|(f_n(\Omega_\rho))'_-(0) - (f_n(\Omega_n))'_-(0)| \le \\ \le \prod_{i=0}^{q_n-1} (T_0)'_-(x_i(\Omega_\rho)) \left(\exp\left(\sum_{i=0}^{q_n-1} \frac{\max_{x\in[0,1]} |T_0''(x)|}{\min_{x\in[0,1]} T_0'(x)} |I_i(\Omega_n)| \right) - 1 \right). \tag{5.10}$$

Using the Denjoy estimate (A) and Corollary 3.6, we obtain

$$|(f_n(\Omega_\rho))'_-(0) - (f_n(\Omega_n))'_-(0)| \le \left(\exp\left(\frac{\max_{x\in[0,1]} |T_0''(x)|}{\min_{x\in[0,1]} T_0'(x)} C_1 a_n \right) - 1 \right) e^V. \tag{5.11}$$

Since $\Delta_0^{(n)}(\Omega_\rho) \subset \Delta_{q_n}^{(n-1)}(\Omega_\rho) = T_{\Omega_\rho}^{q_n}(\Delta_0^{(n-1)}(\Omega_\rho))$, one has $a_n \le e^V$ and, thus,

$$|(f_n(\Omega_\rho))'_-(0) - (f_n(\Omega_n))'_-(0)| \le \frac{\max_{x\in[0,1]} |T_0''(x)|}{\min_{x\in[0,1]} T_0'(x)} C_1 \exp\left(V + \frac{\max_{x\in[0,1]} |T_0''(x)|}{\min_{x\in[0,1]} T_0'(x)} C_1 e^V \right) a_n. \tag{5.12}$$

The claim follows. **QED**

We can now prove the key lemma. It shows that, for $\rho \in S_{non}$, there are pairs of circle maps with a break of the same size and with the same rotation number ρ, for which the one-sided derivatives of the $(\sigma_n - 1)$-th renormalizations at the break point are different by at least an exponentially small quantity in σ_n.

Lemma 5.6 *For every C^r-smooth ($r > 2$) circle map T_0 with a break of size $c \in \mathbb{R}_+ \setminus \{1\}$ at the fixed point, there exist $\mu \in (0,1)$ and $N_2, A_2 \in \mathbb{N}$ such that the following holds. For every $\rho \in S_{non}$, there exists a constant $K_3 > 0$ and for every $A \geq A_2$, there exists a C^r-smooth circle map $\widetilde{T}_0$ with a break of size c at the fixed point such that, for every $n \geq N_2$,*

$$\left|(f_{\sigma_n-1}(\Omega_\rho))'_-(0) - (\widetilde{f}_{\sigma_n-1}(\widetilde{\Omega}_\rho))'_-(0)\right| \geq K_3 \mu^{\sigma_n} . \quad (5.13)$$

Proof. It follows from Lemma 5.4 and Lemma 5.5 that there exist $\mu \in (0,1)$ and constants $N_1, A_1 \in \mathbb{N}$ and, for every $\rho \in S_{non}$, a constant $K > 0$ such that, for $A \geq A_1$, there exist a sequence σ_n (associated to ρ and A), a C^r-smooth circle map $\widetilde{T}_0$ with a break of size c at the fixed point and a constant $K_4 > 0$ such that, for $n \geq N_1$,

$$\left|(f_{\sigma_n-1}(\Omega_\rho))'_-(0) - (\widetilde{f}_{\sigma_n-1}(\widetilde{\Omega}_\rho))'_-(0)\right| \geq K \min\{c^2, c^{-2}\} \mu^{\sigma_n} - K_4(a_{\sigma_n-1} + \widetilde{a}_{\sigma_n-1}) . \quad (5.14)$$

By Proposition 5.1, for every $\epsilon_0 > 0$, there is a constant $K_5 > 0$ such that, for $k_{\sigma_n} \geq A\sigma_n$, we have $a_{\sigma_n-1}, \widetilde{a}_{\sigma_n-1} \leq K_5(\min\{c, c^{-1}\} + \epsilon_0)^{\frac{1}{2}k_{\sigma_n}}$. If $A \geq A_2 \geq A_1 \geq A_0$ and A_2 has been chosen such that

$$(\min\{c, c^{-1}\} + \epsilon_0)^{\frac{1}{2}A_2} < \mu, \quad (5.15)$$

there exists $K_3 > 0$ and $N_2 \geq N_1$ such that (5.13) holds for $n \geq N_2$. **QED**

On the other hand, we have the following necessary condition for a $C^{1+\varepsilon}$-smooth conjugacy between circle maps with a break.

Lemma 5.7 *If T and $\widetilde{T}$ are two C^r-smooth ($r > 2$) circle maps with breaks of size $c \in \mathbb{R}_+ \setminus \{1\}$ at 0, with the same irrational rotation number ρ, which are conjugate to each other via a C^1-smooth diffeomorphism h which is $C^{1+\varepsilon}$-smooth at 0, then there is a constant $K_6 > 0$ such that*

$$|(f_n)'_-(0) - (\widetilde{f}_n)'_-(0)| \leq K_6 a_n^{\varepsilon}. \quad (5.16)$$

Proof. By differentiating $h \circ T^{q_n} = \widetilde{T}^{q_n} \circ h$ at 0, taking into account $h(0) = 0$, we obtain

$$h'(x_{q_n})(T^{q_n})'(0) = (\widetilde{T}^{q_n})'(0)h'(0). \quad (5.17)$$

Since h is $C^{1+\varepsilon}$ at zero, for every $x \in \mathbb{S}^1$, (1.1) holds. Using (5.17) and Denjoy estimate (A), we obtain

$$\begin{aligned} |(f_n)'_-(0) - (\widetilde{f}_n)'_-(0)| = |(T^{q_n})'_-(0) - (\widetilde{T}^{q_n})'_-(0)| = (T^{q_n})'_-(0) \left| 1 - \frac{h'(x_{q_n})}{h'(0)} \right| \\ = \frac{(T^{q_n})'_-(0)}{|h'(0)|} |h'(0) - h'(x_{q_n})| \leq \frac{e^V}{|h'(0)|} |\Delta_0^{(n)}|^\varepsilon, \end{aligned} \tag{5.18}$$

Since $h'(0) \neq 0$, the claim follows. **QED**

Proof of Theorem 1.1. Part (i) of the claim holds for the set S defined in Section 4, as follows from the results of [12] and [13] and the fact that S_{non} is of full Lebesgue measure. To prove part (ii), let $\varepsilon > 0$. Let T_0 be a C^r-smooth ($r > 2$) circle map with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$ at the fixed point 0 and let $N_2, A_2 \in \mathbb{N}$ be constants whose existence is guaranteed by Lemma 5.6. By the same lemma, there exist a constant $\mu \in (0,1)$ and, for every $\rho \in S$ a constant $K_3 > 0$ such that the following holds. For every $A \geq A_2$, there exist a sequence σ_n and a circle map $\widetilde{T}_0$ with a break of size c at the fixed point 0 such that the renormalizations of the corresponding maps $T = T_0 + \Omega_\rho$ and $\widetilde{T} = \widetilde{T}_0 + \widetilde{\Omega}_\rho$, with rotation number ρ, satisfy (5.13), for $n \geq N_2$.

By part (i) of the claim, T and $\widetilde{T}$ are C^1-smoothly conjugate to each other via a diffeomorphism h which satisfies $h(0) = 0$. Assume that h is $C^{1+\varepsilon}$-smooth at 0. By Lemma 5.7, there exists $K_6 > 0$ such that,

$$\left| (f_{\sigma_n - 1}(\Omega_\rho))'_-(0) - (\widetilde{f}_{\sigma_n - 1}(\widetilde{\Omega}_\rho))'_-(0) \right| \leq K_6 a^\varepsilon_{\sigma_n - 1}\,. \tag{5.19}$$

This inequality, (5.13) and Proposition 5.1 imply that, for $\epsilon_0 > 0$ and some $K_9 > 0$,

$$K_3 \mu^{\sigma_n} \leq K_6 a^\varepsilon_{\sigma_n - 1} \leq K_9 (\min\{c, c^{-1}\} + \epsilon_0)^{\frac{1}{2}\varepsilon A \sigma_n}. \tag{5.20}$$

If $A \geq A_3$, and $\epsilon_0 > 0$ and $A_3 \geq A_2$ have been chosen such that

$$(\min\{c, c^{-1}\} + \epsilon_0)^{\frac{1}{2}\varepsilon A_3} < \mu, \tag{5.21}$$

for sufficiently large n, this leads to a contradiction. This proves part (ii) of the claim. **QED**

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