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A renormalization approach to lower-dimensional tori with Brjuno frequency vectors

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ABSTRACT

We extend the renormalization scheme for vector fields on $\mathbb{T}^d \times \mathbb{R}^m$ in order to construct lower-dimensional invariant tori with Brjuno frequency vectors for near-integrable Hamiltonian flows. For every Brjuno frequency vector $\omega \in \mathbb{R}^d$ and every vector $\Omega \in \mathbb{R}^D$ satisfying a Diophantine condition with respect to ω, there exists an analytic manifold $\mathcal{W}$ of infinitely renormalizable Hamiltonian vector fields; each vector field on $\mathcal{W}$ is shown to have an analytic invariant torus with frequency vector ω.

1. Introduction

It is well known that in near-integrable Hamiltonian systems one has the persistence of analytic invariant tori on which motion is conjugate to a rotation, with frequency vectors satisfying a condition weaker than Diophantine, initially considered by Kolmogorov [19]. Conditions of this type, such as the one introduced by Brjuno [2,3] that we will refer to as the Brjuno condition, were extensively studied by Rüssmann [26]. In the Siegel problem [28,30] and the case of circle diffeomorphisms [31], the Brjuno condition is known to be the optimal condition under which such an analytic conjugacy is guaranteed to exist. In higher-dimensional problems, the question of the weakest possible condition is a fundamental open problem. Recently, we have developed a renormalization group approach to the construction of invariant tori with Brjuno frequency vectors for vector fields on $\mathbb{T}^d \times \mathbb{R}^m$ [16], using two general theorems, a normal form theorem and a stable manifold theorem, that we proved in [15]. The main objective of this paper is to extend the renormalization methods developed in [16] in order to construct analytic lower-dimensional tori of near-integrable Hamiltonian systems.

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The term lower-dimensional torus in Hamiltonian systems refers to a torus of dimension lower than the number of degrees of freedom. The problem of persistence of lower-dimensional tori, under the conditions stated by Melnikov [23], has already been considered by Moser [24] in the case of dimension lower by one than the number of degrees of freedom. The proof of existence of elliptic (stable) lower-dimensional tori in the general case was given by Eliasson [6]. For an overview of the results on lower-dimensional tori, the reader is referred to [4,6–10,22,24,25,27,29,32]. Lower-dimensional tori also appear in and are particularly relevant for PDEs (see e.g. [1,5,20]).

Renormalization group methods have previously been applied to construction of analytic maximal-dimensional tori of near-integrable Hamiltonian systems (see e.g. [12,17]). They have also been applied to non-perturbative problems in Hamiltonian dynamics. In the case of two degree-of-freedom Hamiltonians and golden mean frequency ratio a computer-assisted proof of the existence of the non-trivial renormalization group fixed point was given in [13], and the corresponding non-smooth invariant tori have been constructed in [14].

We consider Hamiltonians $H = H_0 + h$ which are small perturbations of the integrable Hamiltonian

$$H_0(q, p, \xi, \eta) = \omega \cdot p + \frac{1}{2}\sum_{j=1}^{D} \Omega_j\big(\xi_j^2 + \eta_j^2\big). \tag{1.1}$$

The corresponding dynamics on the phase space $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^D \times \mathbb{R}^D$ is given by Hamilton's equations

$$\begin{aligned}
\frac{dq_l}{dt} &= \omega_l + \partial_{p_l} h, \\
\frac{dp_l}{dt} &= -\partial_{q_l} h, \\
\frac{d\xi_j}{dt} &= \Omega_j \eta_j + \partial_{\eta_j} h, \\
\frac{d\eta_j}{dt} &= -\Omega_j \xi_j + \partial_{\xi_j} h,
\end{aligned} \tag{1.2}$$

for $l = 1, \ldots, d$ and $j = 1, \ldots, D$.

For $h = 0$, the system is integrable and the dynamics are constrained to invariant tori. These tori are contained in the level sets of H_0. The maximal dimension of these tori is $d + D$, and in that case the motion on these tori are characterized by frequencies $\omega_1, \ldots, \omega_d$ and $\Omega_1, \ldots, \Omega_D$. The hyper-surface $p = 0$, $\xi = \eta = 0$, contained in the level set $H_0 = 0$, is a lower-dimensional torus of dimension d. The motion on this torus is a linear flow $q \mapsto q + \omega t$ characterized by internal frequencies $\omega_1, \ldots, \omega_d$. The normal space is described by coordinates ξ, η, whose origin is an elliptic fixed point characterized by the normal frequencies $\Omega_1, \ldots, \Omega_D$. The problem of persistence of such elliptic lower-dimensional invariant tori under small perturbations of Hamiltonians of the form (1.1) has been considered for example in [25,27].

After the change of variables

$$u_j = \frac{1}{\sqrt{2}}(\xi_j + \mathrm{i}\eta_j), \qquad w_j = \frac{1}{\sqrt{2}}(\xi_j - \mathrm{i}\eta_j), \tag{1.3}$$

the Hamiltonian H_0 is transformed into

$$H_0(q, p, u, w) = \omega \cdot p - \mathrm{i}\sum_{j=1}^{D} \Omega_j u_j w_j. \tag{1.4}$$

To simplify the notation, we have denoted this function by H_0 also.

The symplectic form $dq \wedge dp + d\xi \wedge d\eta$ gives rise to the new symplectic form $dq \wedge dp + du \wedge dw$. The corresponding operator $\hat{H}_0$, defined by the Poisson bracket $\hat{H}_0 \Psi = \{\Psi, H_0\}$, is given by

$$\hat{H}_0 = \omega \cdot \nabla_q - \mathrm{i} \sum_j \Omega_j (u_j \partial_{u_j} - w_j \partial_{w_j}). \tag{1.5}$$

If $\Psi_{\nu,\kappa,\alpha,\beta}(q,p,u,w) = e^{\mathrm{i}\nu\cdot q} p^\kappa u^\alpha w^\beta$, where $(\nu,\kappa,\alpha,\beta) \in \mathbb{Z}^d \times \mathbb{N}_0^d \times \mathbb{N}_0^D \times \mathbb{N}_0^D$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$$\hat{H}_0 \Psi_{\nu,\kappa,\alpha,\beta} = \mathrm{i}\big[\omega \cdot \nu + \Omega \cdot (\beta - \alpha)\big] \Psi_{\nu,\kappa,\alpha,\beta}. \tag{1.6}$$

The moduli of the eigenvalues $|\omega \cdot \nu + \Omega \cdot (\beta - \alpha)|$ of $\hat{H}_0$ are precisely the small denominators that appear in the problem of construction of lower-dimensional tori. In the renormalization approach that we pursue here, the small divisors are transformed into "large divisors" by a scaling of the phase space. The terms containing "large divisors" are then eliminated by a coordinate change.

Given $\mu > 0$, we will perform the following scaling of the non-toral part of the phase space

$$S_\mu(q,p,u,w) = \big(q, \mu p, \mu^{1/2} u, \mu^{1/2} w\big). \tag{1.7}$$

Under such a transformation, a Hamiltonian H is transformed as $H \mapsto \frac{1}{\mu} H \circ S_\mu$. Notice that the Hamiltonian H_0 is invariant under this transformation.

Given a matrix $T \in \mathrm{GL}(d,\mathbb{R})$, we will also perform the canonical scaling

$$\mathcal{T}(q,p,u,w) = \big(Tq, \bar{T}^{-1} p, u, w\big), \tag{1.8}$$

where $\bar{T} = T^T$ is the transpose of T. This scaling is an essential part of the method and the choice of matrices T has played an important role in previous schemes. In [15], we constructed a renormalization scheme using a sequence of scaling matrices $T \in \mathrm{SL}(d,\mathbb{Z})$ generated by a multidimensional continued fraction algorithm introduced by Khanin, Lopes Dias and Marklof [11], and based on an algorithm by Lagarias [21]. The results of [11] provide good bounds in the case of Diophantine vectors ω, and allowed for the extension of the results of [17] for two degree of freedom Hamiltonians to higher dimensions [12]. In order to extend the results to Brjuno frequency vectors, we developed a different method [16,18], which uses non-integer matrices in $\mathrm{GL}(d,\mathbb{R})$. In the present paper, we will perform this scaling with non-integer matrices in $\mathrm{SL}(d,\mathbb{R})$. On a technical level, the problem at hand is more complicated than the one considered in [16], due to the existence of finitely many additional "resonances" (planes in $\mathbb{R}^d$ whose any open neighborhood contains small denominators).

Our matrices T will have the property that $T\omega = \eta^{-1}\omega$, where $0 < \eta < 1$. With an additional time rescaling the Hamiltonians H_0 are transformed as

$$H_0 \mapsto \eta^{-1}\mu^{-1} H_0 \circ \mathcal{T} \circ S_\mu(q,p,u,w) = \omega \cdot p - \mathrm{i} \sum_j \eta^{-1} \Omega_j u_j w_j. \tag{1.9}$$

We assume that ω satisfies the following Brjuno condition [2,3]:

Definition 1.1. $\omega \in \mathbb{R}^d$ is Brjuno if

$$\sum_{n=1}^{\infty} 2^{-n} \ln\left(\frac{1}{\min_{0<|\nu|\leqslant 2^n} |\omega \cdot \nu|}\right) < \infty, \tag{1.10}$$

where the minimum is taken over lattice vectors ν in $\mathbb{Z}^d$.

We remark that the set of all Brjuno vectors has full Lebesgue measure. It particular, it contains all Diophantine vectors.

We will further assume that the frequencies $\Omega_1, \ldots, \Omega_D$ are real, nonzero and all different and that $\Omega = (\Omega_1, \ldots, \Omega_D)$ is Diophantine with respect to ω.

Definition 1.2. We say that $\Omega = (\Omega_1, \ldots, \Omega_D) \in \mathbb{R}^D$ is Diophantine with respect to $\omega \in \mathbb{R}^d$ if there exist constants $\tau > 0$ and $\mathcal{C} > 0$ such that

$$|\omega \cdot \nu + \Omega \cdot V| > \mathcal{C}|\nu|^{-\tau}, \tag{1.11}$$

for all $\nu \in \mathbb{Z}^d \setminus \{0\}$, and all $V \in \mathbb{Z}^D$ with $0 < |V| \leqslant 2$.

The objective of this paper is to consider analytic Hamiltonians, close to H_0, that can be expanded in Fourier–Taylor series as

$$H(q, p, u, w) = \sum_{\nu,\kappa,\alpha,\beta} H_{\nu,\kappa,\alpha,\beta} e^{\mathrm{i}q\cdot\nu} p^\kappa u^\alpha w^\beta. \tag{1.12}$$

Instead of working with Hamiltonian functions, we will work with the corresponding Hamiltonian vector fields that they generate. We will consider Banach spaces $\mathcal{A}_\rho$, with $\rho > 0$, of analytic vector fields (see Section 2), close to the vector field K, defined by $K(q, p, u, w) = (\omega, 0, -\mathrm{i}\bar{\Omega}u, \mathrm{i}\bar{\Omega}w)$, where $\bar{\Omega} = \operatorname{diag}[\Omega_1, \ldots, \Omega_D]$.

We are interested in Hamiltonians that are real analytic in the original variables (q, p, ξ, η), i.e. which satisfy $H \circ C^* = C^* \circ H$, where C^* is the complex conjugation. In the new variables, this symmetry implies that the Hamiltonians have the following property

$$H \circ C(q, p, u, w) = C^* \circ H(q, p, u, w), \tag{1.13}$$

where $C(q, p, u, w) = (q^*, p^*, -\mathrm{i}w^*, \mathrm{i}u^*)$. The corresponding Hamiltonian vector fields X then satisfy $C^* Q X \circ C = X$, where Q is the linear transformation $Q(q, p, u, w) = (q, p, -\mathrm{i}w, \mathrm{i}u)$. We will call vector fields which satisfy this property real.

The main result of this paper is the following.

Theorem 1.3. *For every Brjuno vector $\omega \in \mathbb{R}^d$ and $\Omega \in \mathbb{R}^D$ Diophantine with respect to ω with different and nonzero components, there exists an open neighborhood B of real analytic Hamiltonian vector fields $X \in \mathcal{A}_\rho$ around K, and an analytic codimension $d + D$ manifold $\mathcal{W} \subset B$, such that every Hamiltonian vector field $X \in \mathcal{W}$ has an analytic d-dimensional invariant torus with frequency vector ω.*

Remark 1.4. The manifold $\mathcal{W}$ is the stable invariant manifold for the sequence of renormalization operators, that we will construct, associated to the "trivial" fixed point. The unstable directions of the renormalization correspond to changes in frequency vectors ω and Ω. One can use the above result to prove the existence of invariant d-dimensional tori in families of Hamiltonian vector fields intersecting the stable manifold. If one is interested in invariant tori with frequency vectors parallel to ω, the number of the necessary parameters can be reduced by 1. As usual (see e.g. [15]), close to a non-degenerate Hamiltonian vector field, for which the q-component $X_q|_{p=0}$ is transversal to ω, one can further reduce the number of parameters needed to prove the existence of invariant tori by $d - 1$, by considering the translations in the p-variables.

The paper is organized as follows. In Section 2, we define the spaces of Hamiltonian vector fields that we consider. Section 3 contains the formulation and estimates on a single renormalization step. In Section 4, we construct a sequence of renormalization parameters and the stable manifold for the sequence of corresponding renormalization operators. Section 5 contains the construction of analytic lower-dimensional invariant tori for Hamiltonian vector fields on the stable manifold of the renormalization transformations.

2. Spaces of vector fields

Since we will perform the scaling of the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ using non-integer matrices, it will be necessary to consider functions with periodicity of different lattices in $\mathbb{R}^d$. Let $\{e_1, \ldots, e_d\}$ be a basis and $\mathcal{Z} = \{\sum_{i=1}^d z_i e_i \mid z_i \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^d$. Let also $\mathcal{V}$ be its dual lattice, i.e. the set of points $v \in \mathbb{R}^d$ satisfying $\exp(iv \cdot z) = 1$ for all $z \in \mathcal{Z}$. If $\mathcal{Z} = (2\pi\mathbb{Z})^d$, then $\mathcal{V} = \mathbb{Z}^d$.

On $\mathbb{C}^n$ we use norms $\|c\| = \sup_j |c_j|$ and $|c| = \sum_j |c_j|$. For linear operators between normed linear spaces, we will always use the operator norm, unless stated otherwise. Denote by D_ρ, with $\rho > 0$, the set of all points (x, y, u, w) in $\mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^D \times \mathbb{C}^D$ characterized by $\|\operatorname{Im} x\| < \rho$, $\|y\| < \rho$, $\|u\| < \rho$ and $\|w\| < \rho$. Define $\mathcal{A}_\rho(\mathcal{V})$ to be the Banach space of all analytic vector fields X on D_ρ, with frequency module in $\mathcal{V}$, that can be expanding in Fourier–Taylor series

$$X(x, y, u, w) = \sum_{v,\kappa,\alpha,\beta} X_{v,\kappa,\alpha,\beta} e^{\mathrm{i}x\cdot v} y^\kappa u^\alpha w^\beta, \tag{2.1}$$

where $v \cdot x = \sum_i v_i x_i$ and $y^\kappa = \prod_i y_i^{\kappa_i}$, and have the finite norm

$$\|X\|_\rho = \sum_{v,\kappa,\alpha,\beta} \|X_{v,\kappa,\alpha,\beta}\| e^{\rho|v|} \rho^{|\kappa|+|\alpha|+|\beta|}. \tag{2.2}$$

Here, $\kappa \in \mathbb{N}_0^d$, $\alpha, \beta \in \mathbb{N}_0^D$. We will consider only Hamiltonian vector fields, and we will use the same symbol $\mathcal{A}_\rho(\mathcal{V})$ to denote the subspace of Hamiltonian vector fields.

We also define $\mathcal{A}'_\rho(\mathcal{V})$ to be the space of all vector fields $X \in \mathcal{A}_\rho(\mathcal{V})$, whose derivatives DX are bounded linear operators on $\mathcal{A}_\rho(\mathcal{V})$. When the lattice $\mathcal{V}$ is fixed, we will simplify the notation by not explicitly mentioning the dependence on $\mathcal{V}$.

We end this section with the following proposition, whose proof follows directly from the above definitions.

Proposition 2.1. *Let $X \in \mathcal{A}_\rho$ and $Z \in \mathcal{A}_{\rho'}$, with $0 < \rho' \leqslant \rho$. Then*

(i) $\|X(x, y, u, w)\| \leqslant \|X\|_\rho$, *for all* $(x, y, u, w) \in D_\rho$.
(ii) $(DX)Z \in \mathcal{A}_{\rho'}$ *and* $\|(DX)Z\|_{\rho'} \leqslant (\rho - \rho')^{-1} \|X\|_\rho \|Z\|_{\rho'}$, *if* $\rho' < \rho$.
(iii) $X \circ (\mathrm{I} + Z) \in \mathcal{A}_{\rho'}$ *and* $\|X \circ (\mathrm{I} + Z)\|_{\rho'} \leqslant \|X\|_\rho$, *if* $\rho' + \|Z\|_{\rho'} \leqslant \rho$.

3. One renormalization step

The vector $\omega \in \mathbb{R}^d$ satisfying the Brjuno condition (1.10) is assumed to be fixed throughout the paper. We will rotate and scale the coordinate system (by a linear map $\mathcal{N}$) such that ω takes the form $(1, 0, \ldots, 0)$. In the new coordinates, the lattice vectors v do not necessarily have integer components. We will denote such a lattice of vectors v by $\mathcal{V}$. The lattice $\mathcal{V}$ is therefore just the lattice $\mathbb{Z}^d$ expressed in a transformed system of coordinates. Notice that the Brjuno condition (1.10) involves scalar products and is invariant under such a transformation. It can be interpreted just as a condition on the lattice $\mathcal{V}$.

On a neighborhood of K in $\mathcal{A}_\rho(\mathcal{V})$, the one-step renormalization operator is defined by

$$\mathcal{R}(X) = \eta^{-1} \mathcal{S}_\mu^* \mathcal{T}^* \mathcal{U}_X^* X, \tag{3.1}$$

where $\mathcal{U}^*$ denotes the pullback of a vector under a map $\mathcal{U}$, i.e. $\mathcal{U}^* X = (D\mathcal{U})^{-1} X \circ \mathcal{U}$. Here, the map $\mathcal{U}_X$ is chosen such that its pullback brings the vector field X into a normal form, that we call resonant.

The construction of such a map is carried out in Lemma 3.8. The scaling maps $\mathcal{S}_\mu$ and $\mathcal{T}$ are defined as in (1.7) and (1.8), respectively (with the change of notation $(q,p) \mapsto (x,y)$).

We choose the matrix T in the definition of the scaling $\mathcal{T}$ to be a diagonal matrix $T = \operatorname{diag}[\eta^{-1}, \zeta, \ldots, \zeta]$ in the new coordinate system. Here $0 < \eta, \zeta < 1$. Thus, the action of T on an arbitrary vector $x \in \mathbb{R}^d$, that admits a decomposition $x = x_\parallel + x_\perp$ into a component $x_\parallel$ parallel to ω, and a component $x_\perp$ perpendicular to ω, is given by

$$T(x) = \eta^{-1} x_\parallel + \zeta x_\perp. \tag{3.2}$$

We assume that the components of $\omega = (1, 0, \ldots, 0)$ are rationally independent with respect to a given lattice $\mathcal{V}$, in the sense that the first component $v_\parallel$ of any nonzero vector $v \in \mathcal{V}$ is nonzero. Then, given any $L \geqslant 1$, we can find $\ell > 0$ such that

$$|v_\perp| > L \quad \text{or} \quad |v_\parallel| \geqslant \ell, \quad \forall v \in \mathcal{V} \setminus \{0\}. \tag{3.3}$$

In other words, all points in $\mathcal{V}$, except for the origin, lie outside the disk characterized by $|v_\perp| \leqslant L$ and $|v_\parallel| < \ell$. This condition is technically very important and will be used below to replace the trivial bound $\|\nu\| \geqslant 1$, valid for $\nu \in \mathbb{Z}^d \setminus \{0\}$.

We will assume that the parameters $\rho', \eta, \zeta, \sigma, \mu, \ell, L$ are given positive numbers and that the following conditions are satisfied

$$\sigma < 1/2, \qquad 2\sigma L \leqslant \ell, \qquad 0 < \eta \leqslant \zeta < 1, \qquad e^{-\rho'(1-\zeta\eta)L} \leqslant \mu^{3/2}. \tag{3.4}$$

Definition 3.1. Consider the Fourier–Taylor expansion (2.1) of a single component of a vector field. A mode of this component of the vector field is the term in this expansion labeled by $(v, \kappa, \alpha, \beta)$. A mode of a vector field characterized by fixed $(v, \kappa, \alpha, \beta)$, with $v \in \mathcal{V}$, $\kappa \in \mathbb{N}_0^d$ and $\alpha, \beta \in \mathbb{N}_0^D$, is the vector field whose x_i-component is the $(v, \kappa - \delta_i^\ell e_\ell, \alpha, \beta)$ mode of this component of the vector field, whenever $\kappa_i > 0$ or zero otherwise; y_i-component is the $(v, \kappa, \alpha, \beta)$ mode of this component of the vector field; u_j-component is the $(v, \kappa, \alpha, \beta - \delta_j^\ell e_\ell)$ mode of this component of the vector field, whenever $\beta_j > 0$ or zero otherwise; w_j-component is the $(v, \kappa, \alpha - \delta_j^\ell e_\ell, \beta)$ mode of this component of the vector field, whenever $\alpha_j > 0$ or zero otherwise. Here $i = 1, \ldots, d$, $j = 1, \ldots, D$; δ_i^ℓ is the Kronecker delta, i.e. $\delta_i^\ell = 1$ if $i = \ell$ and zero otherwise; $e_\ell = (\delta_1^\ell, \ldots, \delta_{\dim}^\ell)$, where $\dim = d$ or $\dim = D$, is the unit vector in the ℓ-th coordinate direction of the standard basis. We have used Einstein's notation, i.e. we assume the summation over repeated up and down indices.

Remark 3.2. Notice that the operators $-\mathrm{i}\partial_x$ and $\mathcal{S}_\mu^*$ commute on the space of vector fields considered, and that modes of a vector field are the joint eigenvectors of $-\mathrm{i}\partial_x$ and S, where S is the generator of the one-parameter group of scalings $\mathcal{S}_\mu^*$. A mode of a vector field characterized by mode indices $(v, \kappa, \alpha, \beta)$ is the joint eigenvector for $(-\mathrm{i}\partial_x, S)$ corresponding to the eigenvalues (v, k), where $k = \frac{|\alpha|+|\beta|}{2} + |\kappa| - 1$. Sometimes, we will also call (v, k) with $v \in \mathcal{V}$ and $k \in \{-1, -1/2, 0, 1/2, 1, 3/2, \ldots\}$ the mode indices, referring to all the mode indices $(v, \kappa, \alpha, \beta)$, with $\frac{|\alpha|+|\beta|}{2} + |\kappa| - 1$ equal to k.

Remark 3.3. Consider the Fourier–Taylor expansion (1.12) of a Hamiltonian. Each mode of this Hamiltonian, i.e. each term in this expansion characterized by a quadruplet $(v, \kappa, \alpha, \beta) \in \mathcal{V} \times \mathbb{N}_0^d \times \mathbb{N}_0^D \times \mathbb{N}_0^D$, generates a single mode Hamiltonian vector field characterized by $(v, \kappa, \alpha, \beta)$. The $(0, 0, 0, 0)$ mode of any Hamiltonian vector field is zero.

Definition 3.4. Let $I = \mathcal{V} \times \mathbb{N}_0^d \times \mathbb{N}_0^D \times \mathbb{N}_0^D$. In the first renormalization step, let I^+ be the set of all $(v, \kappa, \alpha, \beta) \in I$ satisfying $|\omega \cdot v| \leqslant \sigma |v|$ when $v \neq 0$, or $\alpha = \beta$ when $v = 0$, or $k := \frac{|\alpha|+|\beta|}{2} + |\kappa| - 1 > 0$, and let I^- be the complement of I^+ in I. Define $\mathbb{I}^+$ and $\mathbb{I}^-$ to be the projection operators onto the subsets of *resonant* and *nonresonant* vector fields spanned by modes with mode indices in I^+ and I^-, respectively. The *resonant* and *nonresonant* parts of a vector field $X \in \mathcal{A}_\rho$ are defined as $\mathbb{I}^+X$ and $\mathbb{I}^-X$, respectively. In addition, we define $\mathbb{E}_k$, for $k \in \{-1, -1/2, 0, 1/2, 1, 3/2, \ldots\}$, as the projection operator onto the space spanned by modes with mode indices $(0, \kappa, \alpha, \beta)$, with $\frac{|\alpha|+|\beta|}{2} + |\kappa| - 1$ equal to k. The torus averaging operator is then defined by $\mathbb{E} = \sum_k \mathbb{E}_k$.

Let $\chi^{-1} = \max\{\|\bar{T}\|, \|T^{-1}\|\} > 1$.

Lemma 3.5. *If $0 < \rho'' \leqslant \chi\rho'$ and $\mu < 1$, then $\mathcal{T}^*\mathcal{S}^*_\mu$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}(\mathcal{V})$ to $\mathcal{A}_{\rho''}(T\mathcal{V})$, with the property that*

$$\left\|\mathcal{T}^*\mathcal{S}^*_\mu\mathbb{E}_k X\right\|_{\rho''} \leqslant \chi^{-1}\mu^k\|\mathbb{E}_k X\|_{\rho'},$$
$$\left\|\mathcal{T}^*\mathcal{S}^*_\mu\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\right\|_{\rho''} \leqslant \chi^{-1}\mu^{1/2}\left\|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\right\|_{\rho'}. \tag{3.5}$$

Proof. It suffices to verify the given bounds for a single mode $X_{\nu,k}$ characterized by (ν, k). From the definitions of the scaling maps it follows that

$$\left\|\mathcal{T}^*S^*_\mu X_{\nu,k}\right\|_{\rho''} \leqslant \chi^{-1}e^A\|X_{\nu,k}\|_{\rho'}, \tag{3.6}$$

where $A \leqslant \rho''|T\nu_\parallel| + \rho''|T\nu_\perp| - \rho'|\nu_\parallel| - \rho'|\nu_\perp| + k\ln(\mu) + (k+1)\ln(\chi^{-1}\rho''/\rho')$. Setting $\nu = 0$, and using that $\rho'' \leqslant \chi\rho'$ and $k \geqslant -1$, yields the first bound in (3.5).

In order to prove the second bound, assume that (ν, k) belongs to I^+. Consider first the case $|\nu_\parallel| \leqslant \sigma|\nu|$, with $\nu \neq 0$, and $k \leqslant 0$. This inequality implies $|\nu_\parallel| < 2\sigma|\nu_\perp|$, by using that $\sigma < 1/2$, and does not allow frequencies ν that satisfy $|\nu_\perp| \leqslant L$ and $|\nu_\parallel| \geqslant \ell$, due to the condition (3.4). Thus, we must have $|\nu_\perp| > L$ by condition (3.3). Consequently,

$$A \leqslant -\rho'\left(1 - \frac{\rho''}{\rho'}\zeta\right)|\nu_\perp| + k\ln(\mu) + (k+1)\ln\left(\frac{\chi^{-1}\rho''}{\rho'}\right) \leqslant -\rho'(1-\zeta\eta)L - \ln(\mu), \tag{3.7}$$

where we have again used that $\rho'' \leqslant \chi\rho'$, $\mu < 1$ and $-1 \leqslant k \leqslant 0$. The second bound in (3.5) now follows by using (3.4).

Next, consider the case $k > 0$, i.e. $k \geqslant 1/2$. Since $\rho'' < \chi\rho'$, we have $A \leqslant (1/2)\ln(\mu)$ and thus

$$\left\|\mathcal{T}^*\mathcal{S}^*_\mu X_{\nu,k}\right\|_{\rho''} \leqslant \chi^{-1}\mu^{1/2}\|X_{\nu,k}\|_{\rho'}. \tag{3.8}$$

This also implies the second bound in (3.5). □

Let $J = \{(\nu, \kappa, \alpha, \beta) \in I^-\colon\ |\Omega \cdot V| > (1/2)|\omega \cdot \nu| \text{ for all } V = \beta - \alpha \in \mathbb{Z}^D \text{ with } |V| \leqslant 2\}$, and

$$\gamma^{(1)} = \max_{(\nu,\kappa,\alpha,\beta)\in J}\left\{1, \frac{\sigma}{|\omega\cdot\nu + \Omega\cdot V|}\right\}, \qquad \gamma^{(2)} = \max_{(\nu,\kappa,\alpha,\beta)\in J}\left\{2, \frac{\sigma|\nu|}{|\omega\cdot\nu + \Omega\cdot V|}\right\}.$$

Notice that $\chi^{(1)}, \chi^{(2)} < \infty$, by condition (1.11).

Proposition 3.6. *For all $0 \neq \nu \in \mathcal{V}$ and $V \in \mathbb{Z}^D$ with $|V| \leqslant 2$, if $|\omega\cdot\nu| > \sigma|\nu|$ and $|\omega\cdot\nu + \Omega\cdot V| \neq 0$, then $|\omega\cdot\nu + \Omega\cdot V| \geqslant \sigma/\gamma^{(1)}$ and $|\omega\cdot\nu + \Omega\cdot V| \geqslant \sigma/\gamma^{(2)}|\nu|$.*

Proof. If $|\omega\cdot\nu| > \sigma|\nu|$ and $|\Omega\cdot V| \leqslant (1/2)|\omega\cdot\nu|$ then we have $|\omega\cdot\nu + \Omega\cdot V| \geqslant |\omega\cdot\nu| - |\Omega\cdot V| \geqslant (1/2)|\omega\cdot\nu|$ and thus $|\omega\cdot\nu + \Omega\cdot V| \geqslant (\sigma/2)|\nu|$. Using conditions (3.3) and (3.4) and $L \geqslant 1$, we also obtain $|\omega\cdot\nu| > \sigma$ and thus $|\omega\cdot\nu + \Omega\cdot V| \geqslant \sigma/2$, in that case.

The number of modes with $|\Omega\cdot V| > (1/2)|\omega\cdot\nu| > (1/2)\sigma|\nu|$ and $|V| \leqslant 2$ is finite. So, if $|\omega\cdot\nu + \Omega\cdot V| \neq 0$ then $|\omega\cdot\nu + \Omega\cdot V| \geqslant \sigma/\gamma^{(1)}$ and $|\omega\cdot\nu + \Omega\cdot V| \geqslant \sigma/\gamma^{(2)}|\nu|$. □

Let $\hat{K}Z = [K, Z]$. Let $c_\Omega > 0$ be the smaller of the $\min_i\{|\Omega_i|\}$ and $\min_{i\neq j}\{|\Omega_i - \Omega_j|\}$.

Proposition 3.7. *If $\rho > 0$, and $Z \in \mathbb{I}^-\mathcal{A}'_\rho$, then*

$$\left\|\mathbb{I}^-[Z, K]\right\|_\rho \geqslant \frac{\sigma}{\gamma}\|Z\|'_\rho, \tag{3.9}$$

where $\gamma = \frac{2(\rho+1)}{\rho}\max\{\gamma^{(1)}, \gamma^{(2)}\}$.

Proof. Assume that $(\nu, \kappa, \alpha, \beta)$ belongs to I^-. Then, $k \leqslant 0$ and either $|\omega\cdot\nu| > \sigma|\nu|$ with $\nu \neq 0$ or $\alpha \neq \beta$ with $\nu = 0$. Due to our choice of the norm, it suffices to verify these bounds for a single mode $Z_{\nu,\kappa,\alpha,\beta}$ of vector field $Z \in \mathbb{I}^-\mathcal{A}'_\rho$. Notice that $\hat{K}Z_{\nu,\kappa,\alpha,\beta} = \mathrm{i}(\omega\cdot\ \nu + \Omega\cdot(\beta-\alpha))Z_{\nu,\kappa,\alpha,\beta}$ and that $\hat{K}$ commutes with $\mathbb{I}^-$. If $\alpha \neq \beta$ with $\nu = 0$, then $|\omega\cdot\nu + \Omega\cdot(\beta-\alpha)| > c_\Omega \geqslant \sigma/\gamma^{(1)}$. The previous bounds and Proposition 3.6 show that if $Z \in \mathbb{I}^-\mathcal{A}'_\rho$ and $Y = [Z, K]$, then $\|Z\|_\rho \leqslant \gamma^{(1)}/\sigma\|Y\|_\rho$ and

$$\sum_{j=1}^{d}\|\partial_{x_j}Z\|_\rho \leqslant \frac{\gamma^{(2)}}{\sigma}\|Y\|_\rho, \qquad \sum_{j=1}^{d}\|\partial_{y_j}Z\|_\rho + \sum_{j=1}^{D}\left(\|\partial_{u_j}Z\|_\rho + \|\partial_{w_j}Z\|_\rho\right) \leqslant \frac{2\gamma^{(1)}}{\rho\sigma}\|Y\|_\rho. \tag{3.10}$$

As a result we obtain (3.9). □

This proposition allows us to apply the normal form theorem of [15], which directly implies the following lemma. The positive number $\varrho < 1$ that appears in the following is fixed throughout the paper. It has the meaning of the step-dependent domain parameter ρ of the initial vector fields that we would like to renormalize.

Consider the equations

$$\mathbb{I}^-\left(X + [Z, X]\right) = 0, \qquad \mathbb{I}^-\mathcal{U}_X^* X = 0. \tag{3.11}$$

Constants that we call universal, do not depend on any renormalization parameters.

Lemma 3.8. *Let $\rho > 0$, and let $\rho' = \rho - (\sigma/\gamma)\varrho$. There exist universal constants C_1 and C_2 such that for every vector field $X \in \mathcal{A}'_\rho$, if*

$$\|X - K\|'_\rho \leqslant C_1(\sigma/\gamma), \qquad \left\|\mathbb{I}^-X\right\|_\rho \leqslant C_1(\sigma/\gamma)^2, \tag{3.12}$$

then there exists a vector field $Z \in \mathbb{I}^-\mathcal{A}'_\rho$ and a change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, solving Eq. (3.11)*, such that the vector field $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$, and*

$$\begin{aligned}\|Z\|'_\rho, \|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\leqslant C_2(\gamma/\sigma)\left\|\mathbb{I}^-X\right\|_\rho,\\ \left\|\mathcal{U}_X^*X - X\right\|_{\rho'} &\leqslant C_2\left(\rho-\rho'\right)^{-1}(\gamma/\sigma)\left\|\mathbb{I}^-X\right\|_\rho,\\ \left\|\mathcal{U}_X^*X - X - [Z, X]\right\|_{\rho'} &\leqslant C_2\left(\rho-\rho'\right)^{-3}(\gamma/\sigma)^3\left\|\mathbb{I}^-X\right\|_\rho^2.\end{aligned} \tag{3.13}$$

The map $X \mapsto \mathcal{U}_X$ is continuous in the region defined by (3.12)*, and analytic in its interior.*

By construction, the map $X \mapsto \mathcal{U}_X^* X$ is type-preserving, in particular Hamiltonian vector fields remain Hamiltonian under this transformation (see the discussion at the end of Section 5 of [15]).

Define the operator $\mathbb{P}$ as the projection operator $\mathbb{P} = \mathbb{E}_{-1} + \mathbb{E}_0^{\alpha=\beta}$ onto the space spanned by resonant vector fields that expand under scaling. Since in this paper we have restricted our consideration to Hamiltonian vector fields, $\mathbb{E}_{-1}$ is simply the zero operator. Let the restriction of $\mathcal{R}$ to this subspace be denoted by $\mathcal{L}$. Here, $\mathbb{E}_0 = \mathbb{E}_0^{\alpha=\beta} + \mathbb{E}_0^{\alpha\neq\beta}$ is the decomposition of $\mathbb{E}_0$ into projection operators onto the spaces characterized by $\alpha = \beta$ and $\alpha \neq \beta$, respectively. We will also define $\mathbb{E}^+ = \mathbb{E}\mathbb{I}^+ = \mathbb{E} - \mathbb{E}\mathbb{I}^-$. Notice that the terms $\mathbb{E}\mathbb{I}^- X$ are eliminated in the first renormalization step once and for all, i.e. after one renormalization step, we have $\mathbb{E}^+ = \mathbb{E}$. In the following, $\mathcal{A}_\rho$ denotes the subspace of Hamiltonian vector fields.

Theorem 3.9. *There exist universal constants $C, R > 0$, such that the following holds, under the given assumptions on $L, \ell, \eta, \zeta, \gamma$ and μ. Let B be the open ball in $\mathcal{A}_\rho(\mathcal{V})$, with $(\sigma/\gamma)\varrho < \rho \leqslant \varrho$, of radius $R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is a bounded analytic map from B to $\mathcal{A}_{\eta\rho-\eta(\sigma/\gamma)\varrho}(T\mathcal{V})$, satisfying $\|\mathcal{L}^{-1}\| \leqslant 1$ and*

$$\begin{aligned}
\big\|\big(\mathbb{I}-\mathbb{E}^+\big)\mathcal{R}(X)\big\|_{\eta\rho-\eta(\sigma/\gamma)\varrho} &\leqslant C\eta^{-2}(\gamma/\sigma)^2\mu^{1/2}\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_\rho, \\
\big\|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\big\|_{\eta\rho-\eta(\sigma/\gamma)\varrho} &\leqslant C\eta^{-2}(\gamma/\sigma)^2\mu^{1/2}\big\|(\mathbb{I}-\mathbb{P})X\big\|_\rho, \\
\big\|\mathbb{P}\mathcal{R}(X)-\mathcal{R}(\mathbb{P}X)\big\|_{\eta\rho-\eta(\sigma/\gamma)\varrho} &\leqslant C\eta^{-2}(\gamma/\sigma)^6\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_\rho\big\|(\mathbb{I}-\mathbb{P})X\big\|_\rho.
\end{aligned} \tag{3.14}$$

Proof. Let $\rho' = \rho - (\sigma/\gamma)\varrho$. There exists a universal constant $R > 0$, such that the conditions (3.12) in Lemma 3.8 hold, whenever X belongs to the domain B, defined by $\|X - K\|_\varrho < R(\sigma/\gamma)^2$.

By Lemma 3.5, we have

$$\begin{aligned}
\big\|\big(\mathbb{I}-\mathbb{E}^+\big)\mathcal{R}(X)\big\|_{\eta\rho'} &= \eta^{-1}\big\|\mathcal{T}^*\mathcal{S}_\mu^*\big(\mathbb{I}-\mathbb{E}^+\big)\mathcal{U}_X^* X\big\|_{\eta\rho'} \\
&\leqslant \eta^{-2}\mu^{1/2}\big[\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_{\rho'} + \big\|\big(\mathbb{I}-\mathbb{E}^+\big)\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'}\big].
\end{aligned} \tag{3.15}$$

Using the bound in (3.13) on the norm of $\mathcal{U}_X^* X - X$, together with the fact that $\mathbb{I}^-\mathbb{E}^+ = 0$, we obtain the first inequality in (3.14). Similarly, Lemma 3.5 implies that

$$\big\|\mathbb{E}_k\mathcal{R}(X)\big\|_{\eta\rho'} \leqslant C_1\eta^{-2}\mu^k\big[\|\mathbb{E}_k X\|_{\rho'} + \big\|\mathbb{E}_k\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'}\big], \tag{3.16}$$

for some universal constant $C_1 > 0$ and for all $k = \frac{|\alpha|+|\beta|}{2} + |\kappa| - 1 \geqslant 1/2$. Summing over all $k \geqslant 1/2$ to get a bound on $\|(\mathbb{E}^+ - \mathbb{P})\mathcal{R}(X)\|_{\eta\rho'}$, and then adding (3.15), yields

$$\big\|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\big\|_{\eta\rho'} \leqslant C_1\eta^{-2}\mu^{1/2}\big[\big\|(\mathbb{I}-\mathbb{P})X\big\|_{\rho'} + \big\|(\mathbb{I}-\mathbb{P})\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'}\big], \tag{3.17}$$

if C_1 is chosen sufficiently large. Using again the second bound in (3.13), and the fact that $\mathbb{I}^-\mathbb{P} = 0$ and $\mathbb{I}^-\mathbb{E}^+ = 0$, we obtain the second inequality in (3.14).

By Lemma 3.5, we also have a bound

$$\big\|\mathbb{P}\mathcal{R}(X) - \mathcal{R}(\mathbb{P}X)\big\|_{\eta\rho'} = \eta^{-1}\big\|\mathcal{T}^*\mathcal{S}_\mu^*\mathbb{P}\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'} \leqslant \eta^{-2}\big\|\mathbb{P}\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'}. \tag{3.18}$$

Using Lemma 3.8, the norm on the right-hand side of (3.18) can be estimated as follows:

$$\big\|\mathbb{P}\big(\mathcal{U}_X^* X - X\big)\big\|_{\rho'} \leqslant C_2(\gamma/\sigma)^6\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_\rho^2 + \big\|\mathbb{P}[Z, X]\big\|_{\rho'}, \tag{3.19}$$

where $Z = \mathbb{I}^- Z$ is the vector field described in (3.11). Hamiltonian vector fields satisfy $\mathbb{P}[Z, \mathbb{P}X] = 0$, and thus

$$\begin{aligned}
\big\|\mathbb{P}[Z, X]\big\|_{\rho'} = \big\|\mathbb{P}\big[Z, (\mathbb{I}-\mathbb{P})X\big]\big\|_{\rho'} &\leqslant C_3(\gamma/\sigma)\|Z\|_\rho'\big\|(\mathbb{I}-\mathbb{P})X\big\|_\rho \\
&\leqslant C_4(\gamma/\sigma)^2\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_\rho\big\|(\mathbb{I}-\mathbb{P})X\big\|_\rho.
\end{aligned} \tag{3.20}$$

Here, we have used the bound on $\|Z\|'_\rho$ from Lemma 3.8. Combining the last three equations yields the third inequality in (3.14).

The bound concerning the restriction $\mathcal{L}$ of $\mathcal{R}$ to $\mathbb{P}\mathcal{A}_\rho(\mathcal{V})$, is obvious if one notices that this restriction is the linear operator $\mathcal{L} = \eta^{-1}\mathcal{T}^*$. $\square$

4. Composed renormalization transformations

We express the Brjuno condition on ω (and thus on $\mathcal{V}$) in terms of the summability of the series of numbers

$$a_n = \sum_{k=n}^{\infty} 2^{n-k}\left[2^{-k-\kappa}\ln\left(1/\Omega'_{k+\kappa}\right) + \left(k+\kappa'\right)^{-2}\right], \qquad \Omega'_n = \min_{0<|\nu_\perp|<2^n} |\nu_\|| , \tag{4.1}$$

for all positive integers n. Here $\kappa, \kappa' > 2$ are two integer constants to be determined later.

It follows from the definition that $a_{n+1}/2 < a_n < 2a_{n+1}$, for all $n \in \mathbb{N}$, and thus $a_{n+1}2^{n+1}/4 < a_n 2^n < a_{n+1}2^{n+1}$. This makes the sequence $a_n 2^n$ increasing and well-controllable.

Define the scaling parameters

$$\eta_n = \left(\frac{A_{n+1}}{A_n}\right)^{\frac{d-1}{d}}, \qquad \zeta_n = \left(\frac{A_{n+1}}{A_n}\right)^{\frac{1}{d}}, \quad \text{where } A_n = \sum_{k=n}^{\infty} a_k, \tag{4.2}$$

for all positive integers n. Since $\{a_n\}$ is a summable sequence of positive numbers, the sequence $\{A_n\}$ is decreasing and converges to zero. Define recursively $\lambda_n = \eta_n \lambda_{n-1}$, with $\lambda_0 = 1$. These definitions imply that $\eta_n < \zeta_n < 1$, for $d \geqslant 2$.

These parameters are used to define the scaling maps T_n and $P_n = T_n \cdots T_1$ at each renormalization step

$$T_n(x) = \eta_n^{-1} x_\| + \zeta_n x_\perp, \qquad P_n(x) = \lambda_n^{-1} x_\| + \left(\prod_{i=1}^{n} \zeta_i\right) x_\perp. \tag{4.3}$$

We also define $T_0 = P_0$ to be the identity. Notice that the determinants $|T_n| = |P_n| = 1$, for all $n \in \mathbb{N}$, by the choice of the scaling parameters.

The geometric data $\mathcal{V}$, L and ℓ used in the n-th renormalization step are

$$\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0, \qquad L_{n-1} = 2^{n+\kappa} \prod_{i=1}^{n-1} \zeta_i, \qquad \ell_{n-1} = \lambda_{n-1}^{-1} e^{-a_n 2^{n+\kappa}}. \tag{4.4}$$

Definition 4.1. Let

$$\sigma_n = (2\lambda_{n-1}L_{n-1})^{-1} e^{-a_n 2^{n+\kappa}} = \frac{A_1}{2A_n} 2^{-(n+\kappa)} e^{-a_n 2^{n+\kappa}}. \tag{4.5}$$

This immediately implies $\sigma_n > 0$ and $2\sigma_n L_{n-1} \leqslant \ell_{n-1}$, for all $n \in \mathbb{N}$.

Proposition 4.2. *For any fixed $\kappa' > 0$ and sufficiently large κ, one has $\sum_{n=1}^{\infty} \sigma_n < 1/2$.*

Proof. Notice that

$$\sigma_n < \frac{A_1}{2a_n} 2^{-(n+\kappa)} e^{-a_n 2^{n+\kappa}}.$$

Since $\{a_n 2^n\}$ is a growing sequence, the sequence $\{\sigma_n\}$ is decreasing. Notice also that for a fixed κ', and sufficiently large κ, we have $2^{n+\kappa} a_n \geqslant 2^{n+\kappa}(n+\kappa')^{-2} \geqslant c' 2^\kappa n$, for some constant $c' > 0$ depending only on κ'. This makes the sum $\sum_{n=1}^\infty \sigma_n$ finite, and by choosing κ sufficiently large, we can make this sum smaller than $1/2$. □

Proposition 4.3. *If $v \in \mathcal{V}_{n-1}$ is nonzero, then either $|v_\parallel| \geqslant \ell_{n-1}$ or $|v_\perp| > L_{n-1}$.*

Proof. Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_\perp| \leqslant L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1} v$ in $\mathcal{V}_0$ satisfies $|\nu_\perp| \leqslant (\prod_{i=1}^{n-1} \zeta_i)^{-1} L_{n-1} = 2^{n+\kappa}$, and thus $|\nu_\parallel| \geqslant \Omega'_{n+\kappa}$ by (4.1). Since we have $\Omega'_{n+\kappa} > e^{-a_n 2^{n+\kappa}}$, this yields

$$|v_\parallel| = \lambda_{n-1}^{-1} |\nu_\parallel| \geqslant \lambda_{n-1}^{-1} \Omega'_{n+\kappa} > \lambda_{n-1}^{-1} e^{-a_n 2^{n+\kappa}} = \ell_{n-1}, \tag{4.6}$$

as claimed. □

Definition 4.4. Let $\Omega_{n-1} := \lambda_{n-1}^{-1} \Omega$, for $n \in \mathbb{N}$. Let also

$$J_{n-1}^- = \bigl\{(v, \kappa, \alpha, \beta) \in I^-(\mathcal{V}_{n-1}){:}\ |\omega \cdot v| < 2|\Omega_{n-1} \cdot V| \text{ for all } V = \beta - \alpha \text{ with } |V| \leqslant 2\bigr\},$$

and

$$\gamma_n := \frac{8}{\varrho \lambda_{n-1}} \max_{(v,\kappa,\alpha,\beta) \in J_{n-1}^-} \left\{2, \frac{\sigma_n \max\{1, |v|\}}{|\omega \cdot v + \Omega_{n-1} \cdot V|}\right\}.$$

Definition 4.5. The domain of analyticity of functions that are going to be renormalized at the n-th step is determined by the parameter

$$\rho_{n-1} = \varrho \lambda_{n-1} \left(1 - \sum_{k=1}^{n-1} \frac{\sigma_k}{\lambda_{k-1} \gamma_k}\right). \tag{4.7}$$

The numbers ρ_n are positive due to Proposition 4.2 and the fact that $\gamma_k \lambda_{k-1} > 1$. Moreover, $\rho_{n-1} > \varrho \lambda_{n-1}/2$ and $\rho'_{n-1} > \rho \lambda_{n-1}/2$, where $\rho'_{n-1} = \rho_{n-1} - \rho \sigma_n / \gamma_n$.

Proposition 4.6. *If Ω is Diophantine with respect to ω, then there exists a universal constant $\xi > 0$, such that for all $n \in \mathbb{N}$,*

$$\gamma_n < \xi \varrho^{-1} \max\bigl\{\bigl(\min |\Omega \cdot V|\bigr)^{-1}, \mathcal{C}^{-1} \max |\Omega \cdot V|^{\tau+1}\bigr\} \lambda_{n-1}^{-(\tau+1)} \left(\sigma_n \prod_{i=1}^{n-1} \zeta_i\right)^{-\tau}. \tag{4.8}$$

Here, the min *and* max *are taken over all $V \in \mathbb{Z}^d$ with $0 < |V| \leqslant 2$.*

Proof. Recall that $\nu = P_{n-1}^{-1} v$. By definition, we have

$$\begin{aligned}\frac{\varrho\lambda_{n-1}}{8}\gamma_n &< \max_{(v,\kappa,\alpha,\beta)\in J_{n-1}^-}\left\{2, \frac{\sigma_n \max\{1,|v|\}}{|\omega\cdot v + \Omega_{n-1}\cdot V|}\right\} \\ &\leqslant \max_{(v,\kappa,\alpha,\beta)\in J_{n-1}^-}\left\{2, \frac{\max\{\sigma_n, |\omega\cdot v|\}}{|\omega\cdot v + \Omega_{n-1}\cdot V|}\right\} \\ &< \max_{(v,\kappa,\alpha,\beta)\in J_{n-1}^-}\left\{2, \frac{\max\{\sigma_n, 2\max|\Omega_{n-1}\cdot V|\}}{\lambda_{n-1}^{-1}|\omega\cdot \nu + \Omega\cdot V|}\right\} \\ &= \max_{(v,\kappa,\alpha,\beta)\in J_{n-1}^-}\left\{2, \frac{\max\{\sigma_n\lambda_{n-1}, 2\max|\Omega\cdot V|\}}{|\omega\cdot \nu + \Omega\cdot V|}\right\},\end{aligned} \tag{4.9}$$

and thus

$$\frac{\varrho\lambda_{n-1}}{8}\gamma_n < \max_{|\nu|<2\max|\Omega\cdot V|(\lambda_{n-1}(\prod_{i=1}^{n-1}\zeta_i)\sigma_n)^{-1}}\left\{2, \frac{\max\{\sigma_n\lambda_{n-1}, 2\max|\Omega\cdot V|\}}{|\omega\cdot \nu + \Omega\cdot V|}\right\}. \tag{4.10}$$

Here, we have used the fact that

$$|v| = |v_\parallel| + |v_\perp| = \lambda_{n-1}^{-1}|\nu_\parallel| + \left(\prod_{i=1}^{n-1}\zeta_i\right)|\nu_\perp| \geqslant \left(\prod_{i=1}^{n-1}\zeta_i\right)|\nu|, \tag{4.11}$$

and that the condition $\sigma_n|v| \leqslant |v_\parallel| < 2\max|\Omega_{n-1}\cdot V|$ implies the following inequality $|\nu| < 2\max|\Omega\cdot V|(\lambda_{n-1}(\prod_{i=1}^{n-1}\zeta_i)\sigma_n)^{-1}$.

Using the Diophantine with respect to ω condition on Ω, we obtain that

$$\gamma_n < \xi\varrho^{-1}\max\bigl\{\bigl(\min|\Omega\cdot V|\bigr)^{-1}, \mathcal{C}^{-1}\max|\Omega\cdot V|^{\tau+1}\bigr\}\lambda_{n-1}^{-(\tau+1)}\left(\sigma_n\prod_{i=1}^{n-1}\zeta_i\right)^{-\tau}, \tag{4.12}$$

where ξ is a universal constant and the min and max are taken over all $V\in\mathbb{Z}^d$ with $0<|V|\leqslant 2$. □

Definition 4.7. Let

$$\mu_n := \exp\left\{-\frac{\varrho}{3}\lambda_{n-1}(1-\zeta_n\eta_n)L_{n-1}\right\} = \exp\left\{-\frac{\varrho}{3A_1}a_n 2^{n+\kappa}\right\}, \quad n\geqslant 1. \tag{4.13}$$

Proposition 4.8. *For all $n\geqslant 1$, $\mu_{n+1} < \mu_n < \mu_{n+1}^{1/4}$. Furthermore, given $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $n\geqslant 1$,*

$$\mu_n \leqslant Ce^{-N2^{n+\kappa}a_n}, \qquad \mu_n \leqslant C2^{-Nn}, \qquad \mu_n \leqslant C\left(\frac{A_n}{A_1}\right)^N. \tag{4.14}$$

Proof. Let $C>0$ and $N>0$ be arbitrary. Since $a_{n+1}/2 < a_n < 2a_{n+1}$, for all $n\in\mathbb{N}$, we have $a_{n+1}2^{n+1}/4 < a_n 2^n < a_{n+1}2^{n+1}$, and thus $\mu_{n+1} < \mu_n < \mu_{n+1}^{1/4}$. By choosing κ and κ' sufficiently large, we have $1/(3A_1)\geqslant N$. Increasing them further, if needed, we obtain the first bound. Keeping κ' fixed, and increasing κ further, if necessary, we obtain the second two bounds in (4.14) by using that

$2^{n+\kappa}a_n \geqslant 2^{n+\kappa}(n+\kappa')^{-2} \geqslant c'2^{\kappa}n$, for some positive constant $c' < 1$ depending only on κ'. The same inequality, together with $A_n/A_1 > a_n/A_1 > (n+\kappa')^{-2}/A_1 > e^{-c'n/A_1} > e^{-2^n/A_1} > e^{-2^{n+\kappa}/(NA_1)}$, where the last inequality is valid for sufficiently large κ, implies the third bound in (4.14). □

Proposition 4.8 directly implies the following.

Corollary 4.9. *Given any $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $n \geqslant 1$,*

$$\mu_n \leqslant C\sigma_n^N, \qquad \mu_n \leqslant C\lambda_n^N \leqslant C\eta_n^N \leqslant C\zeta_n^N, \qquad \mu_n \leqslant C\gamma_n^{-N}. \tag{4.15}$$

At this point we have verified all of the assumptions made in Section 3. We can now apply Theorem 3.9 to the n-th step renormalization operator $\mathcal{R}_n$, defined, by the parameters introduced above, from a subset in $\mathcal{A}_{\rho_{n-1}}(\mathcal{V}_{n-1})$ to $\mathcal{A}_{\rho_n}(\mathcal{V}_n)$. Denote by $\mathcal{L}_n$ the corresponding linear operator from $\mathbb{P}\mathcal{A}_{\rho_{n-1}}(\mathcal{V}_{n-1})$ to $\mathbb{P}\mathcal{A}_{\rho_n}(\mathcal{V}_n)$.

Define $\mathcal{A}_{\rho_n,n} = \mathcal{A}_{\rho_n}(\mathcal{V}_n)$, for all non-negative integers n. From Theorem 3.9 we immediately obtain the following theorem. To simplify the notation, we will not write down explicitly the dependence of the norm on the lattice $\mathcal{V}_n$.

Theorem 4.10. *There exist constants $C, r > 0$, such that the n-th step renormalization operator $\mathcal{R}_n$ is a bounded analytic map from an open ball B_{n-1} in $\mathcal{A}_{\rho_{n-1},n-1}$ of radius $r(\sigma_n/\gamma_n)^2$, centered at K, into $\mathcal{A}_{\rho_n,n}$, satisfying $\|\mathcal{L}_n^{-1}\| \leqslant 1$ and*

$$\begin{aligned}
\big\|\big(\mathbb{I}-\mathbb{E}^+\big)\mathcal{R}_n(X)\big\|_{\rho_n} &\leqslant C\eta_n^{-2}(\gamma_n/\sigma_n)^2\mu_n^{1/2}\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_{\rho_{n-1}},\\
\big\|(\mathbb{I}-\mathbb{P})\mathcal{R}_n(X)\big\|_{\rho_n} &\leqslant C\eta_n^{-2}(\gamma_n/\sigma_n)^2\mu_n^{1/2}\big\|(\mathbb{I}-\mathbb{P})X\big\|_{\rho_{n-1}},\\
\big\|\mathbb{P}\mathcal{R}_n(X)-\mathcal{R}_n(\mathbb{P}X)\big\|_{\rho_n} &\leqslant C\eta_n^{-2}(\gamma_n/\sigma_n)^6\big\|\big(\mathbb{I}-\mathbb{E}^+\big)X\big\|_{\rho_{n-1}}\big\|(\mathbb{I}-\mathbb{P})X\big\|_{\rho_{n-1}}.
\end{aligned} \tag{4.16}$$

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathcal{R}_n$ is a subset of the ball B_{n-1} described in Theorem 4.10, that is open in $\mathcal{A}_{\rho_{n-1},n-1}$ and contains the vector field K. Given a domain $\mathcal{D}_{n-1}$ for each $\mathcal{R}_n$, the domain $\tilde{\mathcal{D}}_n$ of the "composed" renormalization operator $\tilde{\mathcal{R}}_{n+1} = \mathcal{R}_{n+1} \circ \tilde{\mathcal{R}}_n$, for $n \in \mathbb{N}$, with $\tilde{\mathcal{R}}_1 = \mathcal{R}_1$, is defined recursively as the set of all vector fields in the domain of $\tilde{\mathcal{R}}_n$ that are mapped under $\tilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 4.10, these domains are open and non-empty, and the transformations $\tilde{\mathcal{R}}_n$ are analytic on $\tilde{\mathcal{D}}_{n-1}$.

Theorem 4.11. *Let $0 < \epsilon < 1/2$. If κ' and then κ are chosen sufficiently large, then there exist a sequence of domains $\{\mathcal{D}_{n-1}\}$ for the renormalization transformations $\{\mathcal{R}_n\}$, such that the set $\mathcal{W} = \bigcap_n \tilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I}-\mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$. For every $X \in \mathcal{W}$, if $n \geqslant 1$ and $\psi_n = \prod_{i=1}^n \mu_i$, then*

$$\begin{aligned}
\big\|\tilde{\mathcal{R}}_n(X) - K_n\big\|_{\rho_n} &\leqslant \psi_n^{1/2-\epsilon}\big\|(\mathbb{I}-\mathbb{P})X\big\|_{\varrho},\\
\big\|(\mathbb{I}-\mathbb{E})\tilde{\mathcal{R}}_n(X)\big\|_{\rho_n} &\leqslant \psi_n^{1/2-\epsilon}\big\|(\mathbb{I}-\mathbb{E})X\big\|_{\varrho}.
\end{aligned} \tag{4.17}$$

Proof. Our goal is to apply the stable manifold theorem of [15]. To do so, we first rescale our transformations $\mathcal{R}_n$. Let $r_n = r_{-1}(\sigma_{n+1}/\gamma_{n+1})^2$ for every non-negative integer n, with $r_{-1} > 0$ smaller than half the constant r from Theorem 4.10.

Define the transformations R_n by

$$R_n(Z) = r_n^{-1}\big[\mathcal{R}_n(K + r_{n-1}Z) - K\big], \quad n \in \mathbb{N}. \tag{4.18}$$

Define the projection operators $\mathbb{E}_n = \mathbb{E}^+$, $\mathbb{P}_n = \mathbb{P}$, for $n \in \mathbb{N}_0$. The restriction $R_n \mathbb{P}_{n-1}$ defines a linear map from $\mathbb{P}_{n-1}\mathcal{A}_{\rho_{n-1},n-1}$ to $\mathbb{P}_n\mathcal{A}_{\rho_n,n}$, which will be denoted by L_n. By Theorem 4.10, R_n is analytic and bounded on the ball $\|Z\|_\varrho < 2$, and satisfies

$$\begin{aligned} \left\|(\mathbb{I}-\mathbb{E}_n)R_n(Z)\right\|_{\rho_n} &\leqslant \varepsilon_n \left\|(\mathbb{I}-\mathbb{E}_{n-1})Z\right\|_{\rho_{n-1}}, \\ \left\|(\mathbb{I}-\mathbb{P}_n)R_n(Z)\right\|_{\rho_n} &\leqslant \vartheta_n \left\|(\mathbb{I}-\mathbb{P}_{n-1})Z\right\|_{\rho_{n-1}}, \\ \left\|\mathbb{P}_n R_n(Z) - R_n(\mathbb{P}_{n-1}Z)\right\|_{\rho_n} &\leqslant \varphi_n \left\|(\mathbb{I}-\mathbb{E}_{n-1})Z\right\|_{\rho_{n-1}} \left\|(\mathbb{I}-\mathbb{P}_{n-1})Z\right\|_{\rho_{n-1}}, \end{aligned} \tag{4.19}$$

where

$$\varepsilon_n = \vartheta_n = C\eta_n^{-2}(\gamma_n/\sigma_n)^2(\gamma_{n+1}/\sigma_{n+1})^2\mu_n^{1/2} \quad \text{and} \quad \varphi_n = C\eta_n^{-2}(\gamma_n/\sigma_n)^6(\gamma_{n+1}/\sigma_{n+1})^2.$$

Here, $C \geqslant 1$ is a constant that may depend on ϱ, but not on any other renormalization parameters. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict R_n to the domain $D_{n-1} \subset \mathcal{A}_{\rho_{n-1},n-1}$, defined by

$$\|\mathbb{P}_{n-1}Z\|_{\rho_{n-1}} < 1, \qquad \left\|(\mathbb{I}-\mathbb{P}_{n-1})Z\right\|_{\rho_{n-1}} < \delta_{n-1}, \qquad \left\|(\mathbb{I}-\mathbb{E}_{n-1})Z\right\|_{\rho_{n-1}} < \delta_{n-1}, \tag{4.20}$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Corollary 4.9, for any $0 < \epsilon < 1/2$, if κ' and κ are chosen sufficiently large, then $\varepsilon_n \leqslant \mu_{1/2-\epsilon} \leqslant \varepsilon = 3/16$, $\vartheta_n \leqslant \mu_{1/2-\epsilon} \leqslant \vartheta = 1/4$, and

$$C\eta_{n-1}^2\eta_n^{-4}(\gamma_{n-1}/\sigma_{n-1})^{-6}(\gamma_n/\sigma_n)^6(\gamma_{n+1}/\sigma_{n+1})^4\mu_n^{1/2} \leqslant 1,$$

for all positive integers n. The latter inequality implies

$$\varepsilon_n\delta_{n-1} \leqslant \delta_n, \qquad \vartheta_n\delta_{n-1} \leqslant \delta_n, \tag{4.21}$$

for all $n \geqslant 1$. Since, $\delta_{n-1}\varphi_n = 1/6 < \varepsilon$, the last bound in (4.19) implies

$$\left\|\mathbb{P}_n R_n(Z) - R_n(\mathbb{P}_{n-1}Z)\right\|_{\rho_n} \leqslant \varepsilon \left\|(\mathbb{I}-\mathbb{E}_{n-1})Z\right\|_{\rho_{n-1}}. \tag{4.22}$$

The hypotheses of the stable manifold theorem (Theorem 6.1 of [15]) are now verified, with $\vartheta = 1/4$ and $\varepsilon = (1-\vartheta)/4$, and the conclusions of this theorem imply the statements in Theorem 4.11. □

5. Lower-dimensional invariant tori

Let $\mathcal{Z}$ be a lattice in $\mathbb{R}^d$ and, $\mathcal{V}$ its dual lattice. Let $T^d = \mathbb{R}^d/\mathcal{Z}^d$. We say that a vector field $X \in \mathcal{A}_\rho(\mathcal{V})$ has an invariant d-torus with frequency vector ω if there is a continuous embedding Γ from $D_0 = T^d \times \{0\}$ into D_ρ, such that for all $t \in \mathbb{R}$,

$$\Phi_X^t \circ \Gamma = \Gamma \circ \Phi_\omega^t, \tag{5.1}$$

where Φ_ω is the flow of the vector field $(\omega, 0)$. Here 0 is the zero vector in $\mathbb{R}^{d+2D}$.

Consider a one-step renormalization operator $\mathcal{R}$ and a vector field X in the domain of $\mathcal{R}$. If F is any map from D_0 into the domain of $\Lambda_X = \mathcal{U}_X \circ \tilde{\mathcal{T}}$, where $\tilde{\mathcal{T}} = \mathcal{S}_\mu \circ \mathcal{T}$, define the map

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \tilde{\mathcal{T}}. \tag{5.2}$$

Formally, if $\Gamma_{\mathcal{R}(X)}$ is an invariant torus for $\mathcal{R}(X)$, then an invariant torus for X is given by $\Gamma_X = \mathcal{M}_X(\Gamma_{\mathcal{R}(X)})$. This can be seen easily from the identity $\Lambda_X \circ \Phi^{\eta t}_{\mathcal{R}(X)} = \Phi^t_X \circ \Lambda_X$.

Denote by $\mathcal{A}_0(\mathcal{V})$ the Banach space of continuous functions $F : D_0 \to \mathbb{C}^{2d+2D}$, with frequency module in $\mathcal{V}$, for which the norm $\|F\|_{0,\mathcal{V}} = \sum_\nu \|F_\nu\|$ is finite, where $\{F_\nu\}$ are the Fourier coefficients of F.

Consider now a Brjuno frequency vector ω and a fixed but arbitrary vector field X such that its push-forward X_0 under the linear change of coordinates $\mathcal{N}$, described at the beginning of Section 3, belongs to the associated stable manifold $\mathcal{W}$ described in Theorem 4.11. Let $X_n = \mathcal{R}_n(X_{n-1})$ for $n \geqslant 1$. In order to simplify notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions $\Gamma_k \in \mathcal{A}_0(\mathcal{V}_k)$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \tilde{\mathcal{T}}_n^{-1}, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \tilde{\mathcal{T}}_n, \tag{5.3}$$

where $\tilde{\mathcal{T}}_n = \mathcal{T}_n \circ \mathcal{S}_{\mu_n}$, for all $n > 0$. Then we will show that Γ_0 is an invariant torus for X_0. Finally, the pullback of Γ_0 under $\mathcal{N}$ is the desired invariant torus of X.

Let us emphasize several important points here: (i) The original vector field X, for which we will construct an invariant torus, has the periodicity of the lattice $2\pi\mathbb{Z}^d$. The push-forward of this vector field under the linear transformation $\mathcal{N}$ on $\mathbb{R}^d \times \mathbb{R}^{d+2D}$ is the vector field X_0 with the periodicity of a lattice $\mathcal{Z}_0$ which is dual to $\mathcal{V}_0$. (ii) At every step of the renormalization procedure, we have vector fields that are invariant under translations of the lattice $\mathcal{Z}_n$, dual to $\mathcal{V}_n$, that is linearly isomorphic to $\mathbb{Z}^d$, or, equivalently, X_n is a vector field on $T^d_n \times \mathbb{C}^{d+2D}$, where $T^d_n = \mathbb{R}^d/\mathcal{Z}_n$. (iii) Since we use non-integer matrices when scaling, the lattice $\mathcal{Z}_n$ changes with each renormalization step. The n-th step renormalization transformation $\mathcal{R}_n$ maps the vector field X_{n-1} with periodicity of the lattice $\mathcal{Z}_{n-1}$ into a vector field X_n with periodicity of $\mathcal{Z}_n$. (iv) Our construction yields (for each n) an invariant torus for X_n (see Remark 5.5), which is an embedding $\Gamma_n : T^d_n \times \{0\} \to T^d_n \times \mathbb{C}^{d+2D}$. In particular, Γ_0 is an invariant torus with frequency vector $(1, 0, \ldots, 0)$ for X_0. The desired invariant torus Γ with Brjuno frequency vector ω, for the original vector field X, is the pullback of Γ_0 under $\mathcal{N}$, which is an embedding $\Gamma : \mathbb{T}^d \times \{0\} \to \mathbb{T}^d \times \mathbb{C}^{d+2D}$.

Now, let us define $\mathcal{B}_n$, for every $n \geqslant 0$, to be the vector space $\mathcal{A}_0(\mathcal{V}_n)$, equipped with the norm

$$\|f\|'_n = r_n^{-1}\|f\|_{0,\mathcal{V}_n} = r_n^{-1}\sum_{\nu\in\mathcal{V}_n}\|f_\nu\|, \qquad r_n = \frac{\varrho}{4^n}\lambda_n. \tag{5.4}$$

Denote by B_n the unit open ball in $\mathrm{I} + \mathcal{B}_n$, centered at the identity function I, and by $B_n/2$ the ball of radius $1/2$ in the same space.

Proposition 5.1. *If κ' and then κ are chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geqslant 1$, the map $\mathcal{M}_n$ is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$. Furthermore, $\mathcal{M}_n$ takes values in $B_{n-1}/2$, and $\|D\mathcal{M}_n(F)\| \leqslant 1/3$, for all $F \in B_n$.*

Proof. Clearly, $\mathcal{M}_n$ is well defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g), \qquad g = \tilde{\mathcal{T}}_n \circ f \circ \tilde{\mathcal{T}}_n^{-1}, \tag{5.5}$$

where $f = F - \mathrm{I}$. In order to estimate the norm of $\mathcal{U}_{n-1} - \mathrm{I}$, we can apply Lemma 3.8, with ρ'_{n-1} equal to $\rho_{n-1} - (\sigma_{n-1}/\gamma_{n-1})\varrho$. By Lemma 3.8 and Theorem 4.11, there exists a constant $C > 0$, such that

$$\begin{aligned}\|\mathcal{U}_{n-1} - \mathrm{I}\|_{\rho'_{n-1}} &\leqslant C(\gamma_n/\sigma_n)\big\|\mathbb{I}^- X_{n-1}\big\|_{\rho_{n-1}} \leqslant C(\gamma_n/\sigma_n)\psi_{n-1}^{1/2-\epsilon}\big\|\big(\mathbb{I} - \mathbb{E}^+\big)X\big\|_\varrho \\ &\leqslant \psi_{n-1}^{1/2-2\epsilon}\big\|\big(\mathbb{I} - \mathbb{E}^+\big)X\big\|_\varrho \leqslant \psi_n^{1/9},\end{aligned} \tag{5.6}$$

for some small $\epsilon > 0$, for all $n > 1$, and for all $X \in \mathcal{W} \cap B$. Here, we have used Proposition 4.8 and assumed that κ' and then κ have been chosen sufficiently large, and that the neighborhood B of K has been chosen sufficiently small (depending on κ' and κ). Though all steps in (5.6) cannot be carried through if $n = 1$, the final estimate is also valid in that case.

The composition with $\mathrm{I} + g$ in Eq. (5.5) is controlled by Proposition 2.1, using the fact that $\|g\|_{0,\mathcal{V}_{n-1}} \leqslant \eta_n^{-1} r_n \|f\|'_n$ is less than $\rho'_{n-1}/2$, since we assume that $F \in B_n$. Using $r_n/r_{n-1} = \eta_n/4$, we obtain $\|g\|'_{n-1} \leqslant \eta_n^{-1}\eta_n/4 \leqslant 1/4$. From (5.6) we obtain $\|\mathcal{U}_{n-1} - \mathrm{I}\|'_{n-1} \leqslant r_{n-1}\psi_n^{1/16} \leqslant 1/2$, if κ' and κ have been chosen sufficiently large. These estimates show that $\mathcal{M}_{n-1}$ maps B_n into $B_{n-1}/2$.

Now, we obtain a bound on the norm of the derivative map

$$D\mathcal{M}_n(F)\bar{f} = \bar{g} + D(\mathcal{U}_{n-1} - \mathrm{I}) \circ (I + g)\bar{g}, \tag{5.7}$$

where $\bar{g} = \tilde{\mathcal{T}}_n \circ \bar{f} \circ \tilde{\mathcal{T}}_n^{-1}$. Since $\|g\|_{0,\mathcal{V}_{n-1}} \leqslant \rho_{n-1}/2$, and

$$\left\| D(\mathcal{U}_{n-1} - \mathrm{I}) \right\|_{\rho'_{n-1}/2} \leqslant \frac{2}{\rho'_{n-1}} \|\mathcal{U}_{n-1} - \mathrm{I}\|_{\rho'_{n-1}}, \tag{5.8}$$

we obtain a bound on this derivative norm analogous to (5.6). This, together with the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\eta_n/4$, shows that $\|D\mathcal{M}_n(F)\| \leqslant 1/3$, for all $n \geqslant 1$, and for all $F \in B_n$. □

Below, we will make use of the following estimate on the difference between the flow for X and the flow for the constant vector field $(\omega, 0)$.

Proposition 5.2. *(See [15].) Let τ be a positive real number and X a vector field in $\mathcal{A}_\varrho$, such that $\tau\|X - \omega\|_\varrho < r < \varrho$. Then for all times t in the interval $[-\tau, \tau]$,*

$$\left\| \Phi_X^t - \Phi_\omega^t \right\|_{\varrho - r} \leqslant \left\| t(X - \omega) \right\|_\varrho. \tag{5.9}$$

Here and in what follows, we write ω for the constant vector field $(\omega, 0)$, in order to simplify the notation.

Let Φ_n be the flow for the vector field X_n. In order to prove that a solution to (5.3) yields an invariant torus Γ_0 for X, we will use the identity

$$\Phi_{n-1}^t \circ \mathcal{M}_n(F) \circ \Phi_\omega^{-t} = \mathcal{M}_n\big(\Phi_n^{\eta_n t} \circ F \circ \Phi_\omega^{-\eta_n t}\big), \tag{5.10}$$

which follows from the relation described after (5.2), between the flow for a vector field and the flow for the corresponding renormalized vector field. This requires an estimate of the following type.

Proposition 5.3. *Under the same assumptions as in Proposition 5.1, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geqslant 1$, the function $\Phi_n^s \circ F \circ \Phi_\omega^{-s}$ belongs to B_n, whenever $F \in B_n/2$ and $|s| \leqslant \lambda_n/(4\max\{\|\Omega\|, 1\})$.*

Proof. We will use the following easily verifiable identity

$$\Phi_n^s \circ F \circ \Phi_\omega^{-s} = \mathrm{I} + f \circ \Phi_\omega^{-s} + \big[\Phi_n^s \circ \Phi_\omega^{-s} - \mathrm{I}\big] \circ \big(\mathrm{I} + f \circ \Phi_\omega^{-s}\big). \tag{5.11}$$

Since $\|f\|'_n \leqslant 1/2$, and $\|f \circ \Phi_\omega^{-s}\|_{0,\mathcal{V}_n} = \|f\|_{0,\mathcal{V}_n}$, we have $\|f \circ \Phi_\omega^{-s}\|_{0,\mathcal{V}_n} \leqslant r_n/2$. The composition in (5.11) is well defined since $r_n/2 < \rho_n$ (see Definition 4.5).

By Proposition 5.2, we have the bound

$$\left\|\Phi_n^s \circ \Phi_\omega^{-s} - \mathrm{I}\right\|_{r_n/2} \leqslant \left\|s(X_n - \omega)\right\|_{r_n} \leqslant \left\|s(X_n - K_n)\right\|_{r_n} + \left\|s(K - \omega)\right\|_{r_n}, \tag{5.12}$$

provided that the right-hand side of this inequality is less than $r_n/2$. Since $X \in \mathcal{W}$, using Theorem 4.11 we can bound the first term as $\|s(X_n - K_n)\|_{r_n} \leqslant |s|\psi_n^{1/2-\epsilon}\|X - K\|_\varrho$ which can be made smaller than $r_n/4$, for any n, if $\|X - K\|_\varrho$ is chosen sufficiently small. The second term can be bounded as $\|s(K_n - \omega)\|_{r_n} \leqslant |s|\|\Omega_n\|r_n = |s|\|\Omega\|\lambda_n^{-1} r_n \leqslant r_n/4$. Thus, the sum on the right-hand side of inequality (5.12) is indeed smaller than $r_n/2$. Finally,

$$\left\|\Phi_n^s \circ F \circ \Phi_\omega^{-s} - \mathrm{I}\right\|'_{n-1} \leqslant r_{n-1}^{-1} r_n \leqslant \eta_n/4, \tag{5.13}$$

which proves the claim. □

Now we are ready to construct invariant tori. A function f defined on $\mathcal{W}$ is said to be analytic if $f \circ W$ is analytic on the domain of W.

Theorem 5.4. *Under the same assumptions as in Proposition* 5.1, *there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds. Given any $X \in \mathcal{W} \cap B$, and any sequence of functions $F_k \in B_k$, define*

$$\Gamma_{n,k} = (\mathcal{M}_{n+1} \circ \cdots \circ \mathcal{M}_k)(F_k), \quad 0 \leqslant n < k. \tag{5.14}$$

Then the limits $\Gamma_n = \lim_{k\to\infty} \Gamma_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots,$ and satisfy the identities (5.3)*. Furthermore, Γ_0 is an invariant torus for X, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $\mathcal{W} \cap B$.*

Proof. By Proposition 5.1, the map $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least 1/2. Thus, if $1 \leqslant n < k < k'$, then the difference $\Gamma_{n,k'} - \Gamma_{n,k}$ is bounded in norm by 2^{n-k+1}. This shows that the sequence $k \mapsto \Gamma_{n,k}$ converges in $\mathcal{B}_n$ to a limit Γ_n, which is independent of the choice of the functions F_k. By choosing $F_k = \Gamma_k$ for all k, we obtain the identities (5.3). The analyticity of $X \mapsto \Gamma_0$ follows via the chain rule from the analyticity of the maps used in our construction, and from uniform convergence.

In order to prove that Γ_0 is an invariant torus for X, we will use the identity (5.10). To be more precise, given a real number t, with $|t| < (4\max\{\|\Omega\|, 1\})^{-1}$, define $t_n = \lambda_n t$ for all $n \geqslant 0$. Proposition 5.3 allows us to iterate the identity (5.10), and get the identity

$$\Phi_0^t \circ \Gamma_{0,k} \circ \Phi_\omega^{-t} = (\mathcal{M}_1 \circ \cdots \circ \mathcal{M}_k)\big(\Phi_k^{t_k} \circ \Phi_\omega^{-t_k}\big), \tag{5.15}$$

for all $k > 0$. As proved above, the right- (and thus left)-hand side of this equation converges in $\mathcal{A}_0$ to Γ_0. In addition, $\Gamma_{0,k} \to \Gamma_0$ in $\mathcal{A}_0$, and the convergence is pointwise as well, by part (i) of Proposition 2.1. Thus, since the flow Φ_0^t is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Phi_\omega^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$, due to the group property of the flow, and the fact that composition with Φ_ω^s is an isometry on $\mathcal{A}_0$. □

Remark 5.5. It easily follows from Theorem 5.4 that Γ_n is an invariant torus for $X_n = \mathcal{R}_n(X_{n-1})$, which is an embedding from the torus $T_n^d \times \{0\}$ into $T_n^d \times \mathbb{C}^{d+2D}$.

Let $\mathcal{A}_\rho^0$ be the subspace of functions $\mathcal{A}_\rho$ which do not depend on the variables y, u, w. In what follows, the torus Γ_0 associated with a vector field $X \in \mathcal{W}$ will be denoted by Γ_X. For convenience, we extend the map $X \mapsto \Gamma_X$ to an open neighborhood of K, by setting $\Gamma_X = \Gamma'_X$, where $X' = (\mathbb{I} + W)(X - \mathbb{P}X) \in \mathcal{W}$.

Theorem 5.6. *Let $\rho > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Proposition* 5.1, *there exists an open neighborhood B of K in $\mathcal{A}_\rho(\mathcal{V}_0)$, such that Γ_X has an analytic continuation to $\|\operatorname{Im} x\| < \delta$, for each $X \in B$. The map $X \mapsto \Gamma_X$ defines* (*via the above extension*) *a bounded analytic map from B to $\mathcal{A}^0_\delta(\mathcal{V}_0)$.*

Proof. The proof of this theorem is completely analogous to the proof of Theorem 4.5 of [15]. For that reason, we will give only a sketch here.

Consider the translations $R_q(x, y, u, w) = (x + q, y, u, w)$. By examining the construction of $\mathcal{W}$ and Γ_X, one verifies that for any $q \in \mathbb{R}^d$, the translated vector field $R_q^* X$ belongs to $\mathcal{W}$ whenever X does, and that

$$\Gamma_X(q, 0, 0, 0) = (R_q \circ \Gamma_{R_q^* X})(0, 0, 0, 0). \tag{5.16}$$

The idea now is to use the analyticity of map $X \mapsto \Gamma_X$, to extend the right-hand side of Eq. (5.16) to the complex domain $\|\operatorname{Im} q\| < \delta$. This yields the desired analytic continuation of Γ_X. The remaining parts of Theorem 5.6 are proved by using the fact that the right-hand side of (5.16) is jointly analytic in X and q. $\square$

The proof of Theorem 1.3 follows from Theorems 4.11, 5.4 and 5.6. Our construction in Theorems 5.4 and 5.6 yields an invariant torus Γ_0 for X_0, which is an embedding from $T_0^d \times \{0\}$ into $T_0^d \times \mathbb{C}^{d+2D}$. The invariant torus with the Brjuno frequency vector ω for the vector field X on the pullback of $\mathcal{W}$ under $\mathcal{N}$ (denoted by $\mathcal{W}$ in Theorem 1.3) is the pullback of Γ_0 under $\mathcal{N}$.

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