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# A renormalization group approach to quasiperiodic motion with Brjuno frequencies

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*Abstract*. We introduce a renormalization group scheme that applies to vector fields on $\mathbb{T}^d \times \mathbb{R}^m$ with frequency vectors that satisfy a Brjuno condition. Earlier approaches were restricted to Diophantine frequencies, owing to a limited control of multidimensional continued fractions. We get around this restriction by avoiding the use of a continued-fractions expansion. Our results concerning invariant tori generalize those of [H. Koch and S. Kocić, Renormalization of vector fields and Diophantine invariant tori. *Ergod. Th. & Dynam. Sys.* **28** (2008), 1559–1585] from Diophantine- to Brjuno-type frequency vectors. In particular, each Brjuno vector $\omega \in \mathbb{R}^d$ determines an analytic manifold $\mathcal{W}$ of infinitely renormalizable vector fields, and each vector field on $\mathcal{W}$ is shown to have an elliptic invariant $d$-torus with frequencies $\omega_1, \omega_2, \ldots, \omega_d$.

## 1. *Introduction and main results*

The renormalization of Hamiltonian flows and related maps has a relatively long history; see, e.g., [**7, 16, 21**] and references therein. The approach we consider here was motivated originally by the problem of describing the breakup of invariant tori for Hamiltonian systems with two degrees of freedom [**1–11**]. Some of these questions have been answered in [**10, 11**]. The same methods also apply to the study of near-integrable Hamiltonians or near-linear flows. In such a regime, the renormalization group (RG) approach can be viewed as an alternative to purely perturbative methods based on KAM theory or Lindstedt series. Over the past few years, its scope has been extended from a small set of 'self-similar' frequency vectors [**8**] to arbitrary Diophantine frequencies [**13–18**], and from Hamiltonian flows to a large class of vector fields [**17**]. As far as the construction of smooth invariant tori is concerned, the works cited above cover the classical KAM results but not the later extensions to Brjuno-type frequency vectors [**22–30**]. This is due to the fact that the current approach requires good bounds on a continued-fractions expansion, such as the ones obtained in [**15**] for Diophantine frequency vectors. Unfortunately, there seem to be significant obstacles to obtaining such bounds for Brjuno vectors in dimensions $d > 2$.

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This has motivated us to develop a renormalization scheme that does not rely on continued fractions. As it turns out, our scheme applies quite naturally to rotation vectors $\omega \in \mathbb{R}^d$ that satisfy the following Brjuno condition [22]:

$$\sum_{n=1}^{\infty} 2^{-n} \ln(1/\Omega_n) < \infty, \qquad \Omega_n = \min_{0<|\nu|\leq 2^n} |\omega \cdot \nu|, \tag{1.1}$$

where $\nu$ denotes lattice points in $\mathbb{Z}^d$. Our new renormalization group transformations share some important features with those used in [**8–20**]. Thus, before discussing the differences, we shall first describe the transformations used in [**8–17**], starting with some general remarks about renormalization.

Renormalization can be viewed as a tool for classifying systems by the value of a given observable that describes asymptotic properties of the system. A renormalization transformation is a map on the space of systems being considered which contracts directions within the same equivalence class and expands directions along which the observable changes, preferably in a way that induces a natural action on the set of observed values. In the case at hand, the systems are vector fields on $\mathcal{M} = \mathbb{T}^d \times \mathbb{R}^m$, and the observed quantities are the ratios among the $d$ frequencies of rotation. Among the natural actions on frequency vectors $\omega \in \mathbb{R}^d$ are the steps in a continued-fractions algorithm. These typically involve integer matrices with a distinguished expanding direction, such that rational approximants to $\omega$ approach (up to rescaling) the orbit of $\omega$ under iteration of the algorithm.

Consider now a fixed matrix $T$ in $\mathrm{SL}(d, \mathbb{Z})$ whose transpose contracts the orthogonal complement of $\omega \in \mathbb{R}^d$. Its role will be to 'scale' the periodic variable $x \in \mathbb{T}^d$. For a scaling of the non-periodic variable $y \in \mathbb{R}^m$ and of time, we choose two (small) constants $\mu$ and $\eta$, respectively. The simplest renormalization transformation associated with these scaling parameters is $\mathcal{R}(X) = \eta^{-1}\mathcal{T}_\mu^* X$, where $\mathcal{T}_\mu^* X = \mathcal{T}_\mu^{-1} X \circ \mathcal{T}_\mu$ and

$$\mathcal{T}_\mu(x, y) = (Tx, \mu T' y). \tag{1.2}$$

Here, $T'$ denotes either the $m \times m$ identity matrix or, in the case of $m = d$, the inverse of the transpose of $T$ (which makes $\mathcal{T}_1$ symplectic), if desired for the renormalization of Hamiltonian vector fields. This choice of $\mathcal{R}$ would be suitable, e.g., for smooth vector fields of the form $X(x, y) = (w, v(y))$, with $w$ a non-zero constant multiple of $\omega$. In this case, if we take $\eta = \|T^{-1}w\|/\|\omega\|$ and $\mu > 0$ sufficiently small, then $\mathcal{R}$ contracts all directions except those given by the affine part of $v$.

When considering more general vector fields, we also have to achieve contraction within families $\mathcal{U} \mapsto \mathcal{U}^* X$ obtained from changes of coordinates $\mathcal{U} : \mathcal{M} \to \mathcal{M}$ close to the identity. In what follows, $\mathcal{U}^* X$ denotes the pullback of $X$ under $\mathcal{U}$, given by $\mathcal{U}^* X = (D\mathcal{U})^{-1}(X \circ \mathcal{U})$. The above suggests that we define

$$\mathcal{R}(X) = \eta^{-1}\mathcal{T}_\mu^* \mathcal{U}_X^* X, \tag{1.3}$$

where $\mathcal{U}_X$ is a change of coordinates designed to bring the renormalized vector field into some appropriate normal form. If $X$ is already in normal form, then $\mathcal{U}_X$ should be the identity. Our choice of normal form will be described later. It includes the vector fields $X(y) = (w, v(y))$ mentioned above.

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If $\omega \in \mathbb{R}^d$ admits a periodic continued-fractions expansion, then it suffices to work with a single RG transformation [**8–12**]. More general frequency vectors require a sequence of RG transformations $\mathcal{R}_n$, one for each of the matrices $T_n$ in the continued-fractions expansion of $\omega$. In the single frequency case ($d = 2$), such an analysis was carried out for Diophantine- [**13, 14**] and Brjuno-type frequencies [**19, 20**]. What makes this case special is that there is a canonical continued-fractions expansion and the corresponding matrices $T_n$ are known explicitly. More recent results [**15–18**] extend the scope of renormalization to Diophantine vectors $\omega \in \mathbb{R}^d$, for arbitrary $d \geq 2$, using a multidimensional continued-fractions algorithm developed in [**15, 31, 32**]. Here, the matrices $T_n$ are no longer known explicitly; they are the increments $T_n = P_{n-1}^{-1} P_n$ of integer approximants $P_k \in \mathrm{SL}(d, \mathbb{Z})$ to matrices $WE(t_k) \in \mathrm{SL}(d, \mathbb{R})$, where $\{t_k\}$ is some appropriately chosen increasing sequence of positive real numbers, $E(t) = \mathrm{diag}(e^{-t}, \ldots, e^{-t}, e^{(d-1)t})$, and $W$ is a fixed matrix in $\mathrm{SL}(d, \mathbb{R})$ that maps the expanding eigenvector of $E(t)$ to $\omega$. The approximation is well-controlled for Diophantine frequency vectors [**15**], but attempts to extend this to Brjuno vectors have not been successful so far.

The idea pursued here is to avoid the integer approximation and renormalize directly with real matrices. This leads us to consider tori of the form $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$, where $\mathcal{Z}$ is a simple lattice in $\mathbb{R}^d$. A non-singular real matrix $T$ defines a map $x + \mathcal{Z} \mapsto Tx + T\mathcal{Z}$ from the torus $\mathbb{R}^d/\mathcal{Z}$ to the torus $\mathbb{R}^d/(T\mathcal{Z})$. In order to simplify notation, this map will again be denoted by $T$. Furthermore, whenever the lattice $\mathcal{Z}$ is fixed or irrelevant, we will simply write $x$ in place of $x + \mathcal{Z}$.

With this in mind, we choose the matrix $T$ in the definition (1.2) of the scaling $\mathcal{T}_\mu$ to be of the form

$$T(x) = \eta^{-1} x_\parallel + \beta x_\perp, \quad 0 < \eta, \beta < 1, \tag{1.4}$$

where $x = x_\parallel + x_\perp$ is the decomposition of $x \in \mathbb{R}^d$ into a component $x_\parallel$ parallel to $\omega$ and a component $x_\perp$ perpendicular to $\omega$. Notice that if $d = 2$ and $\omega_1/\omega_2 = 1/(k + 1/(k + \ldots))$, then the choice $\eta = \beta = (\omega_1/\omega_2)^2$ makes $T$, in fact, an integer matrix. A matrix of the type (1.4) will be referred to as a *scaling matrix*. The corresponding RG transformation $\mathcal{R}$ is again taken to be of the form (1.3), with $\eta^{-1}$ being the expanding eigenvalue of $T$. Our choice of $\mu$ and $\mathcal{U}_X$ will be described later. We note that by choosing the time scaling $\eta^{-1}$ in (1.3) to be the same as the spatial scaling $\eta^{-1}$ in (1.4), which is independent of the vector field $X$, we can allow $\mathcal{R}$ to have a non-contracting direction. However, since time scaling commutes with our renormalization transformations, this direction can easily be taken care of later (in applications of our main result).

Functions on the torus $\mathbb{R}^d/\mathcal{Z}$ can be identified with functions on $\mathbb{R}^d$ that are invariant under $\mathcal{Z}$-translations or, equivalently, with quasiperiodic functions on $\mathbb{R}^d$ whose frequency module lies in the dual lattice (the set of points $v \in \mathbb{R}^d$ satisfying $\exp(iv \cdot z) = 1$ for all $z \in \mathcal{Z}$). For convenience, we will now perform a linear change of coordinates in $\mathbb{R}^d$, so that $\omega = (1, 0, \ldots, 0)$. The lattice obtained from $2\pi\mathbb{Z}^d$ under this transformation will be denoted by $\mathcal{Z}_0$, and its dual lattice by $\mathcal{V}_0$. The frequencies $\nu$ in (1.1) now range over $\mathcal{V}_0$.

Our analysis applies to vector fields that are close to $K = (\omega, 0)$. We assume analyticity on a complex neighborhood $D_\varrho$ of $D_0 = \mathbb{T}^d \times \{0\}$, characterized by the conditions $|\mathrm{Im}\, x_i| < \varrho$ and $|y_j| < \varrho$. Denote by $t \mapsto \Phi_X^t$ the flow for a vector field $X$. An invariant torus for $X$, with frequency vector $\omega$, is a continuous embedding $\Gamma$ of $D_0$ into the

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domain of $X$ such that $\Gamma \circ \Phi_K^t = \Phi_X^t \circ \Gamma$. Denote by $A^u$ the space of all vector fields $Y(x, y) = (w, My + v)$, with $(w, v)$ a vector in $\mathbb{C}^d \times \mathbb{C}^m$ and $M$ a complex $m \times m$ matrix. In §2, we will introduce Banach spaces $\mathcal{A}_\varrho(\mathcal{V})$ of analytic vector fields on $D_\varrho$ with frequency module in $\mathcal{V}$, as well as a projection operator $\mathbb{P}$ from $\mathcal{A}_\varrho(\mathcal{V})$ onto $A^u$. The subspace of functions in $\mathcal{A}_\varrho(\mathcal{V})$ that do not depend on the coordinate $y \in \mathbb{C}^m$ will be denoted by $\mathcal{A}_\varrho^0(\mathcal{V})$. A function will be called 'real' if it takes real values for real arguments. Our main result is the following.

THEOREM 1.1. *Assume that $\omega$ satisfies the Brjuno condition (1.1). Then there exists a sequence of scaling matrices $T_n$ and a corresponding sequence of RG transformations $\mathcal{R}_n$ of the form (1.3) such that the following holds. Define $\mathcal{V}_n = T_n \mathcal{V}_{n-1}$ for $n = 1, 2, \ldots$. Then $\mathcal{R}_n$ is an analytic map from some open neighborhood $\mathcal{D}_{n-1}$ of $K$ in $\mathcal{A}_\varrho(\mathcal{V}_{n-1})$ to $\mathcal{A}_\varrho(\mathcal{V}_n)$. The set $\mathcal{W}$ of infinitely renormalizable vector fields $X_0$ in $\mathcal{D}_0$, characterized by the property that $X_n = \mathcal{R}_n(X_{n-1})$ belongs to $\mathcal{D}_n$ for $n = 1, 2, \ldots$, is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$ that satisfies $W(0) = K$ and $DW(0) = 0$. If $\rho > \varrho + \delta$ with $\delta > 0$, then every vector field $X \in \mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$ has an elliptic invariant torus $\Gamma_X \in \mathcal{A}_\delta^0(\mathcal{V}_0)$ with frequency vector $\omega$. The map $X \mapsto \Gamma_X$ is real analytic on $\mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$.*

The bounds obtained in the proof of this theorem are uniform within classes of Brjuno vectors $\mathcal{B}(\Omega')$ described at the end of §3. In addition, we obtain results analogous to those in [**17**], concerning the restriction of $W$ to special types of vector fields (Hamiltonian, reversible, divergence free, symmetric) and the reduction of the number of parameters via non-degeneracy conditions. Since the proofs are completely analogous as well, we refer to [**17**] for details.

The main new aspect in this paper is the choice of the scaling matrices $T_n$ and the control of the corresponding sequence of RG transformations $\mathcal{R}_n$. The choice of the coordinate change $\mathcal{U}_X$ is determined by the same considerations as in earlier work [**8–20**]. Its role is to compensate for the loss of analyticity that results from the scaling $X \mapsto \mathcal{T}_\mu^* X$. In this step, we use a normal form theorem proved in [**17**]. Thus, controlling a single RG step is quite simple; see §2. In §3, we define the matrices $T_n$ and give estimates on the transformations $\mathcal{R}_n$; then we apply a stable manifold theorem for sequences of maps [**17**] to obtain the manifold $\mathcal{W}$ described in Theorem 1.1. The construction of invariant tori for vector fields $X \in \mathcal{W}$ is described in §4.

## 2. *A single renormalization step*

As mentioned in the introduction, we work with coordinates where the frequency vector is $\omega = (1, 0, \ldots, 0)$. The torus considered in this section is $\mathbb{T}^d = \mathbb{R}^d / \mathcal{Z}$, where $\mathcal{Z}$ is some simple lattice in $\mathbb{R}^d$. The dual lattice will be denoted by $\mathcal{V}$.

Unless specified otherwise, our norm on $\mathbb{C}^n$ is $\|v\| = \sup_j |v_j|$. Another norm that will be used is $|v| = \sum_j |v_j|$. For linear operators between normed linear spaces, we will always use the operator norm unless stated otherwise. Denote by $D_\rho$ the set of all vectors $(x, y)$ in $\mathbb{C}^d \times \mathbb{C}^m$ characterized by $\|\mathrm{Im}\, x\| < \rho$ and $\|y\| < \rho$. Define $\mathcal{A}_\rho(\mathcal{V})$ to be the space of all analytic vector fields $X$ on $D_\rho$ with frequency module in $\mathcal{V}$ and having a finite

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norm

$$\|X\|_\rho = \sum_{v,\alpha} \|X_{v,\alpha}\| e^{\rho|v|}\rho^{|\alpha|}, \qquad X(x, y) = \sum_{v,\alpha} X_{v,\alpha} e^{iv\cdot x} y^\alpha, \tag{2.1}$$

where $v \cdot x = \sum_j v_j x_j$ and $y^\alpha = \prod_j y_j^{\alpha_j}$. The sums in (2.1) range over all $v \in \mathcal{V}$ and $\alpha \in \mathbb{N}^m$. In §4, we will also use functions with domain $D_0 = \mathbb{T}^d \times \{0\}$. We denote by $\mathcal{A}_0(\mathcal{V})$ the Banach space of continuous functions $F : D_0 \to \mathbb{C}^{d+m}$ with frequency module in $\mathcal{V}$ for which the norm $\|F\|_0 = \sum_v \|F_v\|$ is finite, where $\{F_v\}$ are the Fourier coefficients of $F$. Since the lattice $\mathcal{V}$ is fixed in this section, we will simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(\mathcal{V})$.

PROPOSITION 2.1. *Let $X \in \mathcal{A}_\rho$ and $Z \in \mathcal{A}_{\rho'}$, with $0 \le \rho' \le \rho$. Then:*
(a) $\|X(x, y)\| \le \|X\|_\rho$ *for all* $(x, y) \in D_\rho$*;*
(b) $(DX)Z \in \mathcal{A}_{\rho'}$ *and* $\|(DX)Z\|_{\rho'} \le (\rho - \rho')^{-1}\|X\|_\rho\|Z\|_{\rho'}$ *if* $\rho' < \rho$*;*
(c) $X \circ (\mathrm{I} + Z) \in \mathcal{A}_{\rho'}$ *and* $\|X \circ (\mathrm{I} + Z)\|_{\rho'} \le \|X\|_\rho$ *if* $\rho' + \|Z\|_{\rho'} \le \rho$.

*Proof.* Let us write $\mathcal{A}_\rho^{(n)}$ in place of $\mathcal{A}_\rho$ when describing vector fields (2.1) with $n$ components, and reserve the notation $\mathcal{A}_\rho$ for the case where $n = d + m$. Define $q(x, y) = x$ and $p(x, y) = y$. The functions $e^{iv\cdot q}p^\alpha$, with $v \in \mathcal{V}$, will be referred to as monomials. They allow us to rewrite (2.1), for arbitrary $n$ and $r > 0$, as

$$\|X\|_r = \sum_{v,\alpha} \|X_{v,\alpha}\| \|e^{iv\cdot q}p^\alpha\|_r, \qquad X = \sum_{v,\alpha} X_{v,\alpha} e^{iv\cdot q}p^\alpha. \tag{2.2}$$

The inequality (a) now follows from the fact that the same inequality holds for monomials.

For part (c) we use the fact that $\mathcal{A}_r^{(1)}$ is a Banach algebra: $\|fg\|_r \le \|f\|_r\|g\|_r$ holds for monomials and thus, more generally,

$$\begin{aligned} \|fg\|_r &\le \sum_{v,\alpha,w,\beta} |f_{v,\alpha}g_{w,\beta}| \, \|e^{iv\cdot q}p^\alpha e^{iw\cdot q}p^\beta\|_r \\ &\le \sum_{v,\alpha,w,\beta} |f_{v,\alpha}||g_{w,\beta}|\|e^{iv\cdot q}p^\alpha\|_r\|e^{iw\cdot q}p^\beta\|_r = \|f\|_r\|g\|_r. \end{aligned} \tag{2.3}$$

In particular, if $P \in \mathcal{A}_r^{(m)}$ with $\|P\|_r \le \delta$, then

$$\|(p + P)^\alpha\|_r = \left\| \prod_j (p_j + P_j)^{\alpha_j} \right\|_r \le \prod_j (r + \delta)^{\alpha_j} = (r + \delta)^{|\alpha|}. \tag{2.4}$$

For $Q \in \mathcal{A}_r^{(d)}$ with $\|Q\|_r \le \delta$, we have

$$\|e^{iv\cdot(q+Q)}\|_r \le \|e^{iv\cdot q}\|_r \sum_k \frac{1}{k!}\|v \cdot Q\|_r^k \le e^{r|v|}e^{|v|\|Q\|_r} \le e^{(r+\delta)|v|}. \tag{2.5}$$

Setting $Z = (Q, P)$ and applying the last two inequalities to the corresponding terms in

$$X_j \circ (\mathrm{I} + Z) = \sum_{v,\alpha} X_{v,\alpha,j} \, e^{iv\cdot(q+Q)}(p + P)^\alpha \tag{2.6}$$

gives the bound in (c). Here, and in what follows, $r = \rho'$ and $r + \delta = \rho$.

Claim (b) is proved by using a Cauchy bound. We may assume that $Z \ne 0$. Let $0 < R < \delta\|Z\|_r^{-1}$. Since $s \mapsto X \circ (\mathrm{I} + sZ)$ is analytic in an open neighborhood of the the disk $|s| \le R$, we can represent $(DX)Z$ as

$$(DX)Z = \frac{d}{ds} X \circ (\mathrm{I} + sZ)\bigg|_{s=0} = \frac{1}{2\pi i}\int_\gamma X \circ (\mathrm{I} + sZ)\,\frac{ds}{s^2}, \tag{2.7}$$

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where $\gamma$ is the positively oriented circle $|s| = R$. Thus, using the fact that on $\gamma$ we hve $\|X \circ (\mathrm{I} + sZ)\|_r \leq \|X\|_\rho$, we find that $\|(DX)Z\|_\rho \leq R^{-1}\|X\|_\rho$. The claim (b) is now obtained by taking $R \to \delta\|Z\|_r^{-1}$. □

In what follows, we assume that $\rho > 0$ unless specified otherwise.

Suppose that the components of $\omega$ are rationally independent with respect to $\mathcal{V}$, in the sense that the first component $v_\parallel$ of any non-zero vector $v \in \mathcal{V}$ is non-zero. Then, given any $L \geq 1$, we can find $\ell > 0$ such that

$$|v_\perp| > L \quad \text{or} \quad |v_\parallel| \geq \ell \quad \text{for all } v \in \mathcal{V} \setminus \{0\}. \tag{2.8}$$

In other words, all points in $\mathcal{V}$ except the origin lie outside the rectangle $|v_\perp| \leq L$ and $|v_\parallel| < \ell$. Notice that the scaling (1.4) shrinks the length $L$ of the excluded rectangle, while expanding its width $\ell$. In what follows, the parameters $L$, $\ell$, $\eta$ and $\beta$ are assumed to be given, subject to the conditions (1.4) and (2.8).

*Definition 2.2.* Denote by $S$ the generator of the one-parameter group of scalings $\mu \mapsto \mathcal{S}_\mu^*$ defined by $\mathcal{S}_\mu(x, y) = (x, \mu y)$. Given any subset $J$ of $I = \mathcal{V} \times \{-1, 0, 1, 2, \ldots\}$, define $P(J)$ to be the joint spectral projection in $\mathcal{A}_\rho(\mathcal{V})$ for the operators $(-i\nabla_x, S)$ associated with the eigenvalues $(v, k)$ in $J$. Let $\tau = (1 + \beta)/2$. For $\gamma \geq 1$, to be chosen later, let $I^+$ be the set of pairs $(v, k) \in I$ satisfying $|Tv| \leq \tau|v|$ or $|Tv| \leq \tau(k - \gamma)$, and let $I^-$ be the complement of $I^+$ in $I$. Define $\mathbb{I}^\pm = P(I^\pm)$. The *resonant* and *non-resonant* parts of a vector field $X \in \mathcal{A}_\rho$ are defined as $\mathbb{I}^+X$ and $\mathbb{I}^-X$, respectively. In addition, we define $\mathbb{E}_k = P(\{(0, k)\})$ for each $k \geq -1$. The torus averaging operator is then given by $\mathbb{E} = \sum_k \mathbb{E}_k$.

As we will see later, the scaling $X \mapsto \mathcal{T}_\mu^* X$ is well-behaved when restricted to resonant vector fields. Thus, before applying this scaling, we try to perform a change of variables $X \mapsto \mathcal{U}_X^* X$ that eliminates the non-resonant part of $X$; [**17**, Theorem 5.2] shows that this is possible, provided that the problem can be solved to first order in the size of $X - K$. The equations for the map $\mathcal{U}_X$ and the vector field $Z = \mathbb{I}^- Z$ generating its first-order approximation $\Phi_Z^1$ are

$$\mathbb{I}^-(X + [Z, X]) = 0, \quad \mathbb{I}^- \mathcal{U}_X^* X = 0, \tag{2.9}$$

where $[Z, X] = (DX)Z - (DZ)X$. The following proposition is used to solve the first equation above. Given any positive real number $r$, denote by $\mathcal{A}_r'$ the set of vector fields $X \in \mathcal{A}_r$ whose first-order partial derivatives belong to $\mathcal{A}_r$. Assume that

$$2\sigma L < \ell, \quad \sigma = \tfrac{1}{2}(1 - \beta)\eta. \tag{2.10}$$

PROPOSITION 2.3. *If $r > 0$ and $Z \in \mathcal{A}_r'$ is non-resonant, then*

$$\|[Z, K]\|_r \geq \sigma\|Z\|_r, \quad \|[Z, K]\|_r \geq \frac{\sigma r}{\sigma r + r + \gamma + 2}\|DZ\|_r. \tag{2.11}$$

*Proof.* Assume that $(v, k)$ belongs to $I^-$. In particular, we have $|Tv| > \tau|v|$ or, equivalently, $\eta^{-1}|v_\parallel| + \beta|v_\perp| > \tau|v_\parallel| + \tau|v_\perp|$. This immediately implies that $|v_\parallel| > \sigma|v_\perp|$. Combining this with the condition $|Tv| > \tau(k - \gamma)$, we obtain

$$\sigma^{-1}|v_\parallel| = \tau^{-1}(\eta^{-1}|v_\parallel| + \beta\sigma^{-1}|v_\parallel|) > \tau^{-1}(\eta^{-1}|v_\parallel| + \beta|v_\perp|) = \tau^{-1}|Tv| > k - \gamma.$$

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The inequality $|v_\parallel| > \sigma|v_\perp|$, together with (2.8) and (2.10), also implies that $|v_\parallel| > \sigma$. These bounds show that if $Z \in \mathbb{I}^-\mathcal{A}'_r$ and $Y = [Z, K]$, then $\|Z\|_r \leq \sigma^{-1}\|Y\|_r$ and

$$\sum_{j=2}^{d}\left\|\frac{\partial}{\partial x_j}Z\right\|_r \leq \frac{1}{\sigma}\|Y\|_r, \quad \sum_{j=1}^{m}\left\|\frac{\partial}{\partial y_j}Z\right\|_r \leq \frac{\gamma+2}{\sigma r}\|Y\|_r. \tag{2.12}$$

As a result we obtain (2.11). □

Using the bounds (2.11), we can now solve the linear (first) equation in (2.9) via Neumann series. The full (second) equation is then solved by a Nash–Moser-type iteration. The result is described in the lemma below.

Let $\varrho > 0$ be fixed once and for all. What we will call a 'universal constant' may depend on the choice of $\varrho$ but not on any other parameter.

LEMMA 2.4. *There exist universal constants $C_1$ and $C_2$ such that the following holds. Let $\rho' > 0$ and $\rho \geq \rho' + \sigma\varrho$. If $X$ is any vector field in $\mathcal{A}'_\rho$ that satisfies*

$$\|X - K\|'_\rho \leq C_1(\sigma/\gamma), \quad \|\mathbb{I}^- X\|_\rho \leq C_1(\sigma/\gamma)^2, \tag{2.13}$$

*then there exists a vector field $Z \in \mathbb{I}^-\mathcal{A}_\rho$ and a change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$ solving equation (2.9). The vector field $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$, and*

$$\begin{aligned} \|Z\|_\rho, \|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\leq C_2(\gamma/\sigma)\|\mathbb{I}^- X\|_\rho, \\ \|\mathcal{U}_X^* X - X\|_{\rho'} &\leq C_2(\rho-\rho')^{-1}(\gamma/\sigma)\|\mathbb{I}^- X\|_\rho, \\ \|\mathcal{U}_X^* X - X - [Z, X]\|_{\rho'} &\leq C_2(\rho-\rho')^{-3}(\gamma/\sigma)^3\|\mathbb{I}^- X\|_\rho^2. \end{aligned} \tag{2.14}$$

*The map $X \mapsto \mathcal{U}_X$ is continuous in the region defined by (2.13) and analytic in its interior.*

This lemma is a special case of the normal form theorem in [**17**, §5]. Its assumptions are precisely the bounds (2.11) and the composition estimate (c) in Proposition 2.1. By construction, the map $X \mapsto \mathcal{U}_X^* X$ preserves certain types of vector fields (Hamiltonian, reversible, divergence free, symmetric). This is shown in [**17**, §5] for an application where $\mathcal{Z} = \mathbb{Z}^d$, but the same arguments apply here as well.

Next, we assume that the scaling parameters $\eta$, $\beta$ and $\mu$ satisfy

$$\eta < 1/2, \quad e^{-\varrho((1-\beta)/6)L} \leq (4\mu)^{\gamma+1}, \quad 4\mu \leq e^{-\varrho}. \tag{2.15}$$

These bounds allow us to control the scaling of vector fields as follows.

LEMMA 2.5. *If $\varrho(2+\beta)/3 \leq \rho' \leq \varrho$, then $\mathcal{T}_\mu^*$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}(\mathcal{V})$ to $\mathcal{A}_\varrho(T\mathcal{V})$ with the property that*

$$\begin{aligned} \|\mathcal{T}_\mu^* \mathbb{E}_k X\|_\varrho &\leq 8\eta^{-1}(4\mu)^k\|\mathbb{E}_k X\|_{\rho'}, \\ \|\mathcal{T}_\mu^* \mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho &\leq 2\eta^{-1}(4e^\varrho\mu)^\gamma\|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_{\rho'}. \end{aligned} \tag{2.16}$$

*Proof.* By our choice of norm, (2.1), it suffices to verify the bounds for vector fields $X = P(J)Y$ where $J \subset I^+$ contains a single point, say $J = \{(v, k)\}$. Let $b = \varrho/(\rho'\beta)$. Then it follows essentially from the definitions that

$$\|\mathcal{T}_\mu^* P(J)Y\|_\varrho \leq 2\eta^{-1}e^A\|P(J)Y\|_{\rho'}, \quad A = \varrho|Tv| - \rho'|v| + k\ln(b\mu). \tag{2.17}$$

Setting $v = 0$ and using the fact that $1 < b < 4$ yields the first bound in (2.16).

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In order to prove the second bound, assume that $(v, k)$ belongs to $I^+$ and that $v \neq 0$. Consider first the case $|Tv| \leq \tau|v|$, which leads to $|v_\parallel| < 2\sigma|v_\perp|$ if we use the fact that $\eta\tau < 1/2$. This inequality excludes frequencies $v$ that satisfy $|v_\perp| \leq L$ and $|v_\parallel| \geq \ell$, owing to the condition (2.10). Thus, we must have $|v_\perp| > L$ by condition (2.8). Consequently,

$$A \leq -\varrho\left(\frac{\rho'}{\varrho} - \tau\right)|v| + k\ln(b\mu) \leq -\varrho\frac{1-\beta}{6}L + k\ln(b\mu), \tag{2.18}$$

and the second bound in (2.16) follows from using (2.15).

Next, consider the case $|Tv| \leq \tau(k-\gamma)$. Notice that $k > \gamma$ here, since $v$ is non-zero. By using the bound $A \leq \varrho(k-\gamma) + k\ln(b\mu)$ together with (2.17), we obtain

$$\|\mathcal{T}_\mu^* P(J)Y\|_\varrho \leq 2\eta^{-1}(be^\varrho\mu)^k\|P(J)Y\|_{\rho'}.$$

This again implies the second bound in (2.16). □

Upon combining the preceding two lemmas, we obtain the following theorem. Notice that, by property (2.14), the restriction of $\mathcal{R}$ to the subspace $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ defines a linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ to $\mathbb{P}\mathcal{A}_\varrho(T\mathcal{V})$. This operator will be denoted by $\mathcal{L}$.

THEOREM 2.6. *There exist universal constants $C, R > 0$ such that the following holds under the given assumptions on $L, \ell, \eta, \beta, \gamma$ and $\mu$. Let $B$ be the open ball in $\mathcal{A}_\varrho(\mathcal{V})$ of radius $R(\sigma/\gamma)^2$, centered at $K$. Then $\mathcal{R}$ is a bounded analytic map from $B$ to $\mathcal{A}_\varrho(T\mathcal{V})$ that satisfies $\|\mathcal{L}^{-1}\| \leq 1$ and*

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\leq \eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)(C\mu)^\gamma\|(\mathbb{I}-\mathbb{E})X\|_\varrho,\\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\|_\varrho &\leq C\eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)\mu\|(\mathbb{I}-\mathbb{P})X\|_\varrho,\\
\|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &\leq C\eta^{-2}(1-\beta)^{-3}(\gamma/\sigma)^3\mu^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2.
\end{aligned} \tag{2.19}$$

*Proof.* Let $\rho = \varrho - \varrho(1-\beta)/12$ and $\rho' = \rho - \varrho(1-\beta)/4$. Then there exists a universal constant $R > 0$ such that the conditions (2.13) in Lemma 2.4 hold whenever $X$ belongs to the domain $B$ defined by $\|X - K\|_\varrho < R(\sigma/\gamma)^2$. Here, we have used the fact that $\sigma < (1-\beta)/4$. By Lemma 2.5, we have

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^* X\|_\varrho\\
&\leq 2\eta^{-2}(4e^\rho\mu)^\gamma[\|(\mathbb{I}-\mathbb{E})X\|_{\rho'} + \|\mathcal{U}_X^* X - X\|_{\rho'}].
\end{aligned} \tag{2.20}$$

Using the bound in (2.14) on the norm of $\mathcal{U}_X^* X - X$ together with the fact that $\mathbb{I}^-\mathbb{E} = 0$, we obtain the first inequality in (2.19). Similarly, Lemma 2.5 implies that

$$\|\mathbb{E}_k\mathcal{R}(X)\|_\varrho \leq C_1\eta^{-2}\mu[\|\mathbb{E}_k X\|_{\rho'} + \|\mathbb{E}_k(\mathcal{U}_X^* X - X)\|_{\rho'}] \tag{2.21}$$

for all $k \geq 1$. Here, and in what follows, $C_1, C_2, \ldots$ denote positive universal constants. Summing over $k \geq 1$ to get a bound on $\|(\mathbb{E}-\mathbb{P})\mathcal{R}(X)\|_\varrho$, and then adding (2.20), yields a bound analogous to (2.21) but with $\mathbb{E}_k$ replaced by $\mathbb{I}-\mathbb{P}$. Using again the bound in (2.14) on $\mathcal{U}_X^* X - X$ and the fact that $\mathbb{I}^-\mathbb{P} = 0$, we obtain the second inequality in (2.19).

By Lemma 2.5, we also have a bound

$$\begin{aligned}
\|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*\mathbb{E}(\mathcal{U}_X^* X - X)\|_\varrho\\
&\leq 2\eta^{-2}\mu^{-1}\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'}.
\end{aligned} \tag{2.22}$$

Using Lemma 2.4, the norm on the right-hand side of (2.22) can be estimated as follows:

$$\|\mathbb{E}(\mathcal{U}_X^* X - X)\|_{\rho'} \leq C_2(1-\beta)^{-3}(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_\rho^2 + \|\mathbb{E}[Z, X]\|_{\rho'}, \tag{2.23}$$

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where $Z = \mathbb{I}^{-}Z$ is the vector field described in Lemma 2.4. Since $\mathbb{E}Z = 0$, we have $\mathbb{E}[Z, \mathbb{E}X] = 0$. As a result,

$$\begin{aligned}\|\mathbb{E}[Z, X]\|_{\rho'} &= \|\mathbb{E}[Z, (\mathbb{I}-\mathbb{E})X]\|_{\rho'} \le C_3(1-\beta)^{-1}\|Z\|_\rho \|(\mathbb{I}-\mathbb{E})X\|_\rho \\ &\le C_4(1-\beta)^{-1}(\gamma/\sigma)\|(\mathbb{I}-\mathbb{E})X\|_\rho^2. \end{aligned} \tag{2.24}$$

Here, we have used Proposition 2.1 and the bound on $\|Z\|_\rho$ from Lemma 2.4. Combining the last three equations yields the third inequality in (2.19).

In order to bound the inverse of $\mathcal{L}$, let $Y$ be a vector field in $\mathbb{P}\mathcal{A}_\rho$. Then $Y$ can be written as $Y(x, y) = (w, My + v)$, and the last inequality in (2.19) now follows from the fact that

$$(\mathcal{L}^{-1}Y)(x, y) = \eta(Tw, My + \mu v). \tag{2.25}$$

Here we have used $T' = \mathrm{I}$, except (optionally) for the renormalization of purely Hamiltonian vector fields, where $M$ and $v$ are zero. □

## 3. *Iterated RG transformations*

Now let $\mathcal{V}_0$ be a simple lattice in $\mathbb{R}^d$ such that the Brjuno condition (1.1) holds if the frequencies $\nu$ are chosen from $\mathcal{V}_0$. Using the same set of frequencies, define

$$a_n = \sum_{k=n}^{\infty} 2^{n-k}[2^{-k-\kappa}\ln(1/\Omega'_{k+\kappa}) + (k+\kappa')^{-2}], \quad \Omega'_n = \min_{0<|\nu_\perp|<2^n} |\nu_\parallel|, \tag{3.1}$$

for all positive integers $n$. Here $\kappa, \kappa' > 2$ are two integer constants to be determined later. The Brjuno condition (1.1) is then equivalent to the condition that the resulting sequence $\{a_n\}$ is summable. We remark that the weighted sum has been included in (3.1) to limit the local growth of the sequence $\{a_n\}$, and the term $(k+\kappa')^{-2}$ has been included to avoid sequences $\{a_n\}$ that decrease too rapidly. This allows for a more uniform treatment of all Brjuno vectors.

Define $\lambda_0 = 1$ and

$$\lambda_n = 2^{-n-\kappa} e^{-2^{n+\kappa}a_n}, \quad \eta_n = \frac{\lambda_n}{\lambda_{n-1}}, \quad A_n = \sum_{k=n}^{\infty} a_k, \quad \beta_n = \frac{A_{n+1}}{A_n}, \tag{3.2}$$

for all positive integers $n$. Consider the corresponding scaling transformations

$$P_n(x) = \lambda_n^{-1}x_\parallel + A_1^{-1}A_{n+1}x_\perp, \quad T_n(x) = \eta_n^{-1}x_\parallel + \beta_n x_\perp. \tag{3.3}$$

Notice that $P_n = T_1T_2\cdots T_n$ by (3.2). These quantities will now be used to define the $n$th-step RG transformation $\mathcal{R} = \mathcal{R}_n$. To this end, we need to verify the assumptions made in §2. Clearly, $\beta = \beta_n$ is positive and less than one, since $n \mapsto A_n$ is a decreasing sequence; and the condition on $\eta = \eta_n$ in (2.15) follows from the fact that $a_n > a_{n-1}/2$ for $n > 1$ and that $\lambda_1 < 1/2$.

The geometric data $\mathcal{V}$, $L$ and $\ell$ used in step $n$ are

$$\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0, \quad L_{n-1} = A_1^{-1}A_n 2^{n+\kappa}, \quad \ell_{n-1} = 2^{n+\kappa}\eta_n. \tag{3.4}$$

These definitions immediately imply (2.10). The following proposition shows that the condition (2.8) holds for all $v \in \mathcal{V}$.

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PROPOSITION 3.1. *If $v \in \mathcal{V}_{n-1}$ is non-zero, then either $|v_\parallel| \geq \ell_{n-1}$ or $|v_\perp| > L_{n-1}$.*

*Proof.* Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_\perp| \leq L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1} v$ in $\mathcal{V}_0$ satisfies $|\nu_\perp| \leq A_1 A_n^{-1} L_{n-1} = 2^{n+\kappa}$, and thus $|\nu_\parallel| \geq \Omega'_{n+\kappa}$ by (3.1). Since we have $\lambda_n < 2^{-n-\kappa}\Omega'_{n+\kappa}$, this yields

$$|v_\parallel| = \lambda_{n-1}^{-1}|\nu_\parallel| \geq \eta_n \lambda_n^{-1}\Omega'_{n+\kappa} > \eta_n 2^{n+\kappa} = \ell_{n-1}, \tag{3.5}$$

as claimed. □

The second condition in (2.15) is satisfied simply by choosing $\mu = \mu_n$, where

$$\mu_k = \exp\left\{-\frac{\varrho}{6} \cdot \frac{1-\beta_k}{\gamma+1} L_{k-1}\right\} = \exp\left\{-\frac{\varrho}{6(\gamma+1)A_1} \cdot 2^{k+\kappa} a_k\right\}, \quad k \geq 1. \tag{3.6}$$

Finally, the third inequality in (2.15) is taken care of by choosing $\kappa'$ and $\kappa$ sufficiently large, as the following proposition shows.

PROPOSITION 3.2. *For all $k \geq 1$, $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$. Furthermore, given $\gamma \geq 1$ and $C, N > 0$, if $\kappa'$ and then $\kappa$ are chosen sufficiently large, then for all $k \geq 1$ we have*

$$\mu_k \leq C e^{-N 2^{k+\kappa} a_k}, \quad \mu_k \leq C \eta_k^N, \quad \mu_k \leq C(1-\beta_k)^N. \tag{3.7}$$

*Proof.* The inequality $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$ follows from the fact that $a_{k+1}/2 < a_k < 2a_{k+1}$. Let $c = \varrho/(6(\gamma+1))$. By choosing $\kappa$ and $\kappa'$ sufficiently large, we have $c/A_1 \geq 2N$. Keeping $\kappa'$ fixed and increasing $\kappa$ further, if necessary, we obtain the first two bounds in (3.7) by using the fact that $2^{k+\kappa} a_k \geq 2^{k+\kappa}(k+\kappa')^{-1} \geq c' 2^\kappa k$ for some constant $c' > 0$. The same inequality, together with $1 - \beta_k = a_k/A_k > (k+\kappa')^{-2} 2N/c$, implies the third bound in (3.7). □

Having verified all of the assumptions made in §2, we can now apply Theorem 2.6 to the $n$th-step RG transformation $\mathcal{R}_n$ defined by the parameters introduced above. Denote by $\mathcal{L}_n$ the corresponding linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_{n-1})$ to $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_n)$.

Define $\mathcal{A}_{\varrho,k} = \mathcal{A}_\varrho(\mathcal{V}_k)$, for all non-negative integers $k$. To simplify notation, the norm in $\mathcal{A}_{\varrho,k}$ and the projections $\mathbb{E}$ and $\mathbb{P}$ on this space will not be given indices. From Theorem 2.6 we immediately obtain the following result.

THEOREM 3.3. *Let $\gamma \geq 1$. There exist constants $r, C > 0$ such that the following holds for every positive integer $n$. Let $B_{n-1}$ be the open ball in $\mathcal{A}_{\varrho,n-1}$ of radius $r\sigma_n^2$, centered at $K$, where $\sigma_n = \frac{1}{2}(1-\beta_n)\eta_n$. Then $\mathcal{R}_n$ is a bounded analytic map from $B_{n-1}$ to $\mathcal{A}_{\varrho,n}$ that satisfies $\|\mathcal{L}_n^{-1}\| \leq 1$ and*

$$\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n^\gamma \|(\mathbb{I}-\mathbb{E})X\|_\varrho, \\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n \|(\mathbb{I}-\mathbb{P})X\|_\varrho, \\
\|\mathbb{E}\mathcal{R}_n(X) - \mathcal{R}_n(\mathbb{E}X)\|_\varrho &\leq C\sigma_n^{-6}\mu_n^{-1} \|(\mathbb{I}-\mathbb{E})X\|_\varrho^2.
\end{aligned} \tag{3.8}$$

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathcal{R}_n$ is a subset of the ball $B_{n-1}$ described in Theorem 3.3 which is open in $\mathcal{A}_{\varrho,n-1}$ and contains the vector field $K$. Given a domain $\mathcal{D}_{n-1}$ for each $\mathcal{R}_n$, the domain $\widetilde{\mathcal{D}}_n$ of the combined RG transformation $\widetilde{\mathcal{R}}_{n+1} = \mathcal{R}_{n+1} \circ \mathcal{R}_n \circ \ldots \circ \mathcal{R}_1$ is defined recursively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_n$ that

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are mapped under $\widetilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 3.3, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

To prove the following result, we apply the stable manifold theorem for sequences of mappings, as given in [**17**, §6].

THEOREM 3.4. *Let $\gamma \geq 4$. If $\kappa'$ and then $\kappa$ are chosen sufficiently large, then there exist a sequence of domains $\mathcal{D}_0, \mathcal{D}_1, \ldots$ for the RG transformations $\mathcal{R}_1, \mathcal{R}_2, \ldots$ such that the set $\mathcal{W} = \bigcap_n \widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$ satisfying $W(0) = K$ and $DW(0) = 0$. For every $X \in \mathcal{W}$, if $n \geq 1$ and $\psi_n = \mu_1\mu_2 \cdots \mu_n$, then*

$$\begin{aligned} \|\widetilde{\mathcal{R}}_n(X) - K_n\|_\varrho &\leq \psi_n^{1/2}\|(\mathbb{I} - \mathbb{P})X\|_\varrho, \\ \|\mathbb{P}[\widetilde{\mathcal{R}}_n(X) - K_n]\|_\varrho &\leq \psi_n\|(\mathbb{I} - \mathbb{P})X\|_\varrho^2, \\ \|(\mathbb{I} - \mathbb{E})\widetilde{\mathcal{R}}_n(X)\|_\varrho &\leq \psi_n^{\gamma-1/2}\|(\mathbb{I} - \mathbb{E})X\|_\varrho. \end{aligned} \tag{3.9}$$

*Proof.* We start by rescaling the transformations $\mathcal{R}_n$. Let $r_n = r_{n-1}\sigma_{n+1}^2$ for every positive integer $n$, with $r_0 > 0$ smaller than half the constant $r$ from Theorem 3.3.

Consider the transformations $R_1, R_2, \ldots$ given by the equation

$$R_n(Z) = r_n^{-1}[\mathcal{R}_n(K + r_{n-1}Z) - K], \quad n = 1, 2, \ldots. \tag{3.10}$$

The restriction $R_n\mathbb{P}$ defines a linear map from $\mathbb{P}\mathcal{A}_{\varrho,n-1}$ to $\mathbb{P}\mathcal{A}_{\varrho,n}$, which will be denoted by $L_n$. By Theorem 3.3, $R_n$ is analytic and bounded on the ball $\|Z\|_\varrho < 2$, and it satisfies

$$\begin{aligned} \|(\mathbb{I} - \mathbb{E})R_n(Z)\|_\varrho &\leq \varepsilon_n\|(\mathbb{I} - \mathbb{E})Z\|_\varrho, \\ \|(\mathbb{I} - \mathbb{P})R_n(Z)\|_\varrho &\leq \vartheta_n\|(\mathbb{I} - \mathbb{P})Z\|_\varrho, \\ \|\mathbb{P}R_n(Z) - R_n'(\mathbb{P}Z)\|_\varrho &\leq \varphi_n\|(\mathbb{I} - \mathbb{E})Z\|_\varrho^2, \end{aligned} \tag{3.11}$$

where

$$\varepsilon_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^\gamma, \quad \vartheta_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n, \quad \varphi_n = C\sigma_n^{-6}\sigma_{n+1}^{-2}\mu_n^{-1}. \tag{3.12}$$

Here, $C \geq 1$ is a constant that may depend on $\gamma$ but not on any other RG parameters. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict $R_n$ to the domain $D_{n-1} \subset \mathcal{A}_{\varrho,n-1}$ defined by

$$\|\mathbb{P}Z\|_\varrho < 1, \quad \|(\mathbb{I} - \mathbb{P})Z\|_\varrho < 1, \quad \|(\mathbb{I} - \mathbb{E})Z\|_\varrho < \delta_{n-1}, \tag{3.13}$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Proposition 3.2, if $\kappa'$ and $\kappa$ are chosen sufficiently large, then $C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^{1/2} \leq 1/6$ and $C\mu_n^{\gamma-3} \leq \sigma_{n+1}^6\sigma_{n+2}^2$ for all positive integers $n$. These inequalities imply that

$$\varepsilon_n \leq \mu_n^{\gamma-1/2}/6, \quad \vartheta_n \leq \mu_n^{1/2}/4, \quad \varepsilon_n\delta_{n-1} \leq \delta_n, \tag{3.14}$$

for all $n \geq 1$. The hypotheses of [**17**, Theorem 6.1] are now verified, with $\vartheta = 1/4$, and the conclusions of this theorem imply the statements in Theorem 3.4. □

We note that the 'min' in (3.1) could be replaced by 'a lower bound', as long as $n \mapsto \Omega_n'$ is a non-increasing sequence of positive real numbers that converges to zero and the corresponding numbers $a_n$ are summable. Our estimates are then uniform in the class $\mathcal{B}(\Omega')$ of vectors $\omega \in \mathbb{R}^d$ that admit the same sequence $n \mapsto \Omega_n'$ of lower bounds.

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## 4. *Invariant tori*

Our construction of invariant tori follows closely the ideas used in [**9, 11, 14, 17**].

Consider the RG transformation $\mathcal{R}$ defined in §2 and a vector field $X$ in the domain of $\mathcal{R}$. For any map $F$ from $D_0$ into the domain of $\Lambda_X = \mathcal{U}_X \circ \mathcal{T}_\mu$, define

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}_\mu^{-1}. \tag{4.1}$$

Formally, if $\widetilde{\Gamma}$ is an invariant torus for $\mathcal{R}(X)$, then $\Gamma = \mathcal{M}_X(\widetilde{\Gamma})$ is an invariant torus for $X$. This can be seen easily from the identity $\Lambda_X \circ \Phi^{\eta t}_{\mathcal{R}(X)} = \Phi^t_X \circ \Lambda_X$. In order to make such identities more precise, we estimate the difference between the flow for $X$ and the flow for the constant vector field $K = (\omega, 0)$.

PROPOSITION 4.1. *Let $\tau$ be a positive real number and $X$ a vector field in $\mathcal{A}_\varrho$ such that $\tau\|X - K\|_\varrho < r < \varrho$. Then for all times $t$ in the interval $[-\tau, \tau]$,*

$$\|\Phi^t_X - \Phi^t_K\|_{\varrho - r} \le \|t(X - K)\|_\varrho. \tag{4.2}$$

The proof of this proposition follows standard arguments, using the contraction mapping principle applied to the integral equation

$$Y(t_2) = Y(t_1) + \int_{t_1}^{t_2} [(X - K) \circ \Phi^t_K] \circ [\mathrm{I} + Y(t)]\, dt \tag{4.3}$$

for the difference $Y(t) = \Phi^t_X - \Phi^t_K$. Notice that $\Phi^t_K$ is an isometry; the domain loss in Proposition 4.1 comes from the composition with $\mathrm{I} + Y(t)$, by using Proposition 2.1(c).

Consider now a fixed but arbitrary vector field $X$ on the stable manifold $\mathcal{W}$ described in Theorem 3.4. Let $X_0 = X$ and $X_n = \mathcal{R}_n(X_{n-1})$ for $n \ge 1$. In order to simplify notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions $\Gamma_k \in \mathcal{A}_0(\mathcal{V}_k)$ satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_{\mu_n}^{-1}, \quad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{T}_{\mu_n}, \tag{4.4}$$

for all $n > 0$. Then we will show that $\Gamma_0$ is an invariant torus for $X_0$.

For every $n \ge 0$, define $\mathcal{B}_n$ to be the vector space $\mathcal{A}_0(\mathcal{V}_n)$ equipped with the norm

$$\|f\|'_n = r_n^{-1}\|f\|_0 = r_n^{-1} \sum_{v \in \mathcal{V}_n} \|f_v\|, \quad r_n = \psi_n^{1/3}, \tag{4.5}$$

where $\psi_0 = 1$. Denote by $B_n$ the unit ball in $\mathcal{B}_n$, centered at the identity function I.

PROPOSITION 4.2. *Let $\gamma \ge 5$. If $\kappa'$ and then $\kappa$ are chosen sufficiently large, then there exists an open neighborhood $B$ of $K$ in $\mathcal{A}_\varrho$ such that for every $X \in \mathcal{W} \cap B$ and every $n \ge 1$, the map $\mathcal{M}_n$ is well-defined and analytic as a function from $B_n$ to $\mathcal{B}_{n-1}$. Furthermore, $\mathcal{M}_n$ takes values in $B_{n-1}/2$, and $\|D\mathcal{M}_n(F)\| \le \mu_n^{1/4}$ for all $F \in B_n$.*

*Proof.* Clearly, $\mathcal{M}_n$ is well-defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g), \quad g = \mathcal{T}_{\mu_n} \circ f \circ \mathcal{T}_{\mu_n}^{-1}, \tag{4.6}$$

where $f = F - \mathrm{I}$. To estimate $\mathcal{U}_{n-1} - \mathrm{I}$ we can apply Lemma 2.4, with $\rho'$ equal to $\varrho - \varrho(1 - \beta_n)/3$, as in the proof of Theorem 2.6. We will use Proposition 3.2 and assume

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that $\kappa'$ and then $\kappa$ have been chosen sufficiently large, without always mentioning this. By Lemma 2.4 and Theorem 3.4, there exists a constant $C > 0$ such that

$$\begin{aligned}\|\mathcal{U}_{n-1} - \mathrm{I}\|_{\rho'} &\le C\sigma_n^{-1}\|\mathbb{I}^- X_{n-1}\|_{\varrho} \le C\sigma_n^{-1}\psi_{n-1}^{\gamma-1/2}\|(\mathbb{I}-\mathbb{E})X\|_{\varrho}\\ &\le \psi_{n-1}^{\gamma-1}\|(\mathbb{I}-\mathbb{E})X\|_{\varrho} \le \psi_n^{3/4},\end{aligned} \tag{4.7}$$

for all $n > 1$ and all $X \in \mathcal{W} \cap B$, provided that the neighborhood $B$ of $K$ has been chosen sufficiently small (depending on $\kappa'$ and $\kappa$). The first inequality in (4.7) and the final bound also hold for $n = 1$.

The composition with $\mathrm{I} + g$ in equation (4.6) is controlled by Proposition 2.1, using the fact that $\|g\|_0 \le \eta_n^{-1} r_n \|f\|'_n$ is less than $\varrho/2$. Here, and in what follows, we assume that $F \in B_n$. By using $r_n/r_{n-1} = \mu_n^{1/3}$, we obtain that $\|g\|'_{n-1} \le \eta_n^{-1}\mu_n^{1/3} \le \mu_n^{2/7}$. When combined with (4.7), this shows that $\mathcal{M}_{n-1}$ maps $B_n$ into $B_{n-1}/2$. Using now $\rho' = \varrho/2$, we obtain a bound analogous to (4.7) for the derivative of $\mathcal{U}_{n-1}$. This, together with the fact that the inclusion map from $B_n$ into $B_{n-1}$ is bounded in norm by $\mu_n^{1/3}$, shows that $\|D\mathcal{M}_n(F)\| \le \mu_n^{1/4}$ for all $n \ge 1$ and all $F \in B_n$. □

Denote by $\Phi_n$ and $\Phi_\infty$ the flows for the vector fields $X_n$ and $K$, respectively. In order to prove that a solution to (4.4) yields an invariant torus $\Gamma_0$ for $X$, we will use the identity

$$\Phi_{n-1}^t \circ \mathcal{M}_n(F) \circ \Phi_\infty^{-t} = \mathcal{M}_n(\Phi_n^{\eta_n t} \circ F \circ \Phi_\infty^{-\eta_n t}), \tag{4.8}$$

which follows from the relation described after (4.1) between the flow for a vector field and the flow for the corresponding renormalized vector field. This requires an estimate of the following type.

PROPOSITION 4.3. *Under the same assumptions as in Proposition 4.2, there exists an open neighborhood $B$ of $K$ in $\mathcal{A}_\varrho$ such that for every $X \in \mathcal{W} \cap B$ and every $n \ge 1$, the function $\Phi_n^s \circ F \circ \Phi_\infty^{-s}$ belongs to $B_n$ whenever $F \in B_n/2$ and $|s| \le \psi_n^{-1/6}$.*

*Proof.* We will use the identity

$$\Phi_n^s \circ F \circ \Phi_\infty^{-s} = \mathrm{I} + f \circ \Phi_\infty^{-s} + [\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}] \circ (\mathrm{I} + f \circ \Phi_\infty^{-s}). \tag{4.9}$$

By Proposition 4.1 and Theorem 3.4, we have the bound

$$\|\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}\|_{\varrho/2} \le \|s(X_n - K)\|_\rho \le C\psi_n^{1/3}\|(\mathbb{I}-\mathbb{P})X\|_\varrho, \tag{4.10}$$

provided, for instance, that the right-hand side of this inequality is less than $\varrho/2$. This is certainly the case, for any $n$, if $\|X - K\|_\varrho$ is sufficiently small. The composition by $\mathrm{I} + f \circ \Phi_\infty^{-s}$ in equation (4.9) is controlled in the same way as the composition by $\mathrm{I} + g$ in the proof of Proposition 4.2, using also that $\|f \circ \Phi_\infty^{-s}\|_0 = \|f\|_0$. As a result, the third term on the right-hand side of (4.9) belongs to $\mathcal{B}_n$ and is bounded in norm by $C\|X - K\|_\varrho$, which is less than $1/2$ for any $n \ge 1$ if $X$ is sufficiently close to $K$. □

Now we are ready to construct invariant tori. A function $f$ defined on $\mathcal{W}$ is said to be analytic if $f \circ W$ is analytic on the domain of $W$.

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THEOREM 4.4. *Under the same assumptions as in Proposition 4.2, there exists an open neighborhood $B$ of $K$ in $\mathcal{A}_\varrho$ such that the following holds. Given any $X \in \mathcal{W} \cap B$ and any sequence of functions $F_k \in B_k$, define*

$$\Gamma_{n,k} = (\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_k)(F_k), \quad 0 \le n < k. \tag{4.11}$$

*The limits $\Gamma_n = \lim_{k\to\infty} \Gamma_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots,$ and satisfy the identities (4.4). Furthermore, $\Gamma_0$ is an elliptic invariant torus for $X$, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $\mathcal{W} \cap B$.*

*Proof.* By Propositions 4.2 and 3.2, the map $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least 1/2. Thus, if $1 \le n < k < k'$, then the difference $\Gamma_{n,k'} - \Gamma_{n,k}$ is bounded in norm by $2^{n-k+1}$. This shows that the sequence $k \mapsto \Gamma_{n,k}$ converges in $\mathcal{B}_n$ to a limit $\Gamma_n$, which is independent of the choice of the functions $F_k$. By choosing $F_k = \Gamma_k$ for all $k$, we obtain the identities (4.4). The analyticity of $X \mapsto \Gamma_0$ follows via the chain rule from the analyticity of the maps used in our construction and from uniform convergence.

In order to prove that $\Gamma_0$ is an invariant torus for $X$, we will use the identity (4.8). To be more precise, given a real number $-1 < t < 1$, define $t_n = \lambda_n t$ for all $n \ge 0$. By using the fact that $\lambda_n \le \psi_n^{-1/6}$ independently of $n$, if $\kappa'$ and $\kappa$ have been chosen sufficiently large (which we assume), then Proposition 4.3 allows us to iterate (4.8) to get the identity

$$\Phi_0^t \circ \Gamma_{0,k} \circ \Phi_\infty^{-t} = (\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_k)(\Phi_k^{t_k} \circ \Phi_\infty^{-t_k}), \tag{4.12}$$

for all $k > 0$. As proved above, the right-hand (and thus left-hand) side of this equation converges in $\mathcal{A}_0$ to $\Gamma_0$. In addition, $\Gamma_{0,k} \to \Gamma_0$ in $\mathcal{A}_0$, and the convergence is pointwise as well by part (a) of Proposition 2.1. Thus, since the flow $\Phi_0^t$ is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Phi_\infty^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$, owing to the group property of the flow and the fact that composition with $\Phi_\infty^s$ is an isometry on $\mathcal{A}_0$.

Finally, notice that $\lambda_n \|DX_n\|_{\varrho/2}$ is an upper bound on the modulus of the Lyapunov exponent for the flow of $\lambda_n X_n$ on the range of $\Gamma_n$. Since $X_0$ is obtained from $\lambda_n X_n$ via a change of variables, and $\Gamma_0$ is the corresponding invariant torus for $X_0$, the same upper bound applies to the flow for $X_0$ on the torus $\Gamma_0$. But, by Theorem 3.4, $\lambda_n \|DX_n\|_{\varrho/2} \to 0$ as $n \to \infty$. This shows that the torus $\Gamma_0$ is elliptic. □

In what follows, the torus $\Gamma_0$ associated with a vector field $X \in \mathcal{W}$ will be denoted by $\Gamma_X$. For convenience, we extend the map $X \mapsto \Gamma_X$ to an open neighborhood of $K$ by setting $\Gamma_X = \Gamma_{X'}$, where $X' = (\mathbb{I} + W)(X - \mathbb{P}X)$.

THEOREM 4.5. *Let $\rho > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Proposition 4.2, there exists an open neighborhood $B$ of $K$ in $\mathcal{A}_\rho(\mathcal{V}_0)$ such that $\Gamma_X$ has an analytic continuation to $\|\mathrm{Im}\, x\| < \delta$ for each $X \in B$. With this continuation, $X \mapsto \Gamma_X$ defines a bounded analytic map from $B$ to $\mathcal{A}_\delta^0(\mathcal{V}_0)$.*

*Proof.* Following [**17**], consider translations $R_u(x, y) = (x + u, y)$ with $u \in \mathbb{R}^d$. If $X$ is a vector field on one of the domains $D_r$, denote by $R_u^* X$ the pullback of $X$ under $R_u$. For functions $F : D_0 \to D_r$ we define $R_u^* F = R_u^{-1} \circ F \circ R_u$. An explicit computation shows that the RG transformation $\mathcal{R}$ and the maps $\mathcal{M}_X$ defined in (4.1) satisfy

$$\mathcal{R} \circ R_u^* = R_{T^{-1}u}^* \circ \mathcal{R}, \quad \mathcal{M}_{R_u^* X} = R_u^* \circ \mathcal{M}_X \circ (R_{T^{-1}u}^*)^{-1}. \tag{4.13}$$

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Here, we have used the fact that the translations $R_u^*$ are isometries on the spaces $\mathcal{A}_r(\mathcal{V})$ and that the domain of $\mathcal{R}$ is translation invariant. This also implies that the manifold $\mathcal{W}$ is invariant under translations $R_u^*$, which is used in the second identity in (4.13).

It is convenient to extend the function $X \mapsto \Gamma_X$ to an open neighborhood of $K$ in $\mathcal{A}_\varrho(\mathcal{V}_0)$, by projecting $X$ onto a point $X' \in \mathcal{W}$ and defining $\Gamma_X = \Gamma_{X'}$. More specifically, we take $X' = (\mathbb{I} + W)((\mathbb{I} - \mathbb{P})X)$ where $W$ is the map defining $\mathcal{W}$, as described in Theorem 3.4. If restricted to a sufficiently small open ball $B \subset \mathcal{A}_\varrho(\mathcal{V}_0)$ centered at $K$, the map $X \mapsto \Gamma_X$ is now analytic and bounded on all of $B$.

The construction of $\Gamma_0$ in the proof of Theorem 4.4, together with the identities (4.13) and the invariance property $W = W \circ R_u^*$, shows that $\Gamma_{R_u^* X} = R_u^* \Gamma_X$ for all $X \in B$. Thus, if $u \in \mathbb{R}^d$, then

$$\Gamma_X(u, 0) = (R_u \circ \Gamma_{R_u^* X})(0, 0), \qquad X \in B. \tag{4.14}$$

The idea now is to extend the right-hand side of (4.14) analytically to complex $u$, by using the analyticity of $X \mapsto \Gamma_X$. To this end, choose an open neighborhood $B'$ of $K$ in $\mathcal{A}_\rho(\mathcal{V}_0)$ such that $R_u^* B' \subset B$ for all $u \in \mathbb{C}^d$ of norm $r = \rho - \varrho$ or less. Then the right-hand side of (4.14), regarded as a function of $(X, u)$, is analytic and bounded on the product of $B'$ with the strip $\|\operatorname{Im} u\| < r$. Denoting this function by $G$, we clearly have $G(X, \cdot) \in \mathcal{A}_\delta^0(\mathcal{V}_0)$ for all $X \in B'$. The analyticity of $X \mapsto G(X, \cdot)$ is obtained now by using, for instance, a contour integral formula for $(g(t) - g(0) - tg'(0))/t^2$ with $g(t) = G(X + tZ, \cdot)$. □

This theorem, together with Theorem 3.4, implies Theorem 1.1.

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