C^1-rigidity of circle maps with breaks for almost all rotation numbers

Konstantin Khanin[1]*, Saša Kocić[2]† and Elio Mazzeo[1]‡

[1] Dept. of Math., University of Toronto, 40 St. George St., Toronto, ON, Canada M5S 2E4

[2] Dept. of Math., University of Mississippi, P. O. Box 1848, University, MS 38677-1848, USA

December 29, 2016

Abstract

We prove that, for almost all irrational $\rho \in (0,1)$, every two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphisms with a break point, i.e., a singular point where the derivative has a jump discontinuity, with the same rotation number ρ and the same size of the break $c \in \mathbb{R}_+ \backslash \{1\}$, are C^1-smoothly conjugate to each other.

Résumé

Nous démontrons que pour presque tous les irrationnels $\rho \in (0,1)$, deux difféomorphismes du cercle $C^{2+\alpha}$ lisses, $\alpha \in (0,1)$, avec un point de singularité de type rupture où la dérivée a une discontinuité de saut, avec le même nombre de rotation ρ et la même taille de rupture $c \in \mathbb{R}_+ \backslash \{1\}$, sont C^1-conjugués l'un à l'autre.

2000 Mathematics Subject Classification: 37E10, 37E20.

Keywords and phrases: Rigidity, conjugacy, circle maps, diffeomorphisms with a break.

Mots-clés et phrases: Rigidité, conjugaison, cartes de cercle, difféomorphismes avec des singularités de type rupture.

1 Introduction

This paper establishes generic C^1-rigidity for circle diffeomorphisms with breaks. The result can viewed as a one-parameter extension of Herman's theory on the linearization of circle diffeomorphisms.

*Email: khanin@math.utoronto.ca

†Email: skocic@olemiss.edu (Corresponding author)

‡The main contribution to the paper is due to the first two authors.

The problem of smoothness of a conjugacy to a linear rotation for smooth diffeomorphisms of a circle is a classic problem in dynamical systems. It was proven by Arnol'd [1], using the methods of KAM (Kolmogorov-Arnol'd-Moser) theory, that every analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rigid rotation $R_\rho : x \mapsto x+\rho \mod 1$, is analytically conjugate to R_ρ. A number ρ is called Diophantine if there exists $C > 0$ and $\beta \geq 0$ such that $|\rho - p/q| > C/q^{2+\beta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Arnol'd also conjectured that the result remains true if the requirement of closeness to the rigid rotation is removed. A version of this global rigidity result, for smooth circle diffeomorphisms, was proven by Herman [6], and is the subject of classical Herman's theory. The theory was further developed for C^r-smooth maps, $r \geq 3$, by Yoccoz [20] who established a dependence of the regularity of the conjugacy on the Diophantine properties of the rotation numbers. For more recent work we refer the reader to [7, 11, 13, 18]. In a recent formulation [13], every $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphism, with rotation number ρ Diophantine with exponent $\beta < \alpha$, is $C^{1+\alpha-\beta}$-smoothly conjugate to the rotation R_ρ. Arnol'd also proved that this result cannot be extended to all irrational rotation numbers [1]. He constructed examples of analytic circle diffeomorphisms with irrational rotation numbers for which the invariant measure is singular, which implies that the conjugacy to the rigid rotation is not absolutely continuous.

We use the term *rigidity* for the phenomenon that any two maps within a given equivalence class determined by topological conjugacy are, in fact, C^1-smoothly conjugate to each other. Herman's theory establishes that, in the case of smooth circle diffeomorphisms, rigidity is guaranteed when rotation numbers satisfy a Diophantine condition. Over the last two decades, great effort has been made to understand the rigidity properties of circle diffeomorphisms with a singular point where the diffeomorphism condition is violated. The singular points refer either to points where the derivative vanishes (critical points) or where it has a jump discontinuity (break points). In the case of critical circle maps, i.e., circle maps with a single singular point where the derivative vanishes, the first rigidity results were obtained by de Faria and de Melo [4, 5]. They established the *convergence of renormalizations* — the main technical tool in proving rigidity results — and rigidity for analytic critical circle maps with the same irrational rotation number of bounded type (i.e., with bounded partial quotients) and the same (odd-integer) order of the critical point (i.e., the exponent of the power law behavior of the map in a neighborhood of the critical point). Renormalizations f_n of a circle map T are obtained from the restriction of T^{q_n} to a small interval, by an affine change of coordinates, where q_n is the denominator of the rational convergent p_n/q_n of the rotation number ρ (see next section). The convergence of renormalizations for analytic critical circle maps and for all irrational rotation numbers was later established by Yampolsky [19]. The results of de Faria and de Melo [4] show that even stronger $C^{1+\epsilon}$-rigidity of analytic critical circle maps, for some $\epsilon > 0$, is generic, i.e., it holds for almost all irrational rotation numbers. C^1-rigidity of analytic critical circle maps holds for all irrational rotation numbers, as was shown by Khanin and Teplinsky [12]. This phenomenon, when rigidity holds without any Diophantine-type conditions, is referred to as *robust rigidity.* Rigidity theory of non-analytic critical circle maps, however, remained an open problem since, up to now, there is no proof of the convergence of renormalizations in this case.

The above results for critical circle maps suggested [8] that the rigidity might also be robust in the case of circle diffeomorphisms with a break point. In [8], rigidity was established for a set of rotation numbers of zero Lebesgue measure. However, as was shown by two of us [9], the above conjecture is false — robust rigidity does not hold for circle maps with breaks. We proved in [9] that there are irrational rotation numbers ρ, and pairs of analytic circle diffeomorphisms with breaks, with the same rotation number ρ and the same size of the break (i.e., the square root of the ratio of the left and right derivatives at the break point), for which any conjugacy between them is not even Lipschitz continuous. The question whether rigidity holds for typical rotation numbers, however, remained open. The main result of this paper provides an affirmative answer to this question.

Before we state our main result, let us define precisely the class of maps that we consider. A C^r-smooth circle diffeomorphism (map) with a break is a map $T:\mathbb{T}^1\to\mathbb{T}^1$, $\mathbb{T}^1=\mathbb{R}/\mathbb{Z}$, for which there exists $x_{\rm br}\in\mathbb{T}^1$ such that $T\in C^r([x_{\rm br},x_{\rm br}+1])$; $T'(x)$ is bounded from below by a positive constant on $[x_{\rm br},x_{\rm br}+1]$; the one-sided derivatives of T at $x_{\rm br}$ are such that the size of the break

$$c:=\sqrt{\frac{T'_-(x_{\rm br})}{T'_+(x_{\rm br})}}\neq 1. \tag{1.1}$$

The main result of this paper is based on the following theorem.

Theorem 1.1 ([10]) *Let $\alpha\in(0,1)$ and let $c\in\mathbb{R}^+\backslash\{1\}$. There exists $\lambda\in(0,1)$ such that, for every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same irrational rotation number $\rho\in(0,1)$, and the same size of the break c, there exists $C>0$, such that the renormalizations f_n and $\widetilde{f}_n$ of T and $\widetilde{T}$, respectively, satisfy $\|f_n-\widetilde{f}_n\|_{C^2}\leq C\lambda^n$, for all $n\in\mathbb{N}$.*

Remark 1 This theorem establishes the exponential convergence of renormalizations for circle diffeomorphisms with a break, with a uniform rate λ for all irrational rotation numbers. Moreover, there exists $\mu\in(0,1)$, independent of α, such that $\lambda=\mu^\alpha$. This result is stronger than what is needed for our next theorem. Note that the statement of the theorem remains true if $c=1$. This essentially follows from Herman's theory.

Let $\lambda_1\in(\lambda,1)$ and $C_1>0$. Let $S_e(C_1,\lambda_1)$ and $S_o(C_1,\lambda_1)$ be the sets of all irrational rotation numbers $\rho=[k_1,k_2,\dots]\in(0,1)$ whose subsequence of partial quotients k_{n+1} (see next section) for all n even or odd, respectively, satisfies the bound $k_{n+1}\leq C_1\lambda_1^{-n}$. Let $S_e(\lambda_1):=\cup_{C_1>0}S_e(C_1,\lambda_1)$ and $S_o(\lambda_1):=\cup_{C_1>0}S_o(C_1,\lambda_1)$. We define $S:=S_e(\lambda_1)$, if

$0 < c < 1$, and $S := S_o(\lambda_1)$, if $c > 1$. Theorem 1.1 and Theorem 2.2, proven in this paper, imply the following strong rigidity statement for circle diffeomorphisms with a break.

Theorem 1.2 *Any two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphisms with a break T and $\widetilde{T}$, with the same size of the break $c \in \mathbb{R}_+ \backslash \{1\}$ and the same rotation number $\rho \in S$, are C^1-smoothly conjugate to each other, i.e., there exists a C^1-smooth diffeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $\varphi \circ T \circ \varphi^{-1} = \widetilde{T}$.*

Remark 2 Set S has full Lebesgue measure. One can also see that it contains some strongly Liouville numbers. The difference between the cases of odd and even n is related to a difference in the behavior of the renormalizations f_n, which will be explained in detail in the next section. If $0 < c < 1$ and n is even and sufficiently large or if $c > 1$ and n is odd and sufficiently large, the renormalizations f_n are concave and the renormalization parameter $a^{(n)} = f_n(0)$ (see the next section) can be exponentially small in k_{n+1}. If $0 < c < 1$ and n is odd and sufficiently large or if $c > 1$ and n is even and sufficiently large, the renormalizations f_n are convex and $a^{(n)}$ is bounded away from zero. The imposed condition on the rotation numbers controls the smallness of this parameter. It is not difficult to see that even the set of rotation numbers $\cap_{\lambda_1 \in (0,1)} S_o(\lambda_1) \cap S_e(\lambda_1)$, for which rigidity holds for any $\alpha \in (0,1)$ and any $c \in \mathbb{R}^+ \backslash \{1\}$, has full Lebesgue measure. On the other hand, it is not obvious that the sets $\cup_{\lambda_1 \in (\lambda,1)} S_o(\lambda_1)$ and $\cup_{\lambda_1 \in (\lambda,1)} S_e(\lambda_1)$, for which rigidity holds for some $\alpha \in (0,1)$ and $c \in \mathbb{R}_+ \backslash \{1\}$, can be extended.

Remark 3 It was recently proven by Kocić [16] that the result of Theorem 1.2 cannot be strengthened by requiring that the conjugacy φ is $C^{1+\varepsilon}$-smooth, for some $\varepsilon > 0$. Kocić proved [16] that, for a set of full Lebesgue measure irrational $\rho \in (0,1)$, for every $c \in \mathbb{R}_+ \backslash \{1\}$, every $r > 2$, and every $\varepsilon > 0$, there exists a pair of C^r-smooth circle diffeomorphisms with a break of size c, with the same rotation number ρ, which are not $C^{1+\varepsilon}$-smoothly conjugate to each other. In fact, he proved a stronger result: for a set of full Lebesgue measure irrational $\rho \in (0,1)$, every $c \in \mathbb{R}_+ \backslash \{1\}$ and every $r > 2$, there exists a pair of C^r-smooth circle diffeomorphisms with a break of size c, with rotation number ρ, which are not $C^{1+\varepsilon}$-smoothly conjugate to each other, for any $\varepsilon > 0$.

Remark 4 The main difficulty in the proofs of Theorem 1.1 and Theorem 1.2 is that the geometry is *strongly unbounded* in this case. This means that the ratio of two nearby elements of dynamical partitions $\mathcal{P}_n$ (see next section) may be of the order of $a^{(n)}$ which can be exponentially small with k_{n+1}. This should be compared to algebraic decay with k_{n+1} in the case of circle diffeomorphisms, and the bounded geometry of critical circle maps. Since this ratio plays an important role in analysis of circle diffeomorphisms with breaks with typical rotation numbers, we must deal with quantities which are smaller than exponentially small with n. This creates major difficulties since, in general, renormalizations of these maps converge only exponentially fast [16]. Due to this difficulty, earlier rigidity results on circle maps with breaks were restricted to rotation numbers for which the geometry is bounded. Those include [8], where rigidity was established for a countable set of rotation numbers and [14], where rigidity was established for a larger set of zero measure. The strongly unbounded geometry is also the reason that one cannot obtain robust rigidity in the case of circle diffeomorphisms with breaks [9]. The set of rotation numbers for which C^1-rigidity holds includes those for which the geometry is super-exponentially bounded, i.e., the logarithms of the ratios of nearby elements of dynamical partitions are bounded by an exponential function. Finally, the strongly unbounded geometry is also the reason that circle diffeomorphisms with breaks are, generically, not $C^{1+\varepsilon}$-rigid, for any $\varepsilon > 0$ [16] (see Remark 3).

At the end of this introduction, let us mention that there is a close relationship between circle maps with breaks and nonlinear (generalized) interval exchange transformations. A nonlinear interval exchange transformation (IET) is obtained by replacing the branches of a piecewise-linear map of an IET by smooth nonlinear homeomorphisms. It is well-known that an IET of two intervals (subintervals of $[0, 1]$) can be viewed as a rigid rotation on a circle, if the end points of the interval $[0, 1]$ are identified. Since, in general, the derivatives at the end points of the intervals do not match, a nonlinear IET of two intervals is a circle map with two break points. As the points are on the same orbit of the map, the map can be conjugated piecewise-smoothly to a circle map with one point of break. Theorem 1.2, thus, corresponds to a non-linearizable case of two intervals. The linearizable case of general nonlinear IET has been studied by Marmi, Moussa and Yoccoz in [17]. The case of cyclic permutations, which corresponds to circle maps with more than one point of break, with product of the sizes of breaks being equal to 1, was studied in [2, 3]. Renormalizations of such maps approach the space of piecewise-linear maps. We consider the general case when the renormalized maps are essentially nonlinear. The convergence of renormalizations Theorem 1.1 and the rigidity Theorem 1.2 are currently the only results in the general non-affine case, for generic rotation numbers. They can also be considered a one-parameter extension of Herman's theory, with the parameter being the size of break c.

The paper is organized as follows. In Section 2, we introduce a general renormalization setting for orientation-preserving circle homeomorphisms and formulate regularity conditions and a rigidity theorem (Theorem 2.2) for maps whose renormalizations satisfy these conditions. In Section 3, we formulate a criterion of smoothness of the conjugacy in terms of ratios of the lengths of the corresponding intervals of dynamical partitions. In the same section, we obtain necessary estimates on these ratios on a fundamental interval and prove Theorem 2.2 by spreading them to the whole circle and using the criterion of smoothness. In Section 4, Theorem 1.2 is proven by verifying that the conditions of Theorem 2.2 hold true in the case of circle diffeomorphisms with breaks.

2 Renormalizations of circle homeomorphisms and a rigidity theorem

2.1 Renormalizations of circle homeomorphisms

It has been known since Poincaré that, for every orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, there is a unique rotation number $\rho \in [0, 1)$, which is given by the x-independent limit $\rho := \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is any lift of T to $\mathbb{R}$. If the rotation number $\rho \in (0, 1)$, it can be expressed in the form of a *continued fraction expansion*

$$\rho = \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \dots}}}, \tag{2.1}$$

that we write as $\rho = [k_1, k_2, k_3, \dots]$. The sequence of integers k_n, called *partial quotients*, is infinite and defined uniquely if and only if ρ is irrational. Every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, \dots, k_n]$. It is well-known that this is a sequence of best rational approximates of ρ, i.e., there are no rational numbers with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively as $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the renormalizations, we start with a *marked point* $x_0 \in \mathbb{T}^1$, and consider the *marked semi-orbit* $x_i = T^i x_0$, for $i \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}_0}$, indexed by the denominators of the sequence of rational convergents p_n/q_n of the rotation number ρ, will be called the sequence of *dynamical convergents*. Although $x_{q_{-1}}$ and x_0 coincide on the circle, we formally set $x_{q_{-1}} := x_0 - 1$. The combinatorial equivalence of all circle homeomorphisms with the same irrational rotation number implies that the order of the dynamical convergents of T is the same as the order of the dynamical convergents for the rigid rotation R_ρ. The well-known arithmetic properties of the rational convergents now imply that dynamical convergents alternate their order in the following way:

$$x_{q_{-1}} < x_{q_1} < x_{q_3} < \dots < x_0 < \dots < x_{q_2} < x_{q_0}. \tag{2.2}$$

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$, and called the n-th *renormalization segments*. The n-th renormalization segment associated to the marked point x_i will be denoted by $\Delta_i^{(n)}$. We also define $\bar{\Delta}_0^{(n-1)} := \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$, and $\check{\Delta}_0^{(n-1)} := \Delta_0^{(n-1)} \setminus \Delta_0^{(n+1)}$. In addition to the property (2.2), we also have the following important property: the only points of the orbit $\{x_i : 0 < i \le q_{n+1}\}$ that belong to $\Delta_0^{(n-1)}$ are $\{x_{q_{n-1}+iq_n} : 0 \le i \le k_{n+1}\}$.

Certain images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ cover the whole circle without overlapping beyond the end points, forming the n-th *dynamical partition* of $\mathbb{T}^1$,

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \le i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \le i < q_{n-1}\}. \tag{2.3}$$

The endpoints of the intervals from $\mathcal{P}_n$ form the set

$$\Xi_n := \{x_i : 0 \le i < q_{n-1} + q_n\}. \tag{2.4}$$

We also define the extended partition $\mathcal{P}_n^* := \mathcal{P}_n \cup \{T^{q_n}(\Delta_0^{(n-1)}), T^{q_{n-1}}(\Delta_0^{(n)})\}$ and the extended set $\Xi_n^* := \Xi_n \cup \{x_{q_{n-1}+q_n}\}$.

The following lemma follows directly from the properties of the continued fractions.

Lemma 2.1 *For every $m > n$, we have the following decomposition*

$$\Xi_m \cap \check{\Delta}_0^{(n-1)} = \bigcup_{x_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{x_{q_n}\}} \ \bigcup_{0 \le i < k_{n+1}} \{x_{l+q_{n-1}+iq_n}\}. \tag{2.5}$$

Furthermore, for every $x_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{x_{q_n}\}$, we have $x_{l+q_{n-1}+k_{n+1}q_n} = x_{l+q_{n+1}} \in \Xi_m^* \cap \bar{\Delta}_0^{(n)}$.

The n-th *renormalization* of an orientation-preserving homeomorphism of the circle T, with rotation number $\rho = [k_1, k_2, k_3, \dots]$, with respect to the marked point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$ obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates from $\Delta_0^{(n-1)}$ to $[-1, 0]$ that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}\,. \tag{2.6}$$

If we identify x_0 with zero, then τ_n is exactly a multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here and in what follows, we use $|\cdot|$ to denote the length of an interval. Definition (2.6) is valid for all $n \in \mathbb{N}_0$ if and only if ρ is irrational.

2.2 Renormalizations of circle diffeomorphisms with breaks

In the case of a circle diffeomorphism with a break, we will use the break point $x_{\mathrm{br}} = 0$ as the marked point x_0.

It was shown in [15] that the renormalizations f_n of $C^{2+\alpha}$-smooth circle diffeomorphisms with a break of size $c \in \mathbb{R}_+ \setminus \{1\}$ approach, exponentially fast in the C^2-norm, a particular family of fractional linear transformations

$$F_{a^{(n)},b^{(n)},M^{(n)},c^{(n)}} : z \mapsto \frac{a^{(n)} + (a^{(n)} + b^{(n)}M^{(n)})z}{1 - (M^{(n)} - 1)z}, \tag{2.7}$$

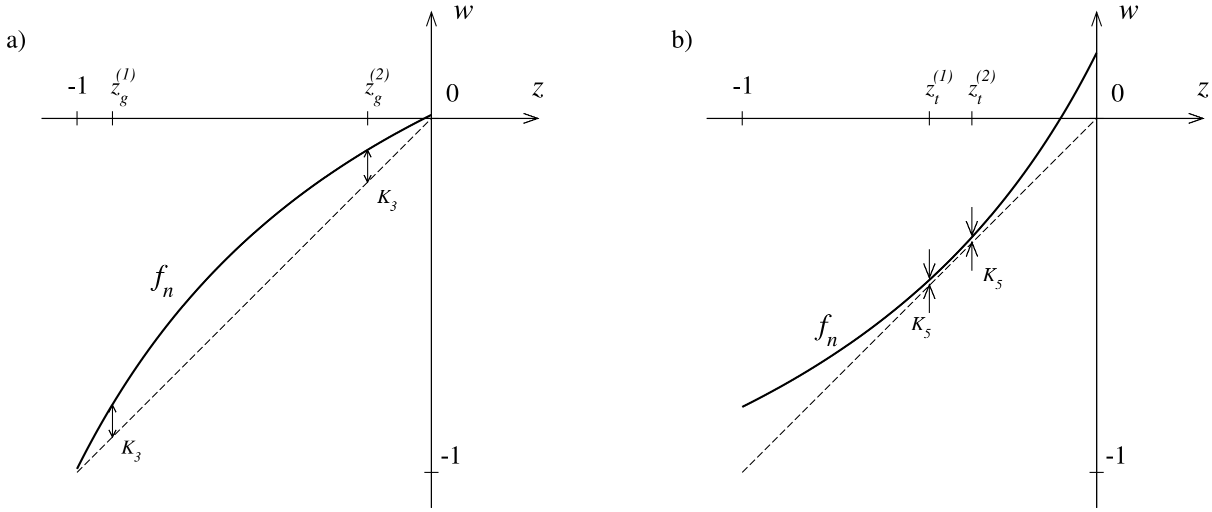

Figure 1: The graph of a renormalized map f_n for sufficiently large n: a) Case $0 < c < 1$ and n even, or $c > 1$ and n odd; b) Case $0 < c < 1$ and n odd, or $c > 1$ and n even.

where $c^{(n)} = c$, if n is even, $c^{(n)} = c^{-1}$, if n is odd, and

$$a^{(n)} = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad b^{(n)} = \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad M^{(n)} = \exp\left((-1)^n \int_{\Delta_0^{(n-1)}} \frac{(T^{q_n})''(z)}{2(T^{q_n})'(z)}\, dz \right). \tag{2.8}$$

The following estimates were also proven in [15]. For every $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphism with a break T, with a break of size $c \in \mathbb{R}_+ \setminus \{1\}$, there exist constants $V := \mathrm{Var}_{x \in \mathbb{T}^1} \ln T' < \infty$, $\mathcal{C} > 0$ and $\lambda \in (0,1)$, such that, for all $n \in \mathbb{N}$, we have

(A) $|\ln(T^{q_n})'(x)| \leq V$, for all $x \in \mathbb{T}^1$ (at points where the derivative has a break, both left and right derivatives are considered),

(B) $\|f_n - F_{a^{(n)},b^{(n)},M^{(n)},c^{(n)}}\|_{C^2} \leq \mathcal{C}\lambda^n$,

(C) $|a^{(n)} + b^{(n)} M^{(n)} - c^{(n)}| \leq \mathcal{C} a^{(n)} \lambda^n$, and

(D) $|M^{(n+1)} - c^{(n+1)}(1 + a^{(n+1)} a^{(n)} (M^{(n)} - 1))| \leq \mathcal{C} a^{(n+1)} a^{(n)} \lambda^n$.

We will refer to (A) as the Denjoy estimate. As we showed in [10], the constant $\lambda \in (0,1)$ can be chosen uniformly for all T with the same size of the break c and Hölder exponent α.

As already mentioned in Remark 2, for maps with breaks, the graphs of the renormalizations f_n look different in the cases of odd and even n (see Figure 1).

The following behavior of renormalizations of circle maps with breaks will be verified in the proof of Theorem 1.2. If $c > 1$, the map f_n is concave for sufficiently large odd n.

Moreover, as $k_{n+1}\to\infty$, the graph of f_n approaches the diagonal $w=z$ at the end points $z=-1$ and $z=0$. Below, we call the small intervals containing these end points the *gates* (the intervals $[-1,z_g^{(1)})$ and $(z_g^{(2)},0]$ on Figure 1 (a)). On the contrary, if n is even and sufficiently large, the map f_n is convex and its graph approaches the diagonal as $k_{n+1}\to\infty$ at a single point of almost-tangency, strictly between -1 and 0. We will later call an interval containing this point of almost-tangency the *tunnel* (the interval $(z_t^{(1)},z_t^{(2)})$ on Figure 1 (b)). The restriction on k_{n+1} in the definition of the set S is related to the concave case. Inside the gates, the distance between successive iterates of f_n grows/decays exponentially, which makes the smallest distance and $a^{(n)}$ to be exponentially small with k_{n+1}. The restriction on the growth rate of k_{n+1} provides a restriction on the rate of decrease of $a^{(n)}$. In the convex case, $a^{(n)}$ is bounded away from zero, and no restriction on k_{n+1} is necessary.

If $0<c<1$, the behavior of the renormalizations f_n is the opposite.

This behavior of renormalizations serves as a motivation for the (more general) regularity conditions introduced in the next section.

2.3 Regularity conditions and a rigidity theorem

Let $\mathfrak{n}:=(\mathfrak{n}_\ell)_{\ell\in\mathbb{N}}$ be an increasing subsequence (infinite, finite or empty) of numbers in $\mathbb{N}_0$. A sequence of functions $g_n:[-1,0]\to\mathbb{R}$, with $n\in\mathbb{N}_0$, will be called *K-regular* with respect to $\mathfrak{n}$, for some vector $K=(K_1,K_2,K_3,K_4,K_5,K_6)\in\mathbb{R}_+^6$, if all g_n satisfy the below conditions (i) and (ii); each g_n, such that $n=\mathfrak{n}_\ell$, for some $\ell\in\mathbb{N}$, satisfies (iii) and (iv); and each g_n, such that $n\neq\mathfrak{n}_\ell$, for any $\ell\in\mathbb{N}$, satisfies (v) and (vi), where:

(i) $\|g_n\|_{C^2}\le K_1$ on $[-1,0]$,

(ii) $g_n'(z)>K_2$ for every $z\in[-1,0]$,

(iii) the set $B_{g_n,K_3}:=\{z\in[-1,0]:g_n(z)-z<K_3\}$ is either empty or consists of one or the union of two disjoint intervals each of which contains one end point, $[-1,z_g^{(1)})$ and $(z_g^{(2)},0]$, where $z_g^{(1)},z_g^{(2)}\in(-1,0)$ (we refer to these intervals as the *gates*),

(iv) $g_n''(z)<-K_4$, for $z\in B_{g_n,K_3}$,

(v) the set B_{g_n,K_5} is either an open interval or empty (we refer to this interval as the *tunnel*; since the points -1 and 0 are outside of the tunnel, this implies $g_n(-1)\ge K_5-1$ and $g_n(0)\ge K_5$),

(vi) $g_n''(z)>K_6$, for $z\in B_{g_n,K_5}$.

Let $\lambda_1\in(0,1)$. For a given subsequence $\mathfrak{n}$ of $\mathbb{N}$, let $\mathcal{S}_\mathfrak{n}=\mathcal{S}_\mathfrak{n}(\lambda_1)$ be the set of $\rho\in(0,1)\backslash\mathbb{Q}$ for which there exists $C_1(\rho)>0$ such that the partial quotients of ρ satisfy $k_{\mathfrak{n}_\ell+1}\le C_1\lambda_1^{-\mathfrak{n}_\ell}$, for every $\ell\in\mathbb{N}$. In the following, $C_1=C_1(\rho)$ is the constant associated to $\rho\in\mathcal{S}_\mathfrak{n}$.

A system of nested partitions $\mathcal{P}_n$, i.e., a sequence of partitions such that each element of a partition $\mathcal{P}_{n+1}$ is contained in an element of a partition $\mathcal{P}_n$, is called *refining* if the maximal length of elements of partition $\mathcal{P}_n$ approaches zero as $n \to \infty$; it is called *exponentially refining* if there exist $C_{\text{ref}} > 0$ and $\lambda_{\text{ref}} \in (0,1)$, such that $|I_m| \leq C_{\text{ref}} \lambda_{\text{ref}}^{m-n} |I_n|$, for any $I_n \in \mathcal{P}_n$ and $I_m \in \mathcal{P}_m$, with $I_m \subset I_n$.

In the following, let $x_0 \in \mathbb{T}^1$ be an arbitrary point on the circle.

Theorem 2.2 *Let $\mathfrak{n} := (\mathfrak{n}_\ell)_{\ell \in \mathbb{N}}$ be an increasing subsequence of $\mathbb{N}_0$. Let T and $\widetilde{T}$ be two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, orientation-preserving circle homeomorphisms that satisfy the following conditions for some $\lambda \in (0,1)$ and $\lambda_1 \in (\lambda, 1)$:*

- *(a)* $\rho(T) = \rho(\widetilde{T}) = \rho \in \mathcal{S}_{\mathfrak{n}}(\lambda_1)$;
- *(b) there exists a vector $K \in \mathbb{R}_+^6$ such that the sequences of renormalizations $(f_n)_{n \in \mathbb{N}_0}$ and $(\widetilde{f}_n)_{n \in \mathbb{N}_0}$ are K-regular with respect to $\mathfrak{n}$;*
- *(c) the systems of dynamical partitions $\mathcal{P}_n$ and $\widetilde{\mathcal{P}}_n$ are exponentially refining;*
- *(d)* $\|f_n - \tilde{f}_n\|_{C^2} \leq C\lambda^n$, *for some $C > 0$ and all $n \in \mathbb{N}_0$.*

Then, there exists a C^1-smooth orientation-preserving circle diffeomorphism φ such that

$$\varphi \circ T \circ \varphi^{-1} = \widetilde{T}. \tag{2.9}$$

Remark 5 As we show in the proof of Theorem 1.2, in the case of circle maps with breaks of size $c \in \mathbb{R}_+ \backslash \{1\}$, condition (b) is satisfied, if x_0 is the break point, for the subsequence $\mathfrak{n}$ consisting of even n, for $0 < c < 1$, and odd n, for $c > 1$.

Remark 6 Conditions (a) and (c) of Theorem 2.2 guarantee that T and $\widetilde{T}$ are topologically conjugate to each other. It is easy to see that, in the case of circle maps with breaks, the conjugacy φ can be C^1-smooth only when it maps the break point x_0 of T into the break point $\widetilde{x}_0$ of $\widetilde{T}$. This condition defines the topological conjugacy φ uniquely.

Under different regularity conditions, valid for renormalizations of critical circle maps, an analogous theorem was proven in [12]. In that case, however, the geometry is bounded, and it requires a much simpler analysis. At present, Theorem 2 in [12] can be viewed as a special case of a more general Theorem 2.2 (with a slightly modified regularity condition (ii)), when the subsequence $\mathfrak{n}$ is empty.

3 A criterion of smoothness and the proof of the main theorem

3.1 A criterion of smoothness

To prove Theorem 2.2, we will use the following criterion of smoothness of φ. It is inspired by a similar criterion in [4] called the "coherence property". For a segment $I \subset \mathbb{T}^1$ or $\mathbb{R}$, we define

$$\sigma(I) := \frac{|\varphi(I)|}{|I|}, \tag{3.1}$$

where $|\cdot|$ is the length of an interval on $\mathbb{T}^1$ or $\mathbb{R}$.

Proposition 3.1 ([12]) *Suppose that the system of partitions $\mathcal{P}_n$ of the circle is refining, and that there exist constants $\bar{C} > 0$ and $\bar{\lambda} \in (0,1)$ such that for any two segments $I, I' \in \mathcal{P}_n$, which are either adjacent or $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$, the following estimate holds*

$$|\ln \sigma(I) - \ln \sigma(I')| \leq \bar{C}\bar{\lambda}^n. \tag{3.2}$$

Then, $\varphi \in C^1(\mathbb{T}^1)$ and $\varphi' > 0$.

Proof. We present the proof for completeness of the argument. Let φ_n be a homeomorphism of $\mathbb{T}^1$ that equals φ on Ξ_n and is linear on each of the segments $I \subset \mathcal{P}_n$. Let further $(\varphi_n)'_+$ be the right derivative of φ_n. It follows from (3.2) that the sequence of differences $\ln((\varphi_n)'_+(x))$ is a Cauchy sequence, uniformly on $\mathbb{T}^1$, and thus converges to some $h(x)$. To see this, notice first that over each $I \subset \mathcal{P}_n$ without the right endpoint, $(\varphi_n)'_+(x) = \sigma(I)$, and that (3.2), for any two intervals $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$, implies that

$$|\ln \sigma(I) - \ln \sigma(J)| \leq \bar{C}\bar{\lambda}^n. \tag{3.3}$$

Now, it is easy to show, using (3.2) for adjacent intervals $I, I' \in \mathcal{P}_n$, that the function h is continuous on $\mathbb{T}^1$. Taking the limit $n \to \infty$ of $\varphi_n(x) = \int_0^x (\varphi_n)'_+(z)\,dz$, we get $\varphi(x) = \int_0^x e^{h(z)}\,dz$. Thus, $\varphi' = e^h$ is continuous and positive on $\mathbb{T}^1$. **QED**

We will also use the ratios of the corresponding rescaled intervals:

$$\mathfrak{s}_n(I) := \frac{|\widetilde{\tau}_n(\varphi(I))|}{|\tau_n(I)|}. \tag{3.4}$$

In addition, we will use the notation

$$r_n^-(I) := \frac{|\widetilde{\tau}_n(\widetilde{\eta}_-) - \tau_n(\eta_-)|}{|\tau_n(I)|}, \qquad r_n^+(I) := \frac{|\widetilde{\tau}_n(\widetilde{\eta}_+) - \tau_n(\eta_+)|}{|\tau_n(I)|}, \tag{3.5}$$

where η_- and η_+ are the end points of I such that $\tau_n(\eta_-) < \tau_n(\eta_+)$; and $\widetilde{\eta}_- = \varphi(\eta_-)$ and $\widetilde{\eta}_+ = \varphi(\eta_+)$ are the end points of $\varphi(I)$ such that $\widetilde{\tau}_n(\widetilde{\eta}_-) < \widetilde{\tau}_n(\widetilde{\eta}_+)$. Clearly,

$$|\mathfrak{s}_n(I) - 1| \leq r_n^-(I) + r_n^+(I). \tag{3.6}$$

To simplify the notation, we will also use $\mathfrak{r}_i := \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+iq_n}) - \tau_n(x_{q_{n-1}+iq_n})$.

3.2 Renormalization graphs concave inside the gates

In this section, we restrict our consideration to subsequences of renormalizations f_n and $\widetilde{f}_n$ of T and $\widetilde{T}$, respectively, for $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, which satisfy the regularity conditions (i), (ii), (iii) and (iv). The graphs of these renormalizations are concave inside the gates.

The following proposition summarizes the main result of this section. We emphasize that the constants C_i that appear in this paper are all independent of n.

Let $\lambda_2 \in (\sqrt{\lambda/\lambda_1}, 1)$ be a fixed number in the given interval.

Proposition 3.2 *Assume that the conditions of Theorem 2.2 hold. There exists $C_2 > 0$ such that for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, and for $0 \leq j \leq k_{n+1}$, we have*

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}), (\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}))^{-1} \leq 1 + C_2\lambda_2^n. \tag{3.7}$$

The proof of this proposition follows directly from Proposition 3.11 and Proposition 3.14 below. Proposition 3.11 establishes that the relative difference of lengths of the renormalized intervals $\tau_n(\Delta^{(n)}_{q_{n-1}})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})$ is exponentially small, i.e, inequalities (3.7), for $j = 0$, while Proposition 3.14 extends this estimate to j satisfying $1 \leq j \leq k_{n+1}$. The proof of Proposition 3.11 is based on Lemma 3.9 and Lemma 3.10. In the proofs of these lemmas, we use Lemma 3.7 and Lemma 3.8, which provide estimates on the iterates of "long" intervals, i.e. intervals whose length is at least of the order of λ_3^n, for some $\lambda_3 \in (\lambda/\lambda_2, \lambda_1\lambda_2)$. Similarly, in the proof of Proposition 3.14, we use Lemma 3.13, which provides the desired estimates on the iterates of the long intervals. To prove Proposition 3.11 and Proposition 3.14, we also use the topological conjugacy of the maps (implied by conditions (a) and (c) of Theorem 2.2) and the exponential convergence of renormalizations (condition (d)). We also use the fact that, for $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, the renormalizations f_n and $\widetilde{f}_n$ satisfy regularity conditions (i), (ii), (iii) and (iv) (condition (b)). In particular, we use the fact that, due to the concavity of renormalizations f_n inside the gates, the intervals between successive iterates of renormalizations, inside the ε-neighborhoods of the end points, are either longer than a constant or their iterates under f_n grow exponentially, as implied by the following proposition.

Proposition 3.3 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be an orientation preserving homeomorphism and let its sequence of renormalizations f_n be K-regular with respect to $\mathfrak{n}$. There exists $B > 1 + K_3/4$ and $0 < \varepsilon < K_3/2$ such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, either $f_n(-1)+1 > K_3/2$ or $f_n'(z) > B$ for $z \in [-1, -1+\varepsilon]$ and either $f_n(0) > K_3/2$ or $f_n'(z) < B^{-1}$ for $z \in [-\varepsilon, 0]$.*

Proof. It follows from the continuity of f_n and regularity condition (iii) (see Section 2.3) that, for $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, if $f_n(-1) + 1 < K_3$, there is $\varkappa_1^{(n)} \in (0,1)$ such that $f_n(-1 + \varkappa_1^{(n)}) + 1 - \varkappa_1^{(n)} = K_3$ and $f_n(z) - z < K_3$ for $z \in [-1, -1 + \varkappa_1^{(n)})$. Since,

$$f_n(z) = f_n(-1) + \int_{-1}^{z} f_n'(\zeta)\, d\zeta = f_n(-1) + \int_{-1}^{z} \left(f_n'(-1) + \int_{-1}^{\zeta} f_n''(\zeta')\, d\zeta' \right) d\zeta, \tag{3.8}$$

using the regularity condition (iv), we obtain

$$f_n(z) \le f_n(-1) + f_n'(-1)(z+1) - \frac{K_4}{2}(z+1)^2. \tag{3.9}$$

Evaluating this expression at $z = -1 + \varkappa_1^{(n)}$, we obtain

$$f_n(-1) + 1 + (f_n'(-1) - 1)\varkappa_1^{(n)} \ge K_3 + \frac{K_4}{2}(\varkappa_1^{(n)})^2 > K_3. \tag{3.10}$$

Therefore, if $f_n(-1) + 1 \le K_3/2$, then $f_n'(-1) > 1 + K_3/2$. Similarly, if $f_n(0) \le K_3/2$, then $f_n'(0) < 1 - K_3/2$. Since the second derivative of f_n is bounded (by regularity condition (i)), in these cases, there exist $\varepsilon > 0$ and $B > 1 + K_3/4$ such that $f_n'(z) > B$, for $z \in [-1, -1+\varepsilon]$, and $f_n'(z) < 2 - B \le B^{-1}$, for $z \in [-\varepsilon, 0]$. The claim follows. In fact, one can choose any $\varepsilon \le K_3/4K_4$. **QED**

The next proposition will be used repeatedly, without always mentioning it explicitly. It implies that the length of the longest of the exponentially growing iterates of an interval is of the order of the sum of their lengths.

Proposition 3.4 *Let $b_0 > 0$, $b_i > B > 1$ for $i \in \mathbb{N}$, and*

$$s_n = \sum_{j=0}^{n} \prod_{i=0}^{j} b_i. \tag{3.11}$$

Then, there exists $A > 0$ such that $\prod_{i=0}^{n} b_i > A s_n$, for all $n \in \mathbb{N}$.

Proof. We can assume, without loss of generality, that $b_0 = 1$. The claim is proven by simple induction. For $n = 1$, the claim is, obviously, true, for any $A < \frac{B}{1+B}$. Assume that the claim is true for some $n \in \mathbb{N}$, with $A < 1 - \frac{1}{B}$. Then,

$$\prod_{i=0}^{n+1} b_i > A s_n b_{n+1} = A b_{n+1}(s_{n+1} - \prod_{i=0}^{n+1} b_i), \tag{3.12}$$

and, thus,

$$\prod_{i=0}^{n+1} b_i > A\frac{b_{n+1}}{1+Ab_{n+1}}s_{n+1} > As_{n+1}. \tag{3.13}$$

The claim follows. **QED**

Proposition 3.5 *Under the assumptions of Proposition 3.3, there exists $\bar{A} > 0$, such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, and, for $0 \le j < k_{n+1}$,*

$$|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| > \bar{A}\min\left\{\sum_{i=0}^{j}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|, \sum_{i=j}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| + |\tau_n(\Delta^{(n+1)}_0)|\right\}. \tag{3.14}$$

Proof. By Proposition 3.3, there exist $\varepsilon > 0$ and $B > 1 + K_3/4$ such that either $f_n(-1) + 1 > K_3/2$ or $f_n'(z) > B$ for $z \in [-1, -1+\varepsilon]$ and either $f_n(0) > K_3/2$ or $f_n'(z) < B^{-1}$ for $z \in [-\varepsilon, 0]$. If $f_n(-1) + 1 > K_3/2$, then $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| > K_3/2$, for any j such that $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \cap [-1, -1+\varepsilon) \neq \emptyset$, since, due to regularity conditions (iii) and (iv), $f_n(z) - z$ is monotone inside the gates and $f_n(z) - z \ge K_3$ outside them; consequently, for any such j, (3.14) holds for any $\bar{A} \le K_3/2$. If $f_n(-1) + 1 \le K_3/2$, then $f_n'(z) > B$, for $z \in [-1, -1+\varepsilon]$ and, by Proposition 3.4 applied to $b_0 = 1$, $b_i = |\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|/|\tau_n(\Delta^{(n)}_{q_{n-1}+(i-1)q_n})|$, for $i > 0$, there exists $A > 0$ such that (3.14) holds with $\bar{A} \le A$, for any j such that $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \cap [-1, -1+\varepsilon) \neq \emptyset$. Similar arguments can be used to show that for sufficiently small $\bar{A}$, (3.14) holds for any j such that $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \cap (-\varepsilon, 0] \neq \emptyset$. It remains to show that (3.14) holds, for sufficiently small $\bar{A}$, and all j such that $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \subset [-1+\varepsilon, \varepsilon]$. This holds since $f_n(-1+\varepsilon) + 1, f_n(-\varepsilon) > (1 - B^{-1})\varepsilon$ and since $f_n(z) - z$ is monotonically increasing in $[-1, z_g^{(1)})$ and monotonically decreasing in $(z_g^{(2)}, 0]$, while $f_n(z) - z \ge K_3$, outside of these intervals. Therefore, the claim follows for sufficiently small $\bar{A}$ satisfying all the upper bounds. **QED**

Corollary 3.6 *Under the assumptions of Proposition 3.3, there exists $\bar{\bar{A}} > 0$, such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, and, for $0 \le j < k_{n+1}$,*

$$|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| > \bar{\bar{A}}\min\left\{\sum_{i=0}^{j+1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|, \sum_{i=j-1}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| + |\tau_n(\Delta^{(n+1)}_0)|\right\}. \tag{3.15}$$

Proof. It follows directly from Proposition 3.5, taking into account that, by regularity conditions (i) and (ii), the lengths of the neighboring intervals $\Delta^{(n)}_{q_{n-1}+iq_n}$ and $\Delta^{(n)}_{q_{n-1}+(i+1)q_n} = T^{q_n}(\Delta^{(n)}_{q_{n-1}+iq_n})$ are of the same order, as $(T^{q_n})'(x) = f_n'(\tau_n(x))$. **QED**

The following two lemmas will be used in the proofs of Lemma 3.9 and Lemma 3.10, respectively.

To simplify the notation $\mathfrak{r}_i := \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+iq_n}) - \tau_n(x_{q_{n-1}+iq_n})$.

Lemma 3.7 *Assume that the conditions of Theorem 2.2 hold. Let $\lambda_4 \in (\lambda, 1)$, $C_3 > 0$ and, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, let $j_l^{(n)}$ be such that $0 \le j_l^{(n)} \le k_{n+1}$. There exists $C_4 > 0$ such that, for all $n = \mathfrak{n}_\ell$ and sufficiently large $\ell \in \mathbb{N}$, if $\mathfrak{r}_{j_l^{(n)}} \ge C_3\lambda_4^n$ then, for all $j \ge j_l^{(n)}$ satisfying $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \cap [-1, -\varepsilon] \neq \emptyset$, we have*

$$\mathfrak{r}_j \ge C_4\lambda_4^n. \tag{3.16}$$

Proof. It follows from the mean value theorem and condition (d) of Theorem 2.2 that

$$\mathfrak{r}_j = \widetilde{f}_n(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n})) - f_n(\tau_n(x_{q_{n-1}+(j-1)q_n})) \ge \widetilde{f}_n'(\xi_{j-1})\mathfrak{r}_{j-1} - C\lambda^n, \tag{3.17}$$

where ξ_{j-1} is a point in the interval $(\tau_n(x_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}))$. By Proposition 3.3, if $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-1, -1+\varepsilon]$, then $\widetilde{f}_n'(\xi_{j-1}) > B > 1 + K_3/4$ and, therefore,

$$\mathfrak{r}_j \ge \mathfrak{r}_{j-1}, \tag{3.18}$$

if $K_3\mathfrak{r}_{j-1}/4 \ge C\lambda^n$. Since $\lambda < \lambda_4$, then this condition is satisfied for $j = j_l^{(n)} + 1$, if n is large enough such that $K_3C_3\lambda_4^n/4 \ge C\lambda^n$. The estimate (3.18) now implies that (3.16) holds for all $j \ge j_l^{(n)}$ satisfying $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-1, -1+\varepsilon]$.

Using the regularity condition (ii), from the inequality (3.17), we also have

$$\mathfrak{r}_j \ge K_2\mathfrak{r}_{j-1} - C\lambda^n. \tag{3.19}$$

This inequality can be iterated a number of times bounded by a constant, if $\lambda < \lambda_4$, to obtain (3.16), with some constant $C_4 > 0$, and all $j \ge j_l^{(n)}$ such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \cap (-1+\varepsilon, -\varepsilon] \neq \emptyset$. This follows from the fact that the number of such indices j is bounded by a constant, independent of n. To see this, notice first that it follows from Corollary 3.6 that the length of all intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$, such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \cap (-1+\varepsilon, -\varepsilon] \neq \emptyset$, is bounded from below by a positive constant proportional to ε. The claim follows. **QED**

Similarly, we have the following.

Lemma 3.8 *Assume that the conditions of Theorem 2.2 hold. Let $C_5 > 0$ and, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, let $j_r^{(n)}$ be such that $0 \le j_r^{(n)} \le k_{n+1}$. There exists $C_6 > 0$ such that, for all $n = \mathfrak{n}_\ell$ and sufficiently large $\ell \in \mathbb{N}$, if $-\mathfrak{r}_{j_r^{(n)}} \ge C_5\lambda_4^n$ then, for all $j \le j_r^{(n)}$ satisfying $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap [-1+\varepsilon, 0] \neq \emptyset$, we have*

$$-\mathfrak{r}_j \ge C_6\lambda_4^n. \tag{3.20}$$

Proof. The mean value theorem and condition (d) of Theorem 2.2 give us that

$$-\mathfrak{r}_j \geq (\widetilde{f}_n'(\xi_j))^{-1}(-\mathfrak{r}_{j+1} - C\lambda^n) \geq \left(1 + \frac{K_3}{4}\right)(-\mathfrak{r}_{j+1} - C\lambda^n) \geq -\mathfrak{r}_{j+1}, \tag{3.21}$$

if $K_3\mathfrak{r}_{j+1}/4 \geq (1 + \frac{K_3}{4})C\lambda^n$. This condition clearly holds for sufficiently large n since $\lambda_4 > \lambda$. As before, ξ_j is a point in the interval $(\tau_n(x_{q_{n-1}+jq_n}), \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}))$. Here, we have used that, by Proposition 3.3, $(\widetilde{f}_n'(\xi_j))^{-1} > B > 1 + K_3/4$, as long as the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ lie inside $[-\varepsilon, 0]$.

Using the regularity condition (i), from the first inequality in (3.21), we have

$$-\mathfrak{r}_j \geq K_1^{-1}(-\mathfrak{r}_{j+1} - C\lambda^n), \tag{3.22}$$

which can be iterated, a number of times bounded by a constant, to obtain (3.20) for some $C_6 > 0$, and all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap [-1+\varepsilon, -\varepsilon) \neq \emptyset$. Here, we have used again that, by Corollary 3.6, the number of such j is bounded, as explained in the proof of Lemma 3.7. The claim follows. **QED**

Throughout the paper, $\lambda_2 \in (\sqrt{\lambda/\lambda_1}, 1)$ and $\lambda_3 \in (\lambda/\lambda_2, \lambda_1\lambda_2)$ are fixed numbers in the given intervals.

Lemma 3.9 *Assume that the conditions of Theorem 2.2 hold. Let $C_7 > 0$. There exists $C_8 > 0$ such that, for $n = \mathfrak{n}_\ell$ and ℓ sufficiently large, if*

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) > 1 + C_7\lambda_2^n, \tag{3.23}$$

then

$$\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n}) \geq C_8\lambda_2^n\lambda_3^n, \tag{3.24}$$

for all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, -\varepsilon) \neq \emptyset$.

Proof. Assume first that $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| \leq \lambda_3^n$. Due to (3.23), $|\tau_n(\Delta^{(n)}_{q_{n-1}})| \leq \lambda_3^n$ as well. Using conditions (b) (regularity conditions (i) and (ii)) and (d) of Theorem 2.2, for $1 \leq j \leq k_{n+1}$ such that $\tau_n(\Delta^{(n)}_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-1, -1+\lambda_3^n]$, we have

$$\begin{aligned}
\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=0}^{j-1}\left(1 + \frac{\widetilde{f}_n'(\widetilde{\zeta}_i) - f_n'(\zeta_i)}{f_n'(\zeta_i)}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) \\
&\geq \prod_{i=0}^{j-1}\left(1 - \frac{|\widetilde{f}_n'(\widetilde{\zeta}_i) - f_n'(\widetilde{\zeta}_i)| + |f_n'(\widetilde{\zeta}_i) - f_n'(\zeta_i)|}{|f_n'(\zeta_i)|}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) \\
&\geq \left(1 - K_2^{-1}(C\lambda^n + K_1\lambda_3^n)\right)^j\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}).
\end{aligned} \tag{3.25}$$

Here, $\zeta_i \in \tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$. Since, by condition (a), $j \leq k_{n+1} \leq C_1\lambda_1^{-n}$, for every $\epsilon_1 > 0$, if n is large enough, we have

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) > 1 + (1-\epsilon_1)C_7\lambda_2^n. \tag{3.26}$$

Here, we have used that $\lambda \leq \lambda_3 < \lambda_1\lambda_2$.

Let j_{λ_3} be the index j of the last interval $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ that is contained in $[-1,-1+\lambda_3^n]$, i.e. such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{\lambda_3}q_n}) \subset [-1,-1+\lambda_3^n]$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j_{\lambda_3}+1)q_n}) \cap (-1+\lambda_3^n, 0] \neq \emptyset$. The previous estimate then implies that

$$\begin{aligned} \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j_{\lambda_3}+1)q_n}) - \tau_n(x_{q_{n-1}+(j_{\lambda_3}+1)q_n}) &= \sum_{j=0}^{j_{\lambda_3}} \left(|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| - |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \right) \\ &> \frac{(1-\epsilon_1)C_7\lambda_2^n}{1+(1-\epsilon_1)C_7\lambda_2^n} \sum_{j=0}^{j_{\lambda_3}} |\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| > C_9\lambda_2^n\lambda_3^n, \end{aligned} \tag{3.27}$$

for some $C_9 > 0$, since $\sum_{j=0}^{j_{\lambda_3}} |\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ is of the order of λ_3^n, as follows from Corollary 3.6. If $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| > \lambda_3^n$, we formally define $j_{\lambda_3} := 0$. In this case, the final estimate in (3.27) follows directly from (3.23).

Since $\lambda < \lambda_2\lambda_3$, Lemma 3.7, with $\lambda_4 = \lambda_2\lambda_3$, gives (3.24), with some constant $C_8 > 0$, for all $j > j_{\lambda_3}$ such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap [-1,-\varepsilon) \neq \emptyset$. Since, for sufficiently large n, $\lambda_3^n < \varepsilon$, this interval of indices includes all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon,-\varepsilon) \neq \emptyset$. The claim follows. **QED**

Lemma 3.10 *Assume that the conditions of Theorem 2.2 hold. Let $C_{10}, C_{11} > 0$. There exists $C_{12} > 0$ such that, for $n = \mathfrak{n}_\ell$ and ℓ sufficiently large, if $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})| \leq C_{10}\lambda_3^n$ and*

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) > 1 + C_{11}\lambda_2^n, \tag{3.28}$$

then

$$\tau_n(x_{q_{n-1}+jq_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) \geq C_{12}\lambda_2^n\lambda_3^n, \tag{3.29}$$

for all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon,-\varepsilon) \neq \emptyset$.

Proof. The estimates in the proof of this claim are similar to the estimates for forward iterations that were used in the proof of the previous lemma. It follows from (3.28) that $|\tau_n(\Delta^{(n)}_{q_{n+1}})| \leq C_{10}\lambda_3^n$. The regularity condition (ii) guarantees that $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}-q_n})| \leq K_2^{-1}C_{10}\lambda_3^n$ and $|\tau_n(\Delta^{(n)}_{q_{n+1}-q_n})| \leq K_2^{-1}C_{10}\lambda_3^n$. Since $\Delta_0^{(n+1)} \subset \Delta^{(n)}_{q_{n+1}}$, there exists $\mathfrak{C} > 0$ such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}-q_n}), \tau_n(\Delta^{(n)}_{q_{n+1}-q_n}) \subset [-\mathfrak{C}\lambda_3^n, 0]$. For $0 \le j < k_{n+1}$ such that the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \subset [-\mathfrak{C}\lambda_3^n, 0]$, we have

$$\begin{aligned}\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=j}^{k_{n+1}-1}\left(1+\frac{f_n'(\zeta_i)-\widetilde{f}_n'(\widetilde{\zeta}_i)}{\widetilde{f}_n'(\widetilde{\zeta}_i)}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}})\\ &= \prod_{i=j}^{k_{n+1}-1}\left(1+\frac{f_n'(\zeta_i)-\widetilde{f}_n'(\zeta_i)+\widetilde{f}_n'(\zeta_i)-\widetilde{f}_n'(\widetilde{\zeta}_i)}{\widetilde{f}_n'(\widetilde{\zeta}_i)}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}})\\ &\ge \prod_{i=j}^{k_{n+1}-1}\left(1-\frac{|f_n'(\zeta_i)-\widetilde{f}_n'(\zeta_i)|+|\widetilde{f}_n'(\zeta_i)-\widetilde{f}_n'(\widetilde{\zeta}_i)|}{|\widetilde{f}_n'(\widetilde{\zeta}_i)|}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}})\\ &\ge \left(1-K_2^{-1}(C\lambda^n+K_1\mathfrak{C}\lambda_3^n)\right)^{k_{n+1}-j}\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}).\end{aligned}\tag{3.30}$$

As before, $\zeta_i \in \tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$. Since, by condition (a) of Theorem 2.2, $j \le k_{n+1} \le C_1\lambda_1^{-n}$ and, since $\lambda \le \lambda_3 < \lambda_1\lambda_2$, using (3.28), for any $\epsilon_2 > 0$, if n is sufficiently large, we obtain

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) > 1+(1-\epsilon_2)C_{11}\lambda_2^n.\tag{3.31}$$

Let $j_{-\lambda_3}$ be the smallest index j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \subset [-\mathfrak{C}\lambda_3^n, 0]$. Then, we have

$$\begin{aligned}\tau_n(x_{q_{n-1}+j_{-\lambda_3}q_n})-\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+j_{-\lambda_3}q_n}) &= |\widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_0)|-|\tau_n(\Delta^{(n+1)}_0)|\\ &\quad+\sum_{j=j_{-\lambda_3}}^{k_{n+1}-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|-\sum_{j=j_{-\lambda_3}}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|\\ &\ge \sum_{j=j_{-\lambda_3}}^{k_{n+1}-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|-\sum_{j=j_{-\lambda_3}}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|-2K_1C\lambda^n\\ &\ge \frac{(1-\epsilon_2)C_{11}\lambda_2^n}{1+(1-\epsilon_2)C_{11}\lambda_2^n}\sum_{j=j_{-\lambda_3}}^{k_{n+1}-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|-2K_1C\lambda^n > C_{13}\lambda_2^n\lambda_3^n,\end{aligned}\tag{3.32}$$

for some $C_{13} > 0$, since $\lambda < \lambda_2\lambda_3$ and, by regularity condition (i), $\sum_{j=-\lambda_3}^{k_{n+1}-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ is of the order of λ_3^n. In the first of these inequalities, we have also used that $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n+1)}_0)|-|\tau_n(\Delta^{(n+1)}_0)| \le 2K_1C\lambda^n$, as follows from the condition (d) and the regularity condition (i).

Using again that $\lambda < \lambda_2\lambda_3$, Lemma 3.8, with $\lambda_4 = \lambda_2\lambda_3$, now gives (3.29), for some $C_{12} > 0$, and all $j \le j_{-\lambda_3}$ such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, 0] \ne \emptyset$. Since, for a sufficiently large n, $\mathfrak{C}\lambda_3^n < \varepsilon$, this interval of indices includes all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, -\varepsilon) \ne \emptyset$ and the claim follows. **QED**

The following proposition shows that the ratio of lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}})$ is exponentially close to 1. The proof is by contradiction. We will show that if the first of these intervals were sufficiently longer, then the corresponding sum $\sum_{j=0}^{k_{n+1}-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ would need to be significantly larger than $\sum_{j=0}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|$ such that it would lead to a contradiction.

Proposition 3.11 *Assume that the conditions of Theorem 2.2 hold. There exists $C_{14}>0$ such that, for all $n=\mathfrak{n}_\ell$, for some $\ell\in\mathbb{N}$, we have*

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}), (\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}))^{-1} \le 1 + C_{14}\lambda_2^n. \tag{3.33}$$

Proof. Notice that it is sufficient to prove the claim for sufficiently large ℓ. If either $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| \ge \lambda_3^n$ or $|\tau_n(\Delta^{(n)}_{q_{n-1}})| \ge \lambda_3^n$, then the claim follows directly from the closeness of renormalizations (condition (d) of Theorem 2.2), since $||\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| - |\tau_n(\Delta^{(n)}_{q_{n-1}})|| = |\widetilde{f}_n(-1) - f_n(-1)| \le C\lambda^n$. In the case when $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| < \lambda_3^n$ and $|\tau_n(\Delta^{(n)}_{q_{n-1}})| < \lambda_3^n$, we will prove the claim by contradiction. To prove the first inequality, let us assume that, for every $C_{14}>0$ and every $n_1\in\mathbb{N}$, there exists $n\ge n_1$ such that $\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) > 1 + C_{14}\lambda_2^n$. The proof of the second inequality is analogous, by exchanging the roles of f_n and $\widetilde{f}_n$. Lemma 3.9 implies that

$$\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n}) \ge C_{15}\lambda_2^n\lambda_3^n, \tag{3.34}$$

for some constant $C_{15}>0$, and all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, -\varepsilon) \ne \emptyset$.

We will now prepare the setting to estimate the same difference starting from the other end of the interval $[-1,0]$. Notice that

$$\begin{aligned}
\mathfrak{s}_n(\Delta^{(n)}_0) &= \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}})\frac{f'_{n-1}(\tau_{n-1}\circ\tau_n^{-1}(\zeta_0))}{\widetilde{f}'_{n-1}(\widetilde{\tau}_{n-1}\circ\widetilde{\tau}_n^{-1}(\widetilde{\zeta}_0))}\\
&= \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}})\frac{f'_{n-1}(-f_{n-1}(0)\zeta_0)}{\widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)}\\
&= \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}})\left(1 + \frac{f'_{n-1}(-f_{n-1}(0)\zeta_0) - \widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)}{\widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)}\right)\\
&\ge \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}})\left(1 - K_2^{-1}(C\lambda^{n-1} + K_1^2C\lambda^{n-1} + K_1^2|\zeta_0-\widetilde{\zeta}_0|)\right)
\end{aligned} \tag{3.35}$$

where $\zeta_0\in\tau_n(\Delta^{(n)}_0)$ and $\widetilde{\zeta}_0\in\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_0)$. We next estimate $|\zeta_0-\widetilde{\zeta}_0|$. Since

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) = \frac{1+\tilde{f}_n(-1)}{1+f_n(-1)} \le 1 + \frac{C\lambda^n}{|\tau_n(\Delta^{(n)}_{q_{n-1}})|} \tag{3.36}$$

and, by assumption $\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) > 1 + C_{14}\lambda_2^n$, we have $C_{14}|\tau_n(\Delta_{q_{n-1}}^{(n)})| < C\lambda_3^n$, since $\lambda < \lambda_3\lambda_2$. Furthermore, since by the regularity conditions (i) and (ii), the length of the interval $\tau_n(\Delta_{q_{n-1}}^{(n)})$, i.e. $|\tau_n(\Delta_{q_{n-1}}^{(n)})| = f'_{n-1}(\tau_{n-1} \circ \tau_n^{-1}(\zeta_0))|\tau_n(\Delta_0^{(n)})|$, is of the same order as $|\tau_n(\Delta_0^{(n)})| = f_n(0)$, we have $f_n(0) \leq C_{16}\lambda_3^n$, for some $C_{16} > 0$. This implies that $|\zeta_0 - \widetilde{\zeta}_0| \leq f_n(0) + C\lambda^n \leq C_{16}\lambda_3^n + C\lambda^n$. Using this estimate and the last inequality in (3.35), we obtain that, for some $\epsilon_3 > 0$, if n is large enough,

$$\mathfrak{s}_n(\Delta_0^{(n)}) \geq 1 + (1 - \epsilon_3)C_{14}\lambda_2^n. \tag{3.37}$$

Notice, further, that

$$\frac{\mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)})}{\mathfrak{s}_n(\Delta_0^{(n)})} = \frac{\tilde{f}_{n+1}(0) - \tilde{f}_{n+1}(-1)}{f_{n+1}(0) - f_{n+1}(-1)}, \tag{3.38}$$

and that the right hand side is bounded from below by $1 - C_{17}\lambda^n$, for some $C_{17} > 0$. Together with (3.37), this implies that, if n is large enough,

$$\mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)}) > 1 + (1 - 2\epsilon_3)C_{14}\lambda_2^n. \tag{3.39}$$

Furthermore, since $\widetilde{\Delta}_{q_{n+1}}^{(n)} = \widetilde{T}^{q_{n+1}}(\widetilde{T}^{-q_{n-1}}(\widetilde{\Delta}_{q_{n-1}}^{(n)}))$, by the regularity conditions (i) and (ii), $|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n+1}}^{(n)})|$ is of the same order as $|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})| \leq \lambda_3^n$. Conditions of Lemma 3.10 are, therefore, satisfied. Applying Lemma 3.10, we obtain

$$\tau_n(x_{q_{n-1}+jq_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) \geq C_{18}\lambda_2^n\lambda_3^n, \tag{3.40}$$

for some $C_{18} > 0$ and all j such that $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+jq_n}^{(n)}) \cap (-1 + \varepsilon, -\varepsilon) \neq \emptyset$. Since, in Proposition 3.3, $\varepsilon > 0$ can be chosen arbitrarily small, due to the regularity condition (i), the set of such indices j is nonempty. By considering this estimate for any such j, we get a contradiction with (3.34). The claim follows. **QED**

Corollary 3.12 *Assume that the conditions of Theorem 2.2 hold. There exists $C_{19} > 0$ such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, we have*

$$\mathfrak{s}_n(\Delta_0^{(n)}), (\mathfrak{s}_n(\Delta_0^{(n)}))^{-1} \leq 1 + C_{19}\lambda_2^n. \tag{3.41}$$

Proof. We will prove the first inequality; the proof of the second is analogous. If $|\tau_n(\Delta_0^{(n)})| > \lambda_3^n$, the claim follows directly from the convergence of renormalizations (condition (d)). Assume that $|\tau_n(\Delta_0^{(n)})| \leq \lambda_3^n$. Using the first three equalities in (3.35), we obtain

$$\mathfrak{s}_n(\Delta_0^{(n)}) \leq \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \left(1 + K_2^{-1}(C\lambda^{n-1} + K_1^2 C\lambda^{n-1} + K_1^2|\zeta_0 - \widetilde{\zeta}_0|)\right), \tag{3.42}$$

where $\zeta_0 \in \tau_n(\Delta_0^{(n)})$ and $\widetilde{\zeta}_0 \in \widetilde{\tau}_n(\widetilde{\Delta}_0^{(n)})$. Since $|\zeta_0 - \widetilde{\zeta}_0| \leq f_n(0) + C\lambda^n$ and $f_n(0) = |\tau_n(\Delta_0^{(n)})| \leq \lambda_3^n$, the claim follows. **QED**

The next lemma deals with the iteration of "long" intervals, i.e., intervals whose lengths are at least of the order of λ_3^n. In Proposition 3.14, it will be applied to long intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$. The number of such intervals is at most of the order of n since, by Proposition 3.3, the length of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ inside of $[-1, -1+\varepsilon)$ grows and inside of $(-\varepsilon, 0]$ decays exponentially under the iteration of f_n, and inside the gates the function $f_n(z) - z$ is monotone and increasing or decreasing, respectively (see the regularity conditions (iii) and (iv)). Lemma 3.13 provides the desired estimates for long intervals. The analysis of "shorter" intervals is more subtle, since one has to deal with quantities that are small on the exponential scale and the convergence of renormalizations (condition (d)) is only exponential.

Let η and $\widetilde{\eta}$ be two corresponding end points of the intervals $I \subset \Delta_0^{(n-1)}$ and $\widetilde{I} \subset \widetilde{\Delta}_0^{(n-1)}$, respectively (either $\eta = \eta_-$ and $\widetilde{\eta} = \widetilde{\eta}_-$ or $\eta = \eta_+$ and $\widetilde{\eta} = \widetilde{\eta}_+$), and let $r_n(I) := \frac{|\widetilde{\tau}_n(\widetilde{\eta}) - \tau_n(\eta)|}{|\tau_n(I)|}$. That is $r_n(I)$ stands for either $r_n^-(I)$ or $r_n^+(I)$ (see (3.5)). Let $I_i := \tau_n^{-1}(f_n^i(\tau_n(I)))$ and $\widetilde{I}_i := \widetilde{\tau}_n^{-1}(\widetilde{f}_n^i(\widetilde{\tau}_n(\widetilde{I})))$.

Lemma 3.13 *Assume that the conditions of Theorem 2.2 hold. Assume that there exist $C_{20}, C_{21}, C_{22} > 0$ such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, there exist intervals I and $\widetilde{I}$ satisfying $I_i \subset \Delta_0^{(n-1)}, \widetilde{I}_i \subset \widetilde{\Delta}_0^{(n-1)}$ and $|\tau_n(I_i)| \geq C_{20}\lambda_3^n$, for all $0 \leq i \leq N_n$, where $N_n \leq C_{21}n$. Assume further that $r_n(I) \leq C_{22}\lambda_2^n$. Then, there exists $C_{23} > 0$ such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, $r_n(I_i) \leq C_{23}\lambda_2^n$, for $0 \leq i \leq N_n$.*

Proof. Notice first that it is sufficient to prove that the claim holds for sufficiently large ℓ. We will assume that η and $\widetilde{\eta}$ are the smaller end points. For the larger end points, the proof is essentially the same. The lemma is proven by induction. For $i = 0$, the claim is true. Assume that for all $0 \leq i \leq j$, $r_n(I_i) \leq C_{23}\lambda_2^n < 1$, for some $C_{23} > 0$ specified below and n large enough.

Using the mean value theorem, condition (d) and regularity conditions (i) and (ii), we obtain

$$
\begin{aligned}
r_n(I_{i+1}) &\leq \frac{f_n'(\xi_{i+1})|\widetilde{\tau}_n(\widetilde{\eta}_i) - \tau_n(\eta_i)| + C\lambda^n}{f_n'(\bar{\zeta}_{i+1})|\tau_n(I_i)|} \\
&\leq \left(1 + \frac{|f_n'(\xi_{i+1}) - f_n'(\bar{\zeta}_{i+1})|}{f_n'(\bar{\zeta}_{i+1})}\right) r_n(I_i) + \frac{C\lambda^n}{K_2 C_{20}\lambda_3^n} \\
&\leq \left(1 + K_1 K_2^{-1}(1 + r_n(I_i))|\tau_n(I_i)|\right) r_n(I_i) + \frac{C\lambda^n}{K_2 C_{20}\lambda_3^n},
\end{aligned}
\tag{3.43}
$$

where $\xi_{i+1} \in (\tau_n(\widetilde{\eta}_i), \tau_n(\eta_i))$ and $\bar{\zeta}_{i+1} \in \tau_n(I_i)$. Here, we have also used that $|\tau_n(I_i)| \geq$

$C_{20}\lambda_3^n$. Applying this inequality recursively from $i=j$ down to $i=0$, we find

$$\begin{aligned} r_n(I_{j+1}) \leq & \prod_{i=0}^{j}\left(1+2K_1K_2^{-1}|\tau_n(I_i)|\right) r_n(I) + CC_{20}^{-1}K_2^{-1}(\lambda/\lambda_3)^n \\ & \cdot\left(1+\sum_{k=1}^{j-1}\prod_{i=1}^{k}\left(1+2K_1K_2^{-1}|\tau_n(I_i)|\right)\right) \\ \leq & e^{2K_1K_2^{-1}} r_n(I) + CC_{20}^{-1}K_2^{-1}(\lambda/\lambda_3)^n(1+C_{21}ne^{2K_1K_2^{-1}}) \\ \leq & C_{23}\lambda_2^n, \end{aligned} \tag{3.44}$$

if $C_{23} \geq C_{22}e^{2K_1K_2^{-1}} + CC_{20}^{-1}K_2^{-1} + CC_{20}^{-1}K_2^{-1}C_{21}e^{2K_1K_2^{-1}} \max_{n\in\mathbb{N}}\left(n\left(\frac{\lambda}{\lambda_3\lambda_2}\right)^n\right)$. For n_2 large enough such that $C_{23}\lambda_2^{n_2} < 1$ and all ℓ such that $n=\mathfrak{n}_\ell \geq n_2$, we, thus, have $r_n(I_{j+1}) \leq C_{23}\lambda_2^n$. The claim follows. **QED**

In the next proposition, we again use $k_{n+1} \leq C_1\lambda_1^{-n}$, for those n considered here. We will show that, under this assumption, if the ratio of lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})$ and $\tau_n(\Delta_{q_{n-1}}^{(n)})$ is exponentially (in n) close to 1 then, due to the convergence of renormalizations (condition (d) in Theorem 2.2), for all $j=1,\dots,k_{n+1}$, the ratios of the lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+jq_n}^{(n)})$ and $\tau_n(\Delta_{q_{n-1}+jq_n}^{(n)})$ are exponentially close to 1.

Proposition 3.14 *Assume that the conditions of Theorem 2.2 hold. Assume that there exists $C_{14}>0$ such that for all $n=\mathfrak{n}_\ell$, $\ell\in\mathbb{N}$, (3.33) is valid. Then, there exists $C_{24}>0$ such that for all $0\leq j\leq k_{n+1}$, we have*

$$\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}), (\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}))^{-1} \leq 1 + C_{24}\lambda_2^n. \tag{3.45}$$

Proof. It suffices to prove that the claim holds for large enough ℓ. Recall that λ_2 and λ_3 have been chosen such that $\lambda < \lambda_2\lambda_3 < \lambda_3 < \lambda_1\lambda_2 < 1$. We will assume first that $|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})| \leq \lambda_3^n$ and $|\tau_n(\Delta_{q_{n-1}}^{(n)})| \leq \lambda_3^n$. For $1\leq j\leq k_{n+1}$ such that both intervals $\tau_n(\Delta_{q_{n-1}+(j-1)q_n}^{(n)}), \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+(j-1)q_n}^{(n)}) \subset [-1,-1+\lambda_3^n]$, we have

$$\begin{aligned} \mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) &= \prod_{i=0}^{j-1}\left(1+\frac{\widetilde{f}_n'(\widetilde{\zeta}_i)-f_n'(\zeta_i)}{f_n'(\zeta_i)}\right)\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \\ &\leq \prod_{i=0}^{j-1}\left(1+\frac{|\widetilde{f}_n'(\widetilde{\zeta}_i)-f_n'(\widetilde{\zeta}_i)|+|f_n'(\widetilde{\zeta}_i)-f_n'(\zeta_i)|}{|f_n'(\zeta_i)|}\right)\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \\ &\leq \left(1+K_2^{-1}(C\lambda^n+K_1\lambda_3^n)\right)^j \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}). \end{aligned} \tag{3.46}$$

Here, $\zeta_i\in\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})$ and $\widetilde{\zeta}_i\in\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})$. We have used condition (d) and regularity conditions (i) and (ii).

Let j_{λ_3} be the largest index j such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ are contained inside the interval $[-1,-1+\lambda_3^n]$. Since, by condition (a), $j \le k_{n+1} \le C_1\lambda_1^{-n}$, and since $\lambda < \lambda_3 < \lambda_1\lambda_2$, by using estimate (3.46) and Proposition 3.11, we obtain the first inequality in (3.45), for $1 \le j \le j_{\lambda_3}+1$. By exchanging the roles of f_n and $\widetilde{f}_n$, we can obtain the second inequality in (3.45), for the specified indices j.

Using the estimates (3.45) for $0 \le j \le j_{\lambda_3}$, we obtain that, for some $C_{25} > 0$,

$$
\begin{aligned}
|\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j+1)q_n}) - \tau_n(x_{q_{n-1}+(j+1)q_n})| &= \left|\mathfrak{s}_n\left(\bigcup_{i=0}^{j}\Delta^{(n)}_{q_{n-1}+iq_n}\right)-1\right| \\
\cdot\sum_{i=0}^{j}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| \le \max_{0\le i\le j}\left|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+iq_n})-1\right| &\sum_{i=0}^{j}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| \le C_{25}\lambda_2^n\lambda_3^n.
\end{aligned}
\tag{3.47}
$$

Corollary 3.6 implies that $\max\{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})|, |\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})|\} \ge C_{26}\lambda_3^n$, for some $C_{26} > 0$, and the length of the longer of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})$ is, thus, of the order of λ_3^n. Together with the estimates (3.45) for $j = j_{\lambda_3}$, this gives that $\min\{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})|, |\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})|\} \ge C_{27}\lambda_3^n$, for some $C_{27} > 0$, and the length of the shorter of these intervals is, thus, also of the same order.

We will now prepare the setting to extend these estimates to j such that, for some $\mathfrak{C} > 0$, both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \subset [-\mathfrak{C}\lambda_3^n, 0]$. It follows from (3.38) and Corollary 3.12 that, for some $C_{28}, C_{29} > 0$,

$$
|\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) - 1| \le |\mathfrak{s}_n(\Delta^{(n)}_0) - 1| + C_{28}\lambda^n \le C_{29}\lambda_2^n. \tag{3.48}
$$

In particular, this implies (3.45) for $j = k_{n+1}$.

We can now perform backward iterations of f_n and $\widetilde{f}_n$, starting from the intervals $\tau_n(\Delta^{(n)}_{q_{n+1}})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})$, respectively. For $1 \le j < k_{n+1}$ such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \subset [-\mathfrak{C}\lambda_3^n, 0]$, we have

$$
\begin{aligned}
\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=j}^{k_{n+1}-1}\left(1+\frac{f_n'(\zeta_i)-\widetilde{f}_n'(\widetilde{\zeta}_i)}{\widetilde{f}_n'(\widetilde{\zeta}_i)}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) \\
&\le \prod_{i=j}^{k_{n+1}-1}\left(1+\frac{|f_n'(\zeta_i)-\widetilde{f}_n'(\zeta_i)|+|\widetilde{f}_n'(\zeta_i)-\widetilde{f}_n'(\widetilde{\zeta}_i)|}{|\widetilde{f}_n'(\widetilde{\zeta}_i)|}\right)\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) \\
&\le \left(1+K_2^{-1}(C\lambda^n+K_1\mathfrak{C}\lambda_3^n)\right)^{k_{n+1}-j}\mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}).
\end{aligned}
\tag{3.49}
$$

Here, we have used again condition (d) and regularity conditions (i) and (ii). Since, by condition (a), $j \le k_{n+1} \le C_1\lambda_1^{-n}$, and since $\lambda < \lambda_3 < \lambda_1\lambda_2$, by using (3.48) and (3.49), we obtain the first inequality in (3.45), for the considered indices j. By exchanging the roles of f_n and $\widetilde{f}_n$, we can obtain the second inequality in (3.45), for the these indices j.

Let $j_{-\lambda_3}$ be the smallest index j satisfying $1 \leq j < k_{n+1}$ such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ are subsets of $[-\mathfrak{C}\lambda_3^n, 0]$. Since $\Delta^{(n)}_{q_{n-1}} = T^{q_{n-1}}(\Delta^{(n)}_0)$, $\Delta^{(n)}_{q_{n+1}} = T^{q_{n+1}}(\Delta^{(n)}_0)$ and $\Delta^{(n)}_{q_{n+1}} = T^{q_n}(\Delta^{(n)}_{q_{n+1}-q_n})$, using regularity conditions (i) and (ii) (these conditions imply that $(T^{q_n})'(x) = f_n'(\tau_n(x))$ is uniformly bounded and bounded away from zero on $\Delta^{(n-1)}_0$), one can easily see that the lengths of the intervals $\Delta^{(n)}_{q_{n-1}}$, $\Delta^{(n)}_0$, $\Delta^{(n)}_{q_{n+1}}$ and $\Delta^{(n)}_{q_{n+1}-q_n}$ are all of the same order. Therefore, if $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| \leq \lambda_3^n$ and $|\tau_n(\Delta^{(n)}_{q_{n-1}})| \leq \lambda_3^n$, there exists $\mathfrak{C} > 0$ such that $\widetilde{\Delta}^{(n)}_{q_{n+1}-q_n}, \Delta^{(n)}_{q_{n+1}-q_n} \subset [-\mathfrak{C}\lambda_3^n, 0]$ (one can choose $\mathfrak{C} = (K_1 + K_1^2)K_2^{-1}$) and $j_{-\lambda_3} \leq k_{n+1} - 1$. As before, Proposition 3.5 implies that there exists $C_{30} > 0$ such that $|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{-\lambda_3}q_n})|, |\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{-\lambda_3}q_n})| \geq C_{30}\lambda_3^n$ and, thus, the lengths of both of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{-\lambda_3}q_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{-\lambda_3}q_n})$ are of the order of λ_3^n.

So far, we have established (3.45) for $0 \leq j \leq j_{\lambda_3}$ and $j_{-\lambda_3} \leq j \leq k_{n+1}$. In order to prove the desired estimates (3.45) for $j_{\lambda_3} < j < j_{-\lambda_3}$, we use Lemma 3.13. We will now verify the assumptions of this lemma. First, for all such j, the lengths of the corresponding intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ are at least of the order of λ_3^n. This follows from the fact that the lengths of $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{-\lambda_3}q_n})$ are of the order of λ_3^n, and that, due to regularity conditions (iii) and (iv), inside the gates $[-1, z_g^{(1)})$ and $(z_g^{(2)}, 0]$, the function $f_n(z) - z$ is monotone, increasing and decreasing, respectively, while $f_n(z) - z \geq K_3$, for $z \in [z_g^{(1)}, z_g^{(2)}]$. To see this, notice that $f_n(z_j) - z_j = |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|$, where $z_j = \tau_n(x_{q_{n-1}+jq_n})$, and that, inside the gates, the derivative $f_n'(z) - 1$ is of definite sign.

We will now show that that the number of indices j satisfying $j_{\lambda_3} < j < j_{-\lambda_3}$ is of the order of n. Proposition 3.3 establishes that (since $f_n(-1) + 1 \leq K_3/2$, for sufficiently large ℓ) inside $[-1, -1+\varepsilon)$ the length of these intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ grows while (since $f_n(0) \leq K_3/2$) inside $(-\varepsilon, 0]$ it decreases exponentially with j. If j_ε and $j_{-\varepsilon}$ are the smallest and largest index j such that the intersection $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \cap [-1+\varepsilon, -\varepsilon] \neq \emptyset$, it follows from Corollary 3.6 that there exists $\varepsilon_1 > 0$, of order ε, such that $|\tau_n(\Delta^{(n)}_{q_{n-1}+j_\varepsilon q_n})|, |\tau_n(\Delta^{(n)}_{q_{n-1}+j_{-\varepsilon}q_n})| > \varepsilon_1$. Furthermore, since, for $j_{\lambda_3} < j \leq j_\varepsilon$, the length of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ grows exponentially from $|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3}q_n})|$, which is of order λ_3^n, to $|\tau_n(\Delta^{(n)}_{q_{n-1}+j_\varepsilon q_n})|$, which is of order 1, the number of such indices, denoted by $N_n^{(l)}$, is of the order of n. Namely, since the rate of this exponential growth is bounded by $B > 1$ (by Proposition 3.3) from below, and K_1 from above (by regularity condition (i)), we have

$B^{N_n^{(l)}} \leq |\tau_n(\Delta^{(n)}_{q_{n-1}+j_\varepsilon q_n})|/|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})| \leq K_1^{N_n^{(l)}}$, and, thus,

$$C_{31}^{-1} n < \frac{\ln \frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+j_\varepsilon q_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})|}}{\ln K_1} \leq N_n^{(l)} \leq \frac{\ln \frac{|\tau_n(\Delta^{(n)}_{q_{n-1}+j_\varepsilon q_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})|}}{\ln B} < C_{31} n, \tag{3.50}$$

for some $C_{31} > 1$. Similarly, the number $N_n^{(r)}$ of indices satisfying $j_{-\varepsilon} \leq j < j_{\lambda_3}$ is of order n as well. Finally, the number $N_n^{(c)}$ of indices satisfying $j_\varepsilon < j < j_{-\varepsilon}$ is bounded by a constant since, due to the above mentioned monotonicity of $f_n(z) - z$ inside the gates, the length of all corresponding intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ is bounded from below by a positive constant. Therefore, the number N_n of indices j that satisfy $j_{\lambda_3} < j < j_{-\lambda_3}$ is of the order of n and, in particular, there exists $C_{32} > 0$ such that $N_n = N_n^{(l)} + N_n^{(c)} + N_n^{(r)} \leq C_{32} n$.

The inequality (3.47) implies that $r_n^-(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})$ and $r_n^+(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})$ are at most of the order of λ_2^n (since the length of $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n})$ is of the order of λ_3^n), i.e., we have $r_n^-(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n}) \leq C_{33}\lambda_2^n$ and $r_n^+(\Delta^{(n)}_{q_{n-1}+j_{\lambda_3} q_n}) \leq C_{33}\lambda_2^n$, for some $C_{33} > 0$. This verifies the assumptions of Lemma 3.13 which implies that, for j satisfying $j_{\lambda_3} < j \leq j_{-\lambda_3}$, we have $r_n^-(\Delta^{(n)}_{q_{n-1}+jq_n}) \leq C_{34}\lambda_2^n$ and $r_n^+(\Delta^{(n)}_{q_{n-1}+jq_n}) \leq C_{34}\lambda_2^n$, for some $C_{34} > 0$. The estimates (3.45) now also follow for $j_{\lambda_3} < j < j_{-\lambda_3}$ since, by inequality (3.6), $|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - 1| \leq C_{35}\lambda_2^n$, where $C_{35} = 2C_{34}$.

To complete the proof, we need to consider the case when either $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| > \lambda_3^n$ or $|\tau_n(\Delta^{(n)}_{q_{n-1}})| > \lambda_3^n$. If either of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}})$ has a length larger than λ_3^n, then the other one has a length which is at least of the order of λ_3^n as well. This follows from the estimate $||\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| - |\tau_n(\Delta^{(n)}_{q_{n-1}})|| = |\widetilde{f}_n(-1) - f_n(-1)| \leq C\lambda^n$. The same lower bound on the lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})$ and $\tau_n(\Delta^{(n)}_{q_{n+1}})$ holds true since, as explained earlier in this proof, due to regularity conditions (i) and (ii), the lengths of these intervals are of the same order as $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}})$, respectively. We will now verify that the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}})$ satisfy the conditions of Lemma 3.13, in this case, and apply this lemma to obtain estimates (3.45) for all j satisfying $0 < j \leq k_{n+1}$. If $f_n(-1) + 1 \leq K_3/2$, using the same arguments as above for $j_{\lambda_3} < j \leq j_{-\lambda_3}$, we obtain that there exists $C_{36} > 0$ such that $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \geq C_{36}\lambda_3^n$, for $0 \leq j \leq k_{n+1}$, and that there exists $C_{37} > 0$ such that $k_{n+1} \leq C_{37} n$. If $f_n(-1) + 1 > K_3/2$, we obtain, using, as above, the monotonicity of $f_n(z) - z$ inside the gates, that the length of all intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$, for $0 \leq j \leq k_{n+1}$, is bounded from below by a positive constant, independent of n, and that, in this case, k_{n+1} is bounded from above, uniformly in n. Since $||\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}})| - |\tau_n(\Delta^{(n)}_{q_{n-1}})|| \leq C\lambda^n$ and $|\tau_n(\Delta^{(n)}_{q_{n-1}})|$ is at least of the order of λ_3^n, we have $r_n^+(\Delta^{(n)}_{q_{n-1}}) \leq C_{38}\lambda_2^n$, for some $C_{38} > 0$. Therefore, the assumptions of Lemma 3.13 are satisfied with $I = \Delta^{(n)}_{q_{n-1}}$ and $N_n = k_{n+1}$, and we can conclude that $r_n^+(\Delta^{(n)}_{q_{n-1}+jq_n}) \leq C_{39}\lambda_2^n$, for some $C_{39} > 0$ and $0 \le j \le k_{n+1}$. The estimates (3.45), in this case, follow directly from this lemma, by using inequality (3.6). **QED**

3.3 Renormalization graphs that are convex inside the tunnels

In this section, we consider the subsequences of renormalizations f_n and $\widetilde{f}_n$ of maps T and $\widetilde{T}$, respectively, for all n such that $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$. These renormalizations satisfy the regularity conditions (i), (ii), (v) and (vi). The graphs of these renormalizations are convex inside the tunnels. The same holds in the case of critical circle maps and some estimates in this section have already been proven in [12]. We include their proofs for completeness of the presentation.

If B_{f_n,K_5} is not empty, let ζ_n^* be a point such that $f_n'(\zeta_n^*) = 1$ (such a point, that we will refer to as the center of the tunnel, exists due to regularity conditions (v) and (vi)). Similarly, if $\widetilde{B}_{\widetilde{f}_n,K_5}$ is not empty, let $\widetilde{\zeta}_n^*$ be a point such that $\widetilde{f}_n'(\widetilde{\zeta}_n^*) = 1$. If $k_{n+1} > 1/K_5$, the tunnel B_{f_n,K_5} is nonempty and ζ_n^* is defined. The affine orientation-preserving change of variables

$$y = h(z) = \frac{1}{2} f_n''(\zeta_n^*)(z - \zeta_n^*) \tag{3.51}$$

maps ζ_n^* into 0 and normalizes the second derivative of f_n there. Under this change of variables f_n is transformed into $g_n = h \circ f_n \circ h^{-1}$ which satisfies $g_n'(0) = 1$ and $g_n''(0) = 2$. We refer to $\kappa = g_n(0) = \min_y\{g_n(y) - y\}$ as the size of the tunnel. Since $f_n(\zeta_n^*) - \zeta_n^* \le k_{n+1}^{-1}$, by regularity condition (vi), we have $0 < \kappa \le \frac{2}{K_6} k_{n+1}^{-1}$. Since f_n is $C^{2+\alpha}$-smooth, it follows from the definition (3.51) that

$$|g_n(y) - (\kappa + y + y^2)| \le C_{40}|y|^{2+\alpha}, \qquad y \in h[-1, 0], \tag{3.52}$$

where $C_{40} > 0$. Similarly, taking into account that $(f_n^{-1})'(\zeta_n^*) = 1$ and $(f_n^{-1})''(\zeta_n^*) = -f_n''(\zeta_n^*)$, we have that, for some $C_{41} > 0$,

$$|g_n^{-1}(y) + (\kappa - y + y^2)| \le C_{41}|y|^{2+\alpha}, \qquad y \in h[f_n(-1), f_n(0)]. \tag{3.53}$$

To estimate the distance of the iterates of the point -1, under f_n, to the center of the tunnel, as well as the distance between its successive iterates, we will use the following two lemmas that were proven in [12]. Namely, it follows from (3.52) and (3.53) that, if $\kappa < \text{const}\,|y|^{2+\alpha}$, the κ term does not influence the asymptotic behavior of g_n while, in the opposite case, it does. The following two lemmas will allow us to obtain two different asymptotic formulas, one for $|y| > \text{const}\,\kappa^{\frac{1}{2+\alpha}}$ and the other for $|y| < \text{const}\,\kappa^{\frac{1}{2+\alpha}}$.

Lemma 3.15 ([12]) *Suppose that, for a sequence of real numbers $\{s_i\}_{i\ge 0}$, there exist $C_{42} > 0$ and $\alpha \in (0,1)$ such that $|s_{i+1} - (s_i - s_i^2)| \le C_{42}|s_i|^{2+\alpha}$, for every $i \ge 0$. Then,* *there exist constants $D_1 > 0$ and $d_1 \in (0,1)$ such that, as long as $s_0 \in (0, d_1]$, the estimate*

$$\left| s_i - \frac{1}{i + s_0^{-1}} \right| \le \frac{D_1}{(i + s_0^{-1})^{1+\alpha}} \tag{3.54}$$

holds, for every $i \ge 0$. Moreover, there exists $D_2 > 0$ such that

$$s_i - s_{i+1} = \frac{1}{(i + s_0^{-1})^2}(1 + \delta_i), \tag{3.55}$$

where $|\delta_i| \le D_2 s_0^\alpha$, for all $i \ge 0$, as long as $s_0 \in (0, d_1]$.

Lemma 3.16 ([12]) *Suppose that, for a sequence of real numbers $\{s_i\}_{i \ge 0}$, there exist $C_{43}, C_{44} > 0$ and $\kappa, \alpha \in (0,1)$ such that*

1. *$|s_0| \le C_{43}\kappa$,*
2. *$|s_{i+1} - (\kappa + s_i + s_i^2)| \le C_{44}|s_i|^{2+\alpha}$, for every $i \ge 0$.*

Fix arbitrary $C_{45} > 0$ and define $N = \kappa^{-1/2}\tan^{-1}(C_{45}\kappa^{-\frac{\alpha}{2(2+\alpha)}})$. Then, there exist constants $D_3 > 0$ and $d_2 \in (0,1)$ such that, as long as $\kappa \in (0, d_2]$, the following estimate holds for every $0 \le i \le N$,

$$|s_i - \sqrt{\kappa}\tan(\sqrt{\kappa} i + a_0)| \le D_3(\sqrt{\kappa}\tan\sqrt{\kappa} i)^{1+\frac{\alpha(\alpha+1)}{2}}, \tag{3.56}$$

where $a_0 = \tan^{-1}(s_0/\sqrt{\kappa})$. Moreover, there exists $D_4 > 0$ such that

$$s_{i+1} - s_i = \frac{\kappa}{(\cos\sqrt{\kappa} i)^2}(1 + \delta_i), \tag{3.57}$$

where $|\delta_i| \le D_4 \kappa^{\frac{\alpha(\alpha+1)}{2(2+\alpha)}}$, for all $0 \le i < N$, as long as $\kappa \in (0, d_2]$.

Lemma 3.15 allows us to establish an upper bound on the distance of the points $f_n^i(-1)$, for $0 \le i \le k_{n+1}$, to the center of the tunnel.

Let $t_0 = h(-1)$ and $t_i = g_n^i(t_0)$, i.e. $t_i = h(f_n^i(-1))$.

Lemma 3.17 *Let $k_{n+1} > 1/K_5$. Let $0 < L_n \le k_{n+1}$ and let $f_n^i(-1) - \zeta_n^* < 0$, for $0 \le i \le L_n$. There exists $C_{46} > 0$ such that $|f_n^i(-1) - \zeta_n^*| \le C_{46} L_n^{-1}$.*

Proof. As long as $t_i < 0$, from (3.52), we have $-t_{i+1} \le -\kappa - t_i - t_i^2 + C_{40}|t_i|^{2+\alpha} \le -t_i - t_i^2 + C_{40}|t_i|^{2+\alpha}$. It is easy to show by induction that if $s_{i+1} = s_i - s_i^2 + C_{40}|s_i|^{2+\alpha}$, $s_0 = -t_j$ and j is large enough such that $-t_j \le 1/2$, then $-t_{i+j} \le s_i$, for all $i \in \mathbb{N}$. It is not difficult to see that there exists $C_{47} > 0$ such that, if $j \ge C_{47}$, then $-t_j \le 1/2$. We will prove the latter claim by contradiction. Namely, since $h^{-1}(t_j) - \zeta_n^* = 2t_j/f_n''(\zeta_n^*)$, if $t_j > 1/2$, then $|h^{-1}(t_j) - \zeta_n^*| \ge K_1^{-1}$, by the regularity condition (i). By the regularity conditions (v) and (vi), we find $f_n(h^{-1}(t_j)) - h^{-1}(t_j) \geq \min\{K_5, K_6(h^{-1}(t_j) - \zeta_n^*)^2/2\} \geq \min\{K_5, K_6K_1^{-1}\}$. It follows, using again the regularity conditions (v) and (vi), that, for $0 \leq i \leq j$, we have $f_n(h^{-1}(t_i)) - h^{-1}(t_i) \geq \min\{K_5, K_6K_1^{-1}\}$. If $C_{47} > 1/\min\{K_5, K_6K_1^{-1}\}$, this leads to a contradiction. Therefore, for $j \geq C_{47}$, we have $-t_j \leq 1/2$ and we can apply Lemma 3.15, with s_i specified above. The claim follows. **QED**

Lemma 3.18 *Let $k_{n+1} > 1/K_5$. Let $0 < L_n \leq k_{n+1}$ and let $f_n^{k_{n+1}-i}(-1) - \zeta_n^* > 0$, for $0 \leq i \leq L_n$. There exists $C_{48} > 0$ such that $|f_n^{k_{n+1}-i}(-1) - \zeta_n^*| \leq C_{48}L_n^{-1}$.*

Proof. The proof of this claim is analogous to the proof of Lemma 3.17. **QED**

We will now estimate some important parameters of the tunnel. Since $\kappa = g_n(0)$, there exists a unique number i_c satisfying $0 < i_c < k_{n+1}$ such that $t_{i_c} \in [0, \kappa)$. Let $i_l = i_c - [\kappa^{-1/2}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}}]$ and $i_r = i_c + [\kappa^{-1/2}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}}]$. The analogous quantities associated to $\widetilde{g}_n$ will be denoted by $\widetilde{\kappa}$, $\widetilde{i}_c$, $\widetilde{i}_l$ and $\widetilde{i}_r$. Combining $\tan^{-1}\frac{1}{x} = \frac{\pi}{2} - \tan^{-1}x$ with the asymptotic formula $\tan^{-1}x = x + \mathcal{O}(x^3)$, $x \to 0$, it is easy to derive

$$\kappa^{-\frac{1}{2}}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}} = \frac{\pi}{2}\kappa^{-\frac{1}{2}} - \kappa^{-\frac{1}{2+\alpha}} + \mathcal{O}(\kappa^{\frac{-1+\alpha}{2+\alpha}}), \quad \kappa \to 0. \tag{3.58}$$

Lemma 3.19 ([12]) *There exist constants $C_{49}, C_{50} > 0$ such that if $k_{n+1} \geq C_{49}$, then*

$$|k_{n+1} - \pi\kappa^{-\frac{1}{2}}| \leq C_{50}\kappa^{\frac{-1+\alpha}{2}}, \tag{3.59}$$

and

$$\left|i_c - \frac{k_{n+1}}{2}\right| \leq C_{50}\kappa^{\frac{-1+\alpha}{2}}. \tag{3.60}$$

Proof. We include the proof for completeness of the presentation. It follows from Lemma 3.15, together with (3.52) and (3.53), that there exists $i_0 \geq 1$ such that,

$$\begin{aligned} \left|t_{i_0+i} + \frac{1}{i - t_{i_0}^{-1}}\right| &\leq \frac{C_{51}}{(i - t_{i_0}^{-1})^{1+\alpha}}, \quad 0 \leq i \leq i_l - i_0, \\ \left|t_{k_{n+1}-i_0-i} - \frac{1}{i + t_{k_{n+1}-i_0}^{-1}}\right| &\leq \frac{C_{51}}{(i + t_{k_{n+1}-i_0}^{-1})^{1+\alpha}}, \quad 0 \leq i \leq k_{n+1} - i_0 - i_r, \end{aligned} \tag{3.61}$$

respectively, for some $C_{51} > 0$. Lemma 3.16, applied to $s_i = -g_n^{-i}(t_{i_c})$ and $s_i = g_n^i(t_{i_c})$ (the assumptions of the lemma are satisfied due to (3.53) and (3.52)), respectively, implies that, for some $C_{52} > 0$,

$$\begin{aligned} \left|t_{i_l} + \kappa^{\frac{1}{2+\alpha}}\right| &\leq C_{52}\kappa^{\frac{1}{2+\alpha} + \frac{\alpha(\alpha+1)}{2(2+\alpha)}}, \\ \left|t_{i_r} - \kappa^{\frac{1}{2+\alpha}}\right| &\leq C_{52}\kappa^{\frac{1}{2+\alpha} + \frac{\alpha(\alpha+1)}{2(2+\alpha)}}. \end{aligned} \tag{3.62}$$

It follows from (3.61), for $i = i_l - i_0$ and $i = k_{n+1} - i_0 - i_r$, respectively, and (3.62) that, for small enough κ,

$$\begin{aligned} \left| i_l - i_0 - t_{i_0}^{-1} - \kappa^{-\frac{1}{2+\alpha}} \right| &\leq C_{53} \kappa^{\frac{-1+\alpha}{2}}, \\ \left| k_{n+1} - i_0 - i_r + t_{k_{n+1}-i_0}^{-1} - \kappa^{-\frac{1}{2+\alpha}} \right| &\leq C_{53} \kappa^{\frac{-1+\alpha}{2}}, \end{aligned} \tag{3.63}$$

where $C_{53} > 0$. Since $k_{n+1} = (k_{n+1} - i_0 - i_r) + (i_r - i_c) + (i_c - i_l) + (i_l - i_0) + 2i_0$, from (3.63), using the asymptotic (3.58) and $\kappa^{\frac{-1+\alpha}{2}} > \kappa^{\frac{-1+\alpha}{2+\alpha}}$, we obtain (3.59). Since $k_{n+1} - 2i_c = k_{n+1} - i_0 - i_r - (i_l - i_0)$ and both $t_{i_0}^{-1}$ and $t_{k_{n+1}-i_0}^{-1}$ are bounded, from (3.63), we also obtain (3.60). **QED**

Corollary 3.20 *There exist $C_{54}, C_{55} > 0$ such that, if $k_{n+1} \geq C_{54}$, then*

$$\left| \frac{\widetilde{\kappa}}{\kappa} - 1 \right| \leq C_{55} \kappa^{\frac{\alpha}{2}}. \tag{3.64}$$

Proof. It follows from Lemma 3.19, by using the corresponding inequalities (3.59) for $\widetilde{\kappa}$ and κ, and the fact that, for x close to 1, $x^2 - 1$ is of the same order as $x - 1$. **QED**

Corollary 3.21 *There exist $C_{56}, C_{57} > 0$ such that, if $k_{n+1} \geq C_{56}$, then*

$$|\widetilde{i}_c - i_c|, |\widetilde{i}_r - i_r|, |\widetilde{i}_l - i_l| \leq C_{57} \kappa^{\frac{-1+\alpha}{2}}. \tag{3.65}$$

Proof. It follows from Lemma 3.19, using estimate (3.60) for i_c and $\widetilde{i}_c$, and the asymptotic formula (3.58). **QED**

Corollary 3.22 *There exist $C_{58}, C_{59} > 0$ such that, if $k_{n+1} \geq C_{58}$, then*

$$\left| \kappa - \frac{\pi^2}{k_{n+1}^2} \right| \leq C_{59} \kappa^{1+\frac{\alpha}{2}}. \tag{3.66}$$

Proof. It follows directly from estimate (3.59) of Lemma 3.19. **QED**

The next lemma gives a lower bound on the distance between the successive iterates of $f_n^i(-1)$, for $0 \leq i \leq k_{n+1}$. To simplify the notation, let $z_i = f_n^i(-1) = \tau_n(x_{q_{n-1}+iq_n})$.

Lemma 3.23 *There exists $C_{60} > 0$ such that, for all $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$,*

$$\begin{aligned} |\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})| &\geq C_{60} i^{-2}, \quad 0 < i \leq i_c, \\ |\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})| &\geq C_{60} (k_{n+1} - i)^{-2}, \quad i_c - 1 \leq i < k_{n+1}. \end{aligned} \tag{3.67}$$

Proof. We will prove the first inequality only. The proof of the second inequality is similar. It follows from the regularity conditions (v) that there exists $C_{61} > 0$ such that $|\tau_n(\Delta^{(n)}_{q_{n-1}})| \ge C_{61}$. Let $J := [1/K_5] + 1$. It follows from the regularity condition (ii) that (3.67) holds for $0 < i \le \min\{J, i_c\}$. If $J \ge i_c$, the claim is proven. In the following, we assume $J < i_c$. The regularity condition (v) implies that, for all the remaining i, satisfying $J < i \le i_c$, $z_i \in B_{f_n,K_5}$. It follows from (3.62) that, there exists $C_{62} > 0$ such that, $\kappa < C_{62}|t_i|^{2+\alpha}$, for $J < i \le i_l$, and we may apply Lemma 3.15 to $s_i = -t_{i+J}$ (the assumptions of the lemma are satisfied due to (3.52)) to obtain the first inequality in (3.67), for $J < i \le i_l$. To compare the length of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $h(\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n}))$, we have also used the fact that, due to regularity condition (i), $|h'(z)| \le \frac{K_1}{2}$. To complete the proof, we need to verify (3.67) for $i_l < i \le i_c$. To do that, we apply Lemma 3.16 with $s_i = -t_{i_c+1-i}$ (the assumptions of the lemma are satisfied due to (3.53)). In particular, for $i_l \le i \le i_c$, Lemma 3.16 implies

$$t_{i+1} - t_i = s_{i_c+1-i} - s_{i_c-i} = \frac{\kappa}{(\cos(\sqrt{\kappa}(i_c - i)))^2}(1 + \delta_{i_c-i}). \tag{3.68}$$

We will now prove that there exists $C_{63} > 0$ such that, for $i_l \le i \le i_c$,

$$t_{i+1} - t_i \ge C_{63} i^{-2}. \tag{3.69}$$

To prove this, we will first verify that the function $\chi(\sqrt{\kappa}i) = \frac{\sqrt{\kappa}i}{\cos(\sqrt{\kappa}(i_c-i))}$ is monotonically increasing. This follows from the fact that the function $\sqrt{\kappa}i\tan(\sqrt{\kappa}(i_c - i))$ has the maximum when $\sqrt{\kappa}i = \frac{\tan(\sqrt{\kappa}(i_c-i))}{1+\tan^2(\sqrt{\kappa}(i_c-i))}$ and, therefore, $\chi'(\sqrt{\kappa}i) = \frac{1-\sqrt{\kappa}i\tan(\sqrt{\kappa}(i_c-i))}{\cos(\sqrt{\kappa}(i_c-i))} \ge (\cos(\sqrt{\kappa}(i_c - i)))^{-1}(1 + \tan^2(\sqrt{\kappa}(i_c - i)))^{-1} > 0$, for $i_l \le i \le i_c$. Thus,

$$t_{i+1} - t_i = (t_{i_l+1} - t_{i_l})\frac{(\cos(\sqrt{\kappa}(i_c - i_l)))^2}{(\cos(\sqrt{\kappa}(i_c - i)))^2}\frac{(1 + \delta_{i_c-i})}{(1 + \delta_{i_c-i_l})} \ge (t_{i_l+1} - t_{i_l})\frac{i_l^2}{i^2}\frac{(1 + \delta_{i_c-i})}{(1 + \delta_{i_c-i_l})}. \tag{3.70}$$

Taking into account that, by the previously established inequality (3.67) for $i = i_l$, $(t_{i_l+1} - t_{i_l})i_l^2$ is bounded from below by a positive constant and that, by Lemma 3.16, $|\delta_{i_c-i}| \le C_{64}\kappa^{\frac{\alpha(\alpha+1)}{2(2+\alpha)}}$, for some $C_{64} > 0$, this proves (3.69). The claim follows. **QED**

The next proposition gives an estimate on the distance between points $\tilde{\tau}_n(\tilde{x}_{q_{n-1}+jq_n})$ and $\tau_n(x_{q_{n-1}+jq_n})$, for $0 \le j \le k_{n+1}$.

Proposition 3.24 *Assume that the conditions of Theorem 2.2 hold. There exists $C_{65} > 0$ such that, for all $n \ne \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, and for $0 \le j \le k_{n+1}$, we have*

$$|\tilde{\tau}_n(\tilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n})| \le C_{65}\lambda^{n/2}. \tag{3.71}$$

Proof. It suffices to prove the claim for sufficiently large $n \neq \mathfrak{n}_\ell$. To simplify the notation, let $z_i = \tau_n(x_{q_{n-1}+iq_n})$ and $\tilde{z}_i = \tilde{\tau}_n(\tilde{x}_{q_{n-1}+iq_n})$. Notice that the "first" pair of points satisfies the desired bound since, by condition (d) of Theorem 2.2,

$$|\tilde{z}_1 - z_1| = |\tilde{f}_n(-1) - f_n(-1)| \leq C\lambda^n. \tag{3.72}$$

The same is true for the "last" pair since

$$|\tilde{z}_{k_{n+1}} - z_{k_{n+1}}| = |\tilde{f}_{n+1}(0)\tilde{f}_n(0) - f_{n+1}(0)f_n(0)| \leq K_1 C(1+\lambda)\lambda^n. \tag{3.73}$$

Let ξ_i be a point between z_i and $\tilde{z}_i$ such that $|f_n(\tilde{z}_i) - f_n(z_i)| = f_n'(\xi_i)|\tilde{z}_i - z_i|$. Then,

$$\begin{aligned} |\tilde{z}_{i+1} - z_{i+1}| &\leq f_n'(\xi_i)|\tilde{z}_i - z_i| + C\lambda^n, \\ |\tilde{z}_{i-1} - z_{i-1}| &\leq (f_n'(\xi_{i-1}))^{-1}(|\tilde{z}_i - z_i| + C\lambda^n). \end{aligned} \tag{3.74}$$

By iterating these two inequalities we obtain

$$\begin{aligned} |\tilde{z}_j - z_j| &\leq |\tilde{z}_1 - z_1| \prod_{i=1}^{j-1} f_n'(\xi_i) + C\lambda^n \left(1 + \sum_{k=2}^{j-1}\prod_{i=k}^{j-1} f_n'(\xi_i)\right), \\ |\tilde{z}_{k_{n+1}-j} - z_{k_{n+1}-j}| &\leq |\tilde{z}_{k_{n+1}} - z_{k_{n+1}}| \prod_{i=k_{n+1}-j}^{k_{n+1}-1} (f_n'(\xi_i))^{-1} + C\lambda^n \sum_{k=k_{n+1}-j}^{k_{n+1}-1} \prod_{i=k}^{k_{n+1}-1} (f_n'(\xi_i))^{-1}. \end{aligned} \tag{3.75}$$

We can now apply these estimates for all $1 \leq j \leq J$, where $J := [1/K_5] + 1$, obtaining $|\tilde{z}_j - z_j| \leq C_{66}\lambda^n$, and $|\tilde{z}_{k_{n+1}-j} - z_{k_{n+1}-j}| \leq C_{66}\lambda^n$, for some $C_{66} > 0$. If $k_{n+1} \leq 2J$, then the claim is proven. Otherwise, all the remaining points z_i and $\tilde{z}_i$ belong to $B_{f_n,K_5} \cap B_{\tilde{f}_n,K_5}$, ζ_n^* and $\widetilde{\zeta}_n^*$ are well-defined and $|\widetilde{\zeta}_n^* - \zeta_n^*| < CK_6^{-1}\lambda^n$, as follows from the regularity condition (vi) and condition (d).

The objective now is to apply the inequalities (3.75) to obtain the desired estimate for all of the remaining points. We will first make at most $L_n := [\lambda^{-n/2}] + 1$ steps from both ends. More precisely, we will make at most L_n steps from the left end, but stop when $\max\{z_j, \tilde{z}_j\} > \zeta_n^*$. From the first of the inequalities (3.75), we obtain $|\tilde{z}_j - z_j| \leq C_{67}\lambda^{n/2}$, for some $C_{67} > 0$, and all $J < j \leq L_l$, where $L_l := \min\{L_n, \min\{k \in \mathbb{N} : \max\{z_k, \tilde{z}_k\} > \zeta_n^*\}\}$. Here, we have used the fact that the products of derivatives in (3.75) are smaller than 1, since all points ξ_i now belong to $B_{f_n,K_5} \cap B_{\tilde{f}_n,K_5}$, and satisfy $\xi_i \leq \zeta_n^*$. The same estimate is obtained for $k_{n+1} - L_r < j < k_{n+1} - J$, such that $L_r := \min\{L_n, k_{n+1} - \max\{k \in \mathbb{N} : \min\{z_k, \tilde{z}_k\} < \zeta_n^*\}\}$, by applying the second inequality in (3.75).

If an early stop did not occur in the previous iterations, i.e., if $L_l = L_r = L_n$, then, for the rest of the points, by Lemma 3.17 and Lemma 3.18, we have $|z_j - \zeta_n^*| \leq C_{68}L_n^{-1} \leq C_{68}\lambda^{n/2}$, and $|\tilde{z}_j - \widetilde{\zeta}_n^*| \leq C_{68}\lambda^{n/2}$, for some $C_{68} > 0$. Together with $|\widetilde{\zeta}_n^* - \zeta_n^*| < CK_6^{-1}\lambda^n$, this completes the proof, in this case. If both the forward and the backward iterations were stopped earlier at $L_l < L_n$ and $L_r < L_n$, respectively, then all the remaining points z_j and $\tilde{z}_j$ are contained in the interval between the leftmost and the rightmost of the points (i.e., the smallest and largest values of) $z_{L_l}, \tilde{z}_{L_l}, z_{k_{n+1}-L_r}, \tilde{z}_{k_{n+1}-L_r}$. The length of this interval, $\mathfrak{d} := \max\{z_{L_l}, \tilde{z}_{L_l}, z_{k_{n+1}-L_r}, \tilde{z}_{k_{n+1}-L_r}\} - \min\{z_{L_l}, \tilde{z}_{L_l}, z_{k_{n+1}-L_r}, \tilde{z}_{k_{n+1}-L_r}\}$, is bounded from above by $2C_{67}\lambda^{n/2}$. If the iteration in one direction was stopped earlier, while in the other was not, then the two arguments above can be easily combined to complete the proof. Namely, if $L_l < L_n$ and $L_r = L_n$ or $L_l = L_n$ and $L_r < L_n$, then $\mathfrak{d} \le C_{67}\lambda^{n/2} + CK_6^{-1}\lambda^n + C_{68}\lambda^{n/2}$. The claim follows. **QED**

Corollary 3.25 *Under the assumptions of Proposition 3.24, there exists* $C_{69} > 0$ *such that, for all* $n \neq \mathfrak{n}_\ell$*, for any* $\ell \in \mathbb{N}$*, and for all* $1 \le j \le [\lambda^{-n/8}] + 1$ *and* $k_{n+1} - [\lambda^{-n/8}] \le j \le k_{n+1}$*, we have*

$$|\tilde{\tau}_n(\tilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n})| \le C_{69}\lambda^{n/4}|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|. \tag{3.76}$$

Proof. For $1 \le j \le \min\{i_c, [\lambda^{-n/8}] + 1\}$ and $\max\{i_c - 1, k_{n+1} - [\lambda^{-n/8}]\} \le j \le k_{n+1}$, the claim follows directly from Proposition 3.24, taking into account that, by Lemma 3.23, there exists $C_{70} > 0$ such that $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \ge C_{70}\lambda^{n/4}$. These estimates can be extended to $i_c < j \le [\lambda^{-n/8}] + 1$ or $k_{n+1} - [\lambda^{-n/8}] \le j < i_c - 1$ since, by the regularity condition (vi), in these two cases $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \ge |\tau_n(\Delta^{(n)}_{q_{n-1}+i_cq_n})|\}$ and $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \ge |\tau_n(\Delta^{(n)}_{q_{n-1}+(i_c-1)q_n})|$, respectively, for sufficiently large n. The claim follows. **QED**

The following lemma shows that, for the values of n considered (corresponding to the renormalization graphs that are convex inside the tunnels), the ratios of lengths of the renormalized intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ are exponentially (in n) close to 1.

Proposition 3.26 *Assume that the conditions of Theorem 2.2 hold. There exists* $C_{70} > 0$*, such that for all* $n \neq \mathfrak{n}_\ell$*, for any* $\ell \in \mathbb{N}$*, and* $0 \le j \le k_{n+1}$*, we have*

$$\left|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - 1\right| \le C_{70}\lambda_5^n, \tag{3.77}$$

with $\lambda_5 := \lambda^{\frac{(1+\alpha)\alpha}{8(2+\alpha)}}$.

Proof. It suffices to prove the claim for sufficiently large n. For $0 \le j \le [\lambda^{-n/8}]$ and $k_{n+1} - [\lambda^{-n/8}] \le j < k_{n+1}$, it follows directly from Corollary 3.25, by using (3.6), that

$$|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - 1| \le C_{71}\lambda^{n/4}, \tag{3.78}$$

for some $C_{71} > 0$. If this constant has been chosen sufficiently large, inequality (3.78) also holds for $j = k_{n+1}$ since, as follows from (3.73) and (3.74), using that $|f_n'(z)| \le K_1$

by the regularity condition (i), $|\tilde{\tau}_n(\tilde{x}_{q_{n+1}+q_n}) - \tau_n(x_{q_{n+1}+q_n})| \le C_{72}\lambda^n$, for some $C_{72} > 0$; and by the regularity conditions (v) and (ii), $|\tau_n(\Delta^{(n)}_{q_{n+1}})| = |\tau_n(T^{q_{n+1}}(\Delta^{(n)}_0))| \ge K_5K_2$, as $(T^{q_{n+1}})'(x) = f'_{n+1}(\tau_{n+1}(x))$.

If $k_{n+1} \le [\lambda^{-n/8}]$, the claim is proven. In the following, we assume $k_{n+1} > [\lambda^{-n/8}]$ and that n is sufficiently large such that $[\lambda^{-n/8}] > 1/K_5$. This latter condition guarantees that for all the remaining indices j, for which (3.77) remains to be proven, we have $z_i, \widetilde{z}_i \in B_{f_n,K_5} \cap B_{\widetilde{f}_n,K_5}$.

To prove (3.77) for $[\lambda^{-n/8}] < j \le \min\{i_l, \widetilde{i}_l\}$, we apply Lemma 3.15 to $s_i = -t_{i+[\lambda^{-n/8}]}$ and $s_i = -\widetilde{t}_{i+[\lambda^{-n/8}]}$, where $i = j - [\lambda^{-n/8}]$. The assumptions of this lemma are satisfied due to (3.52), since it follows from (3.62) that there exists $C_{73} > 0$ such that, $\kappa < C_{73}|t_j|^{2+\alpha}$, for $[\lambda^{-n/8}] < j \le i_l$. We obtain

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) = \frac{(j - [\lambda^{-n/8}] - t^{-1}_{[\lambda^{-n/8}]})^2}{(j - [\lambda^{-n/8}] - \widetilde{t}^{-1}_{[\lambda^{-n/8}]})^2} \frac{(1+\widetilde{\delta}_j)}{(1+\delta_j)} \frac{\widetilde{f}''_n(\widetilde{\zeta}^*_n)}{f''_n(\zeta^*_n)}, \tag{3.79}$$

for $[\lambda^{-n/8}] \le j \le \min\{i_l, \widetilde{i}_l\}$. Since $|\frac{(1+a)^2}{(1+b)^2} - 1| \le |\frac{a^2}{b^2} - 1|$, for $a, b > 0$, from the last equality, we find

$$\left|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \frac{(1+\delta_j)}{(1+\widetilde{\delta}_j)} \frac{f''_n(\zeta^*_n)}{\widetilde{f}''_n(\widetilde{\zeta}^*_n)} - 1\right| \le \left|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+[\lambda^{-n/8}]q_n}) \frac{(1+\delta_{[\lambda^{-n/8}]})}{(1+\widetilde{\delta}_{[\lambda^{-n/8}]})} \frac{f''_n(\zeta^*_n)}{\widetilde{f}''_n(\widetilde{\zeta}^*_n)} - 1\right|. \tag{3.80}$$

Since, by Lemma 3.15, for the considered indices j, $|\delta_j| \le C_{74}t^{\alpha}_{[\lambda^{-n/8}]}$, $|\widetilde{\delta}_j| \le C_{74}\widetilde{t}^{\alpha}_{[\lambda^{-n/8}]}$, with $C_{74} > 0$, by Lemma 3.17, $|\delta_j|, |\widetilde{\delta}_j| \le C_{75}\lambda^{n\alpha/8}$, for some $C_{75} > 0$. Using this estimate, condition (d), $|\widetilde{\zeta}^*_n - \zeta^*_n| < CK_6^{-1}\lambda^n$ and that, by regularity condition (vi), $\widetilde{f}''_n(\widetilde{\zeta}^*_n) > K_6$, inequality (3.80), together with (3.78) for $j = [\lambda^{-n/8}]$, implies that, for $[\lambda^{-n/8}] \le j \le \min\{i_l, \widetilde{i}_l\}$,

$$\left|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - 1\right| \le C_{76}\lambda^{n\alpha/8}, \tag{3.81}$$

where $C_{76} > 0$. One can similarly obtain the same estimate for $\max\{i_r, \widetilde{i}_r\} \le j \le k_{n+1}$.

It remains to prove (3.77) for $\min\{i_l, \widetilde{i}_l\} \le j \le \max\{i_r, \widetilde{i}_r\}$. To estimate the ratio of lengths of the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ and $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$, we will apply Lemma 3.16. Since, $\widetilde{i}_c$ may be different from i_c, we will use the following factorization

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) = \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j+\widetilde{i}_c-i_c)q_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|} \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|}{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j+\widetilde{i}_c-i_c)q_n})|}. \tag{3.82}$$

The second of the ratios in (3.82) can be estimated as follows. To be specific let us assume that $\widetilde{i}_c < i_c$ (in the opposite case, the proof is similar). We have that

$$\left| \frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|}{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j+\widetilde{i}_c-i_c)q_n})|} - 1 \right| = \left| \prod_{i=j+\widetilde{i}_c-i_c}^{j-1} \widetilde{f}'_n(\widetilde{\xi}_i) - 1 \right| \leq C_{77}\widetilde{\kappa}^{\frac{\alpha(1+\alpha)}{2(2+\alpha)}}, \tag{3.83}$$

for some $C_{77} > 0$. Here, $\xi_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ and we have used Corollary 3.21 and that, by (3.62), $|\widetilde{f}'_n(\widetilde{\xi}_i) - 1| \leq C_{78}\widetilde{\kappa}^{\frac{1}{2+\alpha}}$, where $C_{78} > 0$. We will now estimate the first ratio in (3.82). To estimate the lengths of the intervals in the numerator and the denominator, for $j \leq i_c$, we apply Lemma 3.16 to $s_i = -\widetilde{t}_{\widetilde{i}_c+1-i}$ and $s_i = -t_{i_c+1-i}$, respectively. The assumptions of this lemma are satisfied due to (3.53) (condition 2) and the facts that $t_{i_c} \in [0, \kappa)$ and g''_n is bounded, due to the regularity condition (i) (condition 1). Similarly, one can verify the assumptions of this lemma for $j \geq i_c$, using (3.52). By Lemma 3.16,

$$\frac{|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j+\widetilde{i}_c-i_c)q_n})|}{|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|} = \frac{\widetilde{\kappa}}{\kappa}\frac{(\cos(\sqrt{\kappa}(i_c-j)))^2}{(\cos(\sqrt{\widetilde{\kappa}}(i_c-j)))^2}\frac{(1+\widetilde{\delta}_{j+\widetilde{i}_c-i_c})}{(1+\delta_j)}\frac{\widetilde{f}''_n(\widetilde{\zeta}^*_n)}{f''_n(\zeta^*_n)}. \tag{3.84}$$

Notice that this holds for $\min\{i_l, \widetilde{i}_l\} \leq j \leq \max\{i_r, \widetilde{i}_r\}$ since, by Corollary 3.20, we have $|[\widetilde{\kappa}^{-1/2}\tan^{-1}\widetilde{\kappa}^{-\frac{\alpha}{2(2+\alpha)}}] - [\kappa^{-1/2}\tan^{-1}\kappa^{-\frac{\alpha}{2(2+\alpha)}}]| \leq [\kappa^{-1/2}\tan^{-1}(C_{79}\kappa^{-\frac{\alpha}{2(2+\alpha)}})]$, for some $C_{79} > 0$, if κ is small enough (which holds for $k_{n+1} > [\lambda^{-n/8}]$ and sufficiently large n, by Corollary 3.22).

Using Corollary 3.20, $|\delta_i|, |\widetilde{\delta}_{i+\widetilde{i}_c-i_c}| \leq C_{80}\kappa^{\frac{\alpha(1+\alpha)}{2(2+\alpha)}}$ (see Lemma 3.16), where $C_{80} > 0$, condition (d) and the estimate $|\widetilde{\zeta}^*_n - \zeta^*_n| < CK_6\lambda^n$ (that follows from the regularity condition (vi)), it follows from (3.82), (3.83) and (3.84) that

$$\left| \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - \frac{(\cos(\sqrt{\kappa}(i_c-j)))^2}{(\cos(\sqrt{\widetilde{\kappa}}(i_c-j)))^2} \right| \leq C_{81}\lambda^{n\frac{\alpha(1+\alpha)}{8(2+\alpha)}}, \tag{3.85}$$

where $C_{81} > 0$, since, by Corollary 3.22, κ and $\widetilde{\kappa}$ are at most of the order of $\lambda^{n/4}$, due to $k_{n+1} > [\lambda^{-n/8}]$. Using the elementary inequalities (that can be easily verified by taking the derivative with respect to j)

$$\left| \ln\frac{\cos(\sqrt{\kappa}(i_c-j-1))}{\cos(\sqrt{\widetilde{\kappa}}(i_c-j-1))} \right| \leq \left| \ln\frac{\cos(\sqrt{\kappa}(i_c-j))}{\cos(\sqrt{\widetilde{\kappa}}(i_c-j))} \right|, \tag{3.86}$$

for $\min\{i_l, \widetilde{i}_l\} \leq j < i_c$, and

$$\left| \ln\frac{\cos(\sqrt{\kappa}(j-i_c-1))}{\cos(\sqrt{\widetilde{\kappa}}(j-i_c-1))} \right| \leq \left| \ln\frac{\cos(\sqrt{\kappa}(j-i_c))}{\cos(\sqrt{\widetilde{\kappa}}(j-i_c))} \right|, \tag{3.87}$$

for $i_c < j \leq \max\{i_r, \widetilde{i}_r\}$, together with the estimates (3.81) and (3.85) for $j = \min\{i_l, \widetilde{i}_l\}$ and $j = \max\{i_r, \widetilde{i}_r\}$, and the asymptotic formula $\ln(1+x) = x + \mathcal{O}(x^2)$, $x \to 0$, we obtain (3.77), for $\min\{i_l, \widetilde{i}_l\} \leq j \leq \max\{i_r, \widetilde{i}_r\}$. **QED**

3.4 The estimates on the fundamental intervals

In this section, we first show that the ratio of lengths of the renormalized fundamental intervals $\widetilde{\tau}_n(\widetilde{\Delta}_0^{(n)})$ and $\tau_n(\Delta_0^{(n)})$ is exponentially in n close to 1. Then, we show that, after an arbitrarily smooth conjugation of one of the maps, the ratio of lengths of the actual fundamental intervals $\widetilde{\Delta}_0^{(n)}$ and $\Delta_0^{(n)}$ is exponentially close to 1.

Lemma 3.27 *Assume that the conditions of Theorem 2.2 hold. There exists $C_{82} > 0$ such that, for all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$,*

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| \leq C_{82}\lambda_2^n, \tag{3.88}$$

and for all $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$,

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| \leq C_{82}\lambda^n. \tag{3.89}$$

Proof. For all $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, the claim follows directly from Corollary 3.12. The improved estimate for $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, follows from the equality

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| = \left|\frac{\tilde{f}_n(0) - f_n(0)}{f_n(0)}\right|, \tag{3.90}$$

taking into account the convergence of renormalizations (condition (d)) and that, for such n, the regularity condition (v) implies $f_n(0) \geq K_5$. **QED**

Lemma 3.28 *Assume that the conditions of Theorem 2.2 hold. There exists $\sigma_\infty > 0$ and $C_{83} > 0$ such that, for all $n \in \mathbb{N}$, we have*

$$\left|\frac{|\widetilde{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|} - \sigma_\infty\right| \leq C_{83}\lambda_2^n. \tag{3.91}$$

Proof. Let $\sigma_n = \frac{|\widetilde{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|}$. It follows from Lemma 3.27, that

$$\left|\frac{\sigma_n}{\sigma_{n-1}} - 1\right| \leq C_{82}\lambda_2^n, \tag{3.92}$$

and, thus, $|\ln\sigma_n - \ln\sigma_{n-1}| = \epsilon_{n-1}$, where $0 \leq \epsilon_{n-1} \leq C_{84}\lambda_2^n$, for some $C_{84} > 0$. Since the sequence of non-negative numbers ϵ_n decreases at least exponentially fast with n, the sequence $(\ln\sigma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to some $\ell_\infty := \lim_{n\to\infty}\ln\sigma_n$. The sequence σ_n, thus, converges to $\sigma_\infty = e^{\ell_\infty} > 0$.

Furthermore, since

$$|\ln \sigma_\infty - \ln \sigma_n| \leq \sum_{m=n}^{\infty} \epsilon_m \leq C_{84} \frac{\lambda_2^n}{1-\lambda_2}, \tag{3.93}$$

we have

$$\left| \frac{\sigma_\infty}{\sigma_n} - 1 \right| \leq C_{85} \lambda_2^n, \tag{3.94}$$

for some $C_{85} > 0$. The claim follows. **QED**

Without loss of generality, we may assume that $\sigma_\infty = 1$. This follows from the following simple lemma.

Lemma 3.29 *Assume that the conditions of Theorem 2.2 hold. There exists an arbitrarily smooth conjugation $\hat{T}$ of $\widetilde{T}$ and $C_{86} > 0$ such that, for all $n \in \mathbb{N}$, we have $\sigma_\infty(\hat{T}) = 1$ and, thus, the length of the fundamental interval $\hat{\Delta}_0^{(n)}$ of $\hat{T}$ satisfies*

$$\left| \frac{|\hat{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|} - 1 \right| \leq C_{86} \lambda_2^n. \tag{3.95}$$

Proof. It is enough to rescale the intervals of $\widetilde{\Delta}_0^{(n)}$ by means of a smooth change of coordinates affine in a neighborhood of $\widetilde{x}_0 = \varphi(x_0)$. Assume that $\sigma_\infty > 1$. Let ψ be a C^∞-smooth orientation-preserving diffeomorphism of $\mathbb{T}^1$, which is affine on $\widetilde{\Delta}_0^{(1)} \cup \widetilde{\Delta}_0^{(2)}$, with derivative σ_∞^{-1}. Let $\hat{T} = \psi \circ \widetilde{T} \circ \psi^{-1}$. This change of $\widetilde{T}$ will not affect the renormalizations $\tilde{f}_n$, for $n \geq 2$, and they will stay regular uniformly with respect to n, but σ_∞ corresponding to $\hat{T}$ and T will be equal to 1. A similar argument works in the case $\sigma_\infty < 1$. **QED**

In this paper, we assume that T and $\widetilde{T}$ have already been adjusted such that (3.91) holds with $\sigma_\infty = 1$.

3.5 Estimates on the intervals of the partition $\mathcal{P}_m$, inside $\Delta_0^{(n-1)}$, with n a constant fraction of m

In the previous sections, we have obtained the necessary estimates on the ratios of lengths of the rescaled intervals of partition $\mathcal{P}_{n+1}$ inside $\bar{\Delta}_0^{(n-1)}$. We would like to extend these estimates to the whole circle. In order to do that, we need to consider the intervals of higher-level partitions $\mathcal{P}_m$, for $m > n+1$, inside this interval. For $I \subset \Delta_{q_{n-1}}^{(n)}$ and $\widetilde{I} = \varphi(I) \subset \widetilde{\Delta}_{q_{n-1}}^{(n)}$, we define, as before, $I_i := \tau_n^{-1}(f_n^i(\tau_n(I)))$ and $\widetilde{I}_i := \widetilde{\tau}_n^{-1}(\widetilde{f}_n^i(\widetilde{\tau}_n(\widetilde{I})))$.

The following lemma will be used in the case when $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, and $k_{n+1} > [\lambda^{-n/8}]$. It concerns "small" intervals inside the tunnel of the convex renormalization graphs.

Lemma 3.30 *There exists $C_{87} > 0$ such that for all $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, and $[\lambda^{-n/8}] < j \leq k_{n+1} - [\lambda^{-n/8}]$, we have*

$$|\ln \mathfrak{s}_n(I_j)| \leq |\ln \mathfrak{s}_n(I_{[\lambda^{-n/8}]})| + C_{87}\lambda_5^n. \tag{3.96}$$

Proof. Notice first that there exist $\zeta_1 \in \tau_n(I_i)$, $\widetilde{\zeta}_1 \in \widetilde{\tau}_n(\widetilde{I}_i)$, $\zeta_2, \zeta_3 \in \tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $\widetilde{\zeta}_2, \widetilde{\zeta}_3 \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ such that

$$\begin{aligned} \left|\ln\left(\frac{\mathfrak{s}_n(I_{i+1})}{\mathfrak{s}_n(I_i)}\frac{\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+iq_n})}{\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+(i+1)q_n})}\right)\right| &= |\ln \widetilde{f}_n'(\widetilde{\zeta}_1) - \ln f_n'(\zeta_1) - \ln \widetilde{f}_n'(\widetilde{\zeta}_2) + \ln f_n'(\zeta_2)| \\ = |(\ln \widetilde{f}_n')'(\widetilde{\zeta}_3)(\widetilde{\zeta}_2 - \widetilde{\zeta}_1) - (\ln f_n')'(\zeta_3)(\zeta_2 - \zeta_1)| &\leq \frac{K_1}{K_2}|\widetilde{\zeta}_2 - \widetilde{\zeta}_1| + \frac{K_1}{K_2}|\zeta_2 - \zeta_1|. \end{aligned} \tag{3.97}$$

Summing up these inequalities from $i = [\lambda^{-n/8}]$ to $j-1$, for some $[\lambda^{-n/8}] < j \leq k_{n+1} - [\lambda^{-n/8}]$, we obtain that

$$\begin{aligned} \left|\ln\left(\frac{\mathfrak{s}_n(I_j)}{\mathfrak{s}_n(I_{[\lambda^{-n/8}]})}\frac{\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+[\lambda^{-n/8}]q_n})}{\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n})}\right)\right| &\leq \frac{K_1}{K_2}\left(\sum_{i=[\lambda^{-n/8}]}^{j-1}|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})|\right. \\ &\left. + \sum_{i=[\lambda^{-n/8}]}^{j-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|\right) \leq C_{88}\frac{K_1}{K_2}\lambda^{n/8}, \end{aligned} \tag{3.98}$$

for some $C_{88} > 0$. The last inequality follows from Lemma 3.17 and Lemma 3.18. Therefore,

$$|\ln \mathfrak{s}_n(I_j)| \leq |\ln \mathfrak{s}_n(I_{[\lambda^{-n/8}]})| + |\ln \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+[\lambda^{-n/8}]q_n})| + |\ln \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n})| + C_{88}\frac{K_1}{K_2}\lambda^{n/8}, \tag{3.99}$$

and the claim follows from this inequality, by using Proposition 3.26. **QED**

We recall that, for the set of rotation numbers considered here, we have a constraint $k_{\mathfrak{n}_\ell+1} \leq C_1\lambda_1^{-\mathfrak{n}_\ell}$, for all $\ell \in \mathbb{N}$ (by condition (a)). Let $\lambda_6 := \max\{\lambda_2, \lambda_5\}$, and let $S_1 := \max\{C_2, C_{70}\}$.

Proposition 3.31 *Assume that the conditions of Theorem 2.2 hold. There exists $S_2 > 1$ such that the following holds. Assume that there exists $C_{89} > 1$ such that for any sufficiently large $n \in \mathbb{N}$, any $m > n$, and all intervals I, with $\mathcal{P}_m \ni I \subset \Delta^{(n)}_{q_{n-1}}$ and the corresponding intervals $\widetilde{I}$, with $\widetilde{\mathcal{P}}_m \ni \widetilde{I} \subset \widetilde{\Delta}^{(n)}_{q_{n-1}}$, we have $|\mathfrak{s}_n(I) - 1| \leq C_{89}\lambda_6^n$. Then, for all $0 \leq i \leq k_{n+1}$, we have $|\mathfrak{s}_n(I_i) - 1| \leq S_2C_{89}\lambda_6^n$.*

Proof. Let y_i and z_i be the left (i.e., smaller) and right (i.e., larger) end point of the interval $\tau_n(I_i)$, respectively. Analogously, let $\widetilde{y}_i$ and $\widetilde{z}_i$ be the left and right end point of the interval $\widetilde{\tau}_n(\widetilde{I}_i)$, respectively. If $z_i - y_i \le \max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}$, then

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| &= \left| \frac{\widetilde{f}_n'(\widetilde{\zeta}_i)|\widetilde{z}_i - \widetilde{y}_i|}{f_n'(\zeta_i)|z_i - y_i|} - 1 \right| \\ \le \left(C\lambda^n K_2^{-1} + 3K_1K_2^{-1}\max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}\right) &\frac{|\widetilde{z}_i - \widetilde{y}_i|}{|z_i - y_i|} + \left| \frac{|\widetilde{z}_i - \widetilde{y}_i|}{|z_i - y_i|} - 1 \right| \\ &\le \left(2C\lambda^n K_2^{-1} + 6K_1K_2^{-1}\max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}\right) + |\mathfrak{s}_n(I_i) - 1|, \end{aligned} \tag{3.100}$$

if $|\mathfrak{s}_n(I_i)| < 2$. If $z_i - y_i > \max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}$, then

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| &= \left| \frac{\int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n'(\zeta)d\zeta}{\int_{y_i}^{z_i} f_n'(\zeta)d\zeta} - 1 \right| = \left| \frac{\widetilde{f}_n'(\widetilde{z}_i)(\widetilde{z}_i - \widetilde{y}_i) - \int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta}{f_n'(z_i)(z_i - y_i) - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta} - 1 \right| \\ &= \left| \frac{\frac{\widetilde{f}_n'(\widetilde{z}_i)}{f_n'(z_i)}\frac{\widetilde{z}_i - \widetilde{y}_i}{z_i - y_i} - 1 - \frac{1}{f_n'(z_i)(z_i - y_i)}\left(\int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta\right)}{1 - \frac{1}{f_n'(z_i)(z_i - y_i)}\int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta} \right|, \end{aligned} \tag{3.101}$$

and, thus,

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| \le \Bigg[& 2CK_2^{-1}\lambda^n \left(1 + \frac{z_i - y_i}{4}\right) + \frac{3}{2}K_1K_2^{-1}|\widetilde{y}_i - y_i| \\ & + 4K_1K_2^{-1}|\widetilde{z}_i - z_i| + |\mathfrak{s}_n(I_i) - 1| \Bigg] (1 + K_1K_2^{-1}|z_i - y_i|). \end{aligned} \tag{3.102}$$

Here, we have used that

$$\left| \frac{\widetilde{f}_n'(\widetilde{z}_i)}{f_n'(z_i)} \frac{\widetilde{z}_i - \widetilde{y}_i}{z_i - y_i} - 1 \right| \le 2(CK_2^{-1}\lambda^n + K_1K_2^{-1}|\widetilde{z}_i - z_i|) + |\mathfrak{s}_n(I_i) - 1|\,, \tag{3.103}$$

if $|\mathfrak{s}_n(I_i)| < 2$,

$$\begin{aligned} \frac{1}{f_n'(z_i)(z_i - y_i)} & \left| \int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta \right| \\ & \le \frac{1}{K_2(z_i - y_i)} \Bigg[C\lambda^n \frac{(z_i - y_i)^2}{2} + K_1|\widetilde{y}_i - y_i||z_i - y_i| \\ & \qquad + K_1\frac{(\widetilde{y}_i - y_i)^2}{2} + 2K_1|z_i - y_i||\widetilde{z}_i - z_i| \Bigg] \\ & \le \frac{1}{K_2}\left[C\lambda^n \frac{(z_i - y_i)}{2} + \frac{3}{2}K_1|\widetilde{y}_i - y_i| + 2K_1|\widetilde{z}_i - z_i| \right], \end{aligned} \tag{3.104}$$

and

$$\frac{1}{f_n'(z_i)(z_i-y_i)}\int_{y_i}^{z_i}|f_n''(\zeta)|(\zeta-y_i)d\zeta \le \frac{1}{2}K_1K_2^{-1}|z_i-y_i|. \tag{3.105}$$

We have also used that $(1-x)^{-1}\le 1+2|x|$, for $x<1/2$, and assumed $K_1K_2^{-1}|z_i-y_i|<1$.

Therefore, in either case, we have

$$\begin{aligned}|\mathfrak{s}_n(I_{i+1})-1| \le \Bigg[&2CK_2^{-1}\lambda^n\left(1+\frac{z_i-y_i}{4}\right)+6K_1K_2^{-1}\max\{|\widetilde{y}_i-y_i|,|\widetilde{z}_i-z_i|\}\\ &+|\mathfrak{s}_n(I_i)-1|\Bigg](1+K_1K_2^{-1}|z_i-y_i|).\end{aligned} \tag{3.106}$$

Using the estimate

$$\max\{|\widetilde{y}_i-y_i|,|\widetilde{z}_i-z_i|\}\le\left(C_{90}S_1\lambda_6^n+\max_{\mathcal{P}_m\ni I_i'\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i')-1|\right)|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|, \tag{3.107}$$

with $C_{90}>0$, which holds for $n=\mathfrak{n}_\ell$, for any $\ell\in\mathbb{N}$ and $0\le i\le k_{n+1}$, and for $n\ne\mathfrak{n}_\ell$, $\ell\in\mathbb{N}$, and $0\le i\le[\lambda^{-n/8}]$ or $k_{n+1}-[\lambda^{-n/8}]\le i\le k_{n+1}$, we obtain

$$\begin{aligned}|\mathfrak{s}_n(I_{i+1})-1|\le\Bigg[&4CK_2^{-1}\lambda^n+6K_1K_2^{-1}\left(C_{90}S_1\lambda_6^n+\max_{\mathcal{P}_m\ni I_i'\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i')-1|\right)\\ &\cdot|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|+\max_{\mathcal{P}_m\ni I_i'\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i')-1|\Bigg](1+K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|).\end{aligned} \tag{3.108}$$

In the estimate (3.107), we have used the fact that for all $n=\mathfrak{n}_\ell$, for some $\ell\in\mathbb{N}$ (corresponding to renormalization graphs concave inside the gates), the distance of the corresponding endpoints of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ is bounded from above by $C_{90}S_1\lambda_6^n|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|$. This follows from Proposition 3.2 and the fact that, by Corollary 3.6, the sum of the lengths of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ inside the gates is of the order of the longest of them. Since $\lambda^{1/4}<\lambda_5\le\lambda_6$, by Corollary 3.25, estimate (3.107) is valid for $n\ne\mathfrak{n}_\ell$, for any $\ell\in\mathbb{N}$ (corresponding to renormalization graphs convex inside the tunnels), if $0\le i\le[\lambda^{-n/8}]$ or $k_{n+1}-[\lambda^{-n/8}]\le i\le k_{n+1}$.

Taking the maximum of the left hand side of (3.108) over all I_{i+1}, such that $\mathcal{P}_m\ni I_{i+1}\subset\Delta^{(n)}_{q_{n-1}+(i+1)q_n}$, we obtain the inequality

$$M_{i+1}\le P_i+Q_iM_i, \tag{3.109}$$

where $M_i := \max_{\mathcal{P}_m \ni I_i \subset \Delta^{(n)}_{q_{n-1}+iq_n}} |\mathfrak{s}_n(I_i) - 1|$, and

$$\begin{aligned} P_i &:= \left(4CK_2^{-1}\lambda^n + 6K_1K_2^{-1}C_{90}S_1\lambda_6^n|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|\right)(1 + K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|), \\ Q_i &:= \left(1 + 6K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|\right)(1 + K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|). \end{aligned} \tag{3.110}$$

By iterating this inequality from $i = j$ down to $i = 0$, we obtain

$$M_{j+1} \le P_j + \sum_{k=0}^{j-1} P_k \prod_{l=k+1}^{j} Q_l + M_0 \prod_{l=0}^{j} Q_l, \tag{3.111}$$

and, thus,

$$|\mathfrak{s}_n(I_{j+1}) - 1| \le e^{8K_1K_2^{-1}}\left[4CK_2^{-1}\lambda^n(j+1) + 6K_1K_2^{-1}C_{90}S_1\lambda_6^n + \max_{\mathcal{P}_m \ni I' \subset \Delta^{(n)}_{q_{n-1}}} |\mathfrak{s}_n(I') - 1|\right]. \tag{3.112}$$

Here, we have used that $\sum_{i=0}^{j} |\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| < 1$ and the inequality $1 + x < e^x$, for $x > 0$. Using (3.112), we can prove the claim for $n = \mathfrak{n}_\ell$, and $\ell \in \mathbb{N}$ sufficiently large, corresponding to the case of renormalization graphs concave inside the gates, with our standing assumption $k_{\mathfrak{n}_\ell+1} \le C_1\lambda_1^{-\mathfrak{n}_\ell}$, and for $n \ne \mathfrak{n}_\ell$, $\ell \in \mathbb{N}$, corresponding to the case of renormalization graphs convex inside the tunnels, if $k_{n+1} \le [\lambda^{-n/8}]$ and n is sufficiently large. From (3.112), we obtain $|\mathfrak{s}_n(I_j) - 1| < C_{91}C_{89}\lambda_6^n$, for all $0 \le j \le k_{n+1}$, where $C_{91} = e^{8K_1K_2^{-1}}(4CK_2^{-1}C_1C_{89}^{-1} + 6K_1K_2^{-1}C_{90}S_1C_{89}^{-1} + 1)$. If $n \ne \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, and $k_{n+1} > [\lambda^{-n/8}]$, the same analysis leads to the bound $|\mathfrak{s}_n(I_j) - 1| < C_{91}C_{89}\lambda_6^n$, for $0 \le j \le [\lambda^{-n/8}]$. The latter bound for $j = [\lambda^{-n/8}]$ and the estimate

$$|\mathfrak{s}_n(I_j) - 1| < |\mathfrak{s}_n(I_{[\lambda^{-n/8}]}) - 1| + C_{92}C_{87}\lambda_5^n, \tag{3.113}$$

where $C_{92} > 0$, which follows from Lemma 3.30, give $|\mathfrak{s}_n(I_j) - 1| < C_{93}C_{89}\lambda_6^n$, where $C_{93} = C_{91} + C_{92}C_{87}C_{89}^{-1}$, for $[\lambda^{-n/8}] < j \le k_{n+1} - [\lambda^{-n/8}]$. Finally, by iterating the inequality (3.109) from $i = j - 1$ down to $i = k_{n+1} - [\lambda^{-n/8}]$, we obtain an estimate analogous to (3.112) and $|\mathfrak{s}_n(I_j) - 1| < C_{94}C_{89}\lambda_6^n$, where $C_{94} = e^{8K_1K_2^{-1}}(4CK_2^{-1}C_1C_{89}^{-1} + 6K_1K_2^{-1}C_{90}S_1C_{89}^{-1} + C_{93})$, for $k_{n+1} - [\lambda^{-n/8}] \le j \le k_{n+1}$. The claim is proven. **QED**

In the following, let $I' := T^{q_{n-1}}(I)$.

Lemma 3.32 *Assume that the conditions of Theorem 2.2 hold. There exists $S_3 > 1$ such that the following holds. Assume that there exists $C_{95} > 1$ such that for any $n \in \mathbb{N}$ large enough, any $m > n$, and all intervals I, with $\mathcal{P}_m \ni I \subset \Delta_0^{(n)}$ and the corresponding intervals $\widetilde{I}$, with $\widetilde{\mathcal{P}}_m \ni \widetilde{I} \subset \widetilde{\Delta}_0^{(n)}$, we have $|\mathfrak{s}_{n+1}(I) - 1| \le C_{95}\lambda_6^n$. Then, $|\mathfrak{s}_n(I') - 1| \le S_3C_{95}\lambda_6^n$.*

Proof. It follows from Lemma 3.27 that, there exists $C_{96} > 0$, such that

$$|\mathfrak{s}_{n-1}(I) - 1| = |\mathfrak{s}_{n+1}(I)\mathfrak{s}_n(\Delta_0^{(n)})\mathfrak{s}_{n-1}(\Delta_0^{(n-1)}) - 1| \leq C_{96}C_{95}\lambda_6^n. \tag{3.114}$$

Let $y, \widetilde{y}$ and $z, \widetilde{z}$ be the left (i.e., smaller) and right (i.e., larger) end points of the intervals $\tau_{n-1}(I)$ at $\widetilde{\tau}_{n-1}(\widetilde{I})$. Since (3.114) holds for all intervals I such that $\mathcal{P}_m \ni I \subset \Delta_0^{(n)}$, we have

$$\max\{|\widetilde{y} - y|, |\widetilde{z} - z|\} \leq C_{96}C_{95}\lambda_6^n. \tag{3.115}$$

Since $I' = \tau_{n-1}^{-1}(f_{n-1}(\tau_{n-1}(I)))$ and $\widetilde{I}' = \widetilde{\tau}_{n-1}^{-1}(\widetilde{f}_{n-1}(\widetilde{\tau}_{n-1}(\widetilde{I})))$, we can derive, completely analogously to (3.106) (in other words, for $I \subset \Delta_0^{(n)}$, we can apply (3.106) with $n-1$ instead of n and k_n instead of i),

$$\begin{aligned}|\mathfrak{s}_{n-1}(I') - 1| \leq \Bigg[&2CK_2^{-1}\lambda^{n-1}\left(1 + \frac{z-y}{4}\right) + 6K_1K_2^{-1}\max\{|\widetilde{y} - y|, |\widetilde{z} - z|\} \\ &+ |\mathfrak{s}_{n-1}(I) - 1|\Bigg](1 + K_1K_2^{-1}|z - y|).\end{aligned} \tag{3.116}$$

Taking into account that $|z - y| < 1$, this inequality, together with (3.114) and (3.115), gives that, for some $C_{97} > 0$,

$$|\mathfrak{s}_{n-1}(I') - 1| \leq C_{97}C_{95}\lambda_6^n. \tag{3.117}$$

Since $\mathfrak{s}_n(I') = \mathfrak{s}_{n-1}(I')/\mathfrak{s}_{n-1}(\Delta_0^{(n-1)})$, together with Lemma 3.27, this proves the claim. **QED**

Proposition 3.33 *Assume that the conditions of Theorem 2.2 hold. For every* $\lambda_7 \in (\lambda_6, 1)$*, there exists* $\nu > 0$ *and* $C_{98} > 0$*, such that*

$$|\sigma(I) - 1| \leq C_{98}\lambda_7^m, \tag{3.118}$$

for all $I \in \mathcal{P}_m$*, such that* $I \subset \bar{\Delta}_0^{(m-[\nu m])}$*, and all* $m \in \mathbb{N}_0$*.*

Proof. It suffices to prove the claim for sufficiently large m. It follows from Proposition 3.2, Proposition 3.26 and Lemma 3.27 that there exists $C_{99} > 1$ such that, for all intervals $I \in \mathcal{P}_m$ such that $I \subset \bar{\Delta}_0^{(m-2)}$, we have

$$|\mathfrak{s}_{m-1}(I) - 1| \leq C_{99}\lambda_6^m. \tag{3.119}$$

There exist $S_4 > 1$ and $n_3 \in \mathbb{N}$, such that, if all intervals $I \in \mathcal{P}_m$, with $I \subset \bar{\Delta}_0^{(n)}$, where $n_3 \leq n \leq m-2$, satisfy $|\mathfrak{s}_{n+1}(I) - 1| \leq C_{100}\lambda_6^n$, for some $C_{100} > 1$, then all intervals

$I'' \in \mathcal{P}_m$, such that $I'' \subset \bar{\Delta}_0^{(n-1)}$, satisfy $|\mathfrak{s}_n(I'') - 1| \leq S_4 C_{100} \lambda_6^n$. To see this, notice that every interval $I'' \in \mathcal{P}_m$, such that $I'' \subset \bar{\Delta}_0^{(n-1)}$, is either a subset of $\check{\Delta}_0^{(n-1)}$ or a subset of $\bar{\Delta}_0^{(n)}$. If $I'' \subset \check{\Delta}_0^{(n-1)}$ then there exist i, such that $0 \leq i < k_{n+1}$, and intervals $I \subset \Delta_0^{(n)}$ and $I' \subset \Delta_{q_{n-1}}^{(n)}$, such that $I'' = T^{iq_n}(I')$ and $I' = T^{q_{n-1}}(I)$, and the claim follows, with $S_4 \geq S_2 S_3$, if n_3 is large enough, by applying Lemma 3.32 and Proposition 3.31. If $I'' \subset \bar{\Delta}_0^{(n)}$, since $\mathfrak{s}_n(I'') = \mathfrak{s}_{n+1}(I'')\mathfrak{s}_n(\Delta_0^{(n)})$, using Lemma 3.27, we have

$$|\mathfrak{s}_n(I'') - 1| \leq |\mathfrak{s}_{n+1}(I'') - 1| + \mathfrak{s}_{n+1}(I'')|\mathfrak{s}_n(\Delta_0^{(n)}) - 1|, \tag{3.120}$$

and the claim follows with $S_4 \geq 1 + 2C_{82}$.

Applying this spreading of estimates from the intervals of partition $\mathcal{P}_m$ inside $\bar{\Delta}_0^{(n)}$ to intervals of partition $\mathcal{P}_m$ inside $\bar{\Delta}_0^{(n-1)}$ recursively, from $n = m - 2$ down to $n = n' \geq n_3$, and using estimate (3.119), for all $I \in \mathcal{P}_m$ such that $I \subset \bar{\Delta}_0^{(n')}$, we have

$$|\mathfrak{s}_{n'+1}(I) - 1| \leq S_4^{m-n'} C_{99} \lambda_6^{n'}. \tag{3.121}$$

The claim follows from the latter inequality, after we set $n' = m - [\nu m]$, choose $\nu > 0$ small enough such that $S_4^{\nu} \lambda_6^{1-\nu} \leq \lambda_7$, and rescale the intervals, using $\sigma(I) = \mathfrak{s}_{n+1}(I)\sigma(\Delta_0^{(n)})$ and (3.91) with $\sigma_\infty = 1$. **QED**

At the end of this section, let us summarize the relations among various rates λ_i that are used in this paper. λ_1^{-1} is the maximal rate of the exponential growth of the partial quotients. λ_1 can be chosen to be an arbitrary number in $(\lambda, 1)$, where λ is the minimal rate of convergence of renormalizations. The established rate of convergence of ratios of lengths of the renormalized intervals $\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)})$, for $n = \mathfrak{n}_\ell$, for some $\ell \in \mathbb{N}$, is $\lambda_2 \in (\sqrt{\lambda/\lambda_1}, 1)$. To prove this, we use a natural separation of length scales given by the exponential rate $\lambda_3 \in (\lambda/\lambda_2, \lambda_1\lambda_2)$. The distance between the corresponding end points of the intervals $\tau_n(\Delta_{q_{n-1}+jq_n}^{(n)})$ and $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+jq_n}^{(n)})$ is controlled by the rate $\lambda_4 = \lambda_2\lambda_3$. The established rate of convergence of ratios $\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)})$, for $n \neq \mathfrak{n}_\ell$, for any $\ell \in \mathbb{N}$, is $\lambda_5 = \lambda^{\frac{(1+\alpha)\alpha}{8(2+\alpha)}}$. The established rate of convergence of these ratios, for any $n \in \mathbb{N}$, is therefore $\lambda_6 = \max\{\lambda_2, \lambda_5\}$. Any $\lambda_7 \in (\lambda_6, 1)$ is the established rate of convergence of ratios $\sigma(I)$ for intervals of partition $I \in \mathcal{P}_n$ inside $\bar{\Delta}_0^{(n-[\nu n])}$, for some $\nu \in (0, 1)$. Finally, as we will see in the next section, the rate $\bar{\lambda}$ in Proposition 3.1 can be taken to be $\bar{\lambda} = \max\{\lambda_7, \lambda_{\text{ref}}^{\nu}\}$.

3.6 Spreading the estimates to the whole circle

Proof of Theorem 2.2. In order to prove the claim, we will use Proposition 3.1. To verify the assumptions of Proposition 3.1, we need to verify the estimates (3.2) for all intervals $I, I' \subset \mathcal{P}_m$ which are either adjacent or belong to the same element of partition $\mathcal{P}_{m-1}$. Proposition 3.33 implies the estimate

$$|\ln\sigma(I) - \ln\sigma(I')| \le C_{101}\lambda_7^m, \tag{3.122}$$

where $C_{101} > 0$, for all pairs of such intervals I, I' which are both contained in $\bar{\Delta}_0^{(m-[\nu m])}$. We will now spread such an estimate from $\bar{\Delta}_0^{(n)}$ to $\bar{\Delta}_0^{(n-1)}$ in $m-[\nu m]$ steps, starting with $n = m-[\nu m]$, and counting down to $n = 0$. In each step, the new intervals for which we need to show such an estimate appear in threads $I_i = T^{iq_n}(I_0)$ and $I'_i = T^{iq_n}(I'_0)$, for $0 \le i < k_{n+1}$. Let us fix the order of the pairs in such a way that I'_0 is closer to x_0 than I_0. This implies that $I_0 \subset T^{q_{n-1}}(\Delta_0^{(n)})$ and that either I'_0 belongs to $T^{q_{n-1}}(\Delta_0^{(n)})$ as well or is adjacent to it.

We will now show that there exists $C_{102} > 0$ such that, for any two intervals $I_0, I'_0 \in \mathcal{P}_m$, with $I_0 \subset T^{q_{n-1}}(\Delta_0^{(n)})$, $n \in \mathbb{N}_0$, $m > n$, that are either adjacent to each other or belong to the same element of partition $\mathcal{P}_{m-1}$, and for all $0 \le j < k_{n+1}$, we have

$$|\ln\sigma(I_j) - \ln\sigma(I'_j)| \le |\ln\sigma(I_{k_{n+1}}) - \ln\sigma(I'_{k_{n+1}})| + C_{102}\lambda_{\mathrm{ref}}^{m-n}. \tag{3.123}$$

Let

$$\delta_i := |\ln|\tau_n(I_{i+1})| - \ln|\tau_n(I_i)| - \ln|\tau_n(I'_{i+1})| + \ln|\tau_n(I'_i)||, \tag{3.124}$$

and let $\widetilde{\delta}_i$ be the corresponding quantity associated to $\widetilde{T}$.

Clearly, there exist $\bar{\zeta}_i \in \tau_n(I_i)$, $\bar{\zeta}'_i \in \tau_n(I'_i)$ and $\zeta_i \in (\bar{\zeta}_i, \bar{\zeta}'_i)$, such that

$$\delta_i = \left|\frac{f''_n(\zeta_i)}{f'_n(\zeta_i)}\right| |\bar{\zeta}'_i - \bar{\zeta}_i|. \tag{3.125}$$

If I_0 and I'_0 belong to the same element J_0 of $\mathcal{P}_{m-1}$, then there is a thread $J_i = T^{iq_n}(J_0) \in \mathcal{P}_{m-1}$, with $0 \le i < k_{n+1}$, such that $I_i \cup I'_i \subset J_i \subset T^{q_{n-1}+iq_n}(\Delta_0^{(n)})$. Using the estimate (3.125), regularity conditions (i) and (ii), and condition (c), we find that $\delta_i \le K_1K_2^{-1}|\tau_n(J_i)| \le K_1K_2^{-1}C_{\mathrm{ref}}\lambda_{\mathrm{ref}}^{m-n-2}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|$. Since $\sum_{i=j}^{k_{n+1}-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| < 1$, the bound (3.123) follows by summing up the inequalities

$$|\ln\mathfrak{s}_n(I_i) - \ln\mathfrak{s}_n(I'_i)| - |\ln\mathfrak{s}_n(I_{i+1}) - \ln\mathfrak{s}_n(I'_{i+1})| \le \delta_i + \widetilde{\delta}_i, \tag{3.126}$$

from $i = j$ to $i = k_{n+1}-1$, we obtain

$$|\ln\mathfrak{s}_n(I_j) - \ln\mathfrak{s}_n(I'_j)| \le |\ln\mathfrak{s}_n(I_{k_{n+1}}) - \ln\mathfrak{s}_n(I'_{k_{n+1}})| + C_{102}\lambda_{\mathrm{ref}}^{m-n}, \tag{3.127}$$

where $C_{102} > 0$. If I_0 and I'_0 are adjacent to each other, belong to different elements of $\mathcal{P}_{m-1}$, but $I_0, I'_0 \subset \Delta^{(n)}_{q_{n-1}}$, then we similarly have $\delta_i \le 2K_1K_2^{-1}C_{\mathrm{ref}}\lambda_{\mathrm{ref}}^{m-n-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|$, and the bound (3.127) follows, using the same estimates. If $I_0 \subset \Delta^{(n)}_{q_{n-1}}$ and $I'_0 \subset \Delta^{(n)}_{q_{n-1}+q_n}$, then we have $\delta_i \le K_1 K_2^{-1} C_{\rm ref}\lambda_{\rm ref}^{m-n-1}(|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| + |\tau_n(\Delta^{(n)}_{q_{n-1}+(i+1)q_n})|)$, for $0 \le i < k_{n+1}-1$ and $\delta_{k_{n+1}-1} \le K_1 K_2^{-1} C_{\rm ref}\lambda_{\rm ref}^{m-n-1}(|\tau_n(\Delta^{(n)}_{q_{n-1}+(k_{n+1}-1)q_n})|+|\tau_n(\Delta_0^{(n+1)})|)$. In the last estimate, we have used that $\mathcal{P}_m \ni I'_{k_{n+1}-1} \subset \Delta_0^{(n+1)} \in \mathcal{P}_{n+1}$. Since $\sum_{i=j+1}^{k_{n+1}-1} |\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})| + |\tau_n(\Delta_0^{(n+1)})| < 1$, we obtain again estimate (3.127).

Inequality (3.123) follows directly from (3.127), taking into account that $\sigma(I) = \mathfrak{s}_n(I)\sigma(\Delta_0^{(n-1)})$.

Applying (3.123) recursively, from $n = m - [\nu m] - 1$ to $n = 0$, and using the estimate (3.122) for intervals of partition $\mathcal{P}_m$ inside $\bar{\Delta}_0^{(m-[\nu m])}$, we obtain

$$|\ln\sigma(I) - \ln\sigma(I')| \le C_{101}\lambda_7^m + C_{102}\sum_{n=0}^{m-[\nu m]-1} \lambda_{\rm ref}^{m-n} \le C_{103}(\lambda_7^m + \lambda_{\rm ref}^{\nu m}), \tag{3.128}$$

where $C_{103} > 0$, for all pairs of $I, I' \in \mathcal{P}_m$, as in Proposition 3.1. Hence, (3.2) holds with $\bar{\lambda} = \max\{\lambda_7, \lambda_{\rm ref}^{\nu}\}$, and Theorem 2.2 is proven. **QED**

4 Proof of the main theorem

In the proof of Theorem 1.2, we will use the following properties of renormalizations of circle diffeomorphisms with a break that were proven in [10]. Let T be a $C^{2+\alpha}$-smooth circle diffeomorphisms with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ at $x_{\rm br} = x_0$ and an irrational rotation number $\rho \in (0, 1)$.

Lemma 4.1 ([10]) *For sufficiently large $n \in \mathbb{N}$, f''_n is uniformly bounded away from zero and positive if $c^{(n)} > 1$ and negative if $c^{(n)} < 1$.*

We can now show that the renormalizations of circle diffeomorphisms with a break are K-regular.

Lemma 4.2 *There exists $K \in \mathbb{R}^6_+$ such that the sequence of its renormalization $(f_n)_{n\in\mathbb{N}_0}$ is K-regular with respect to the sequence $\mathfrak{n}$ consisting of all odd numbers in $\mathbb{N}_0$, if $c > 1$, and all even numbers in $\mathbb{N}_0$, if $0 < c < 1$.*

Proof. The regularity condition (i) holds, with large enough $K_1 > 0$, by the Denjoy estimate (A) (which implies that the derivative of renormalizations $f'_n(z) = (T^{q_n})'(\tau_n^{-1}(z))$ is uniformly bounded) and the fact that the second derivative of renormalizations f''_n is uniformly bounded. This fact follows from estimate (B) and the explicit form of the second derivative

$$F''_{a^{(n)},b^{(n)},M^{(n)},c^{(n)}}(z) = 2(M^{(n)} - 1)\frac{(a^{(n)} + b^{(n)})M^{(n)}}{(1 - (M^{(n)} - 1)z)^3}, \tag{4.1}$$

taking into account that $a^{(n)} = |\Delta_0^{(n)}|/|\Delta_0^{(n-1)}| \leq |T^{q_n}(\Delta_0^{(n-1)})|/|\Delta_0^{(n-1)}| \leq e^V$ (due to (A)), $b^{(n)} < 1$ and that

$$M^{(n)} = \exp\left((-1)^n \sum_{i=0}^{q_n-1} \int_{\Delta_i^{(n-1)}} \frac{T''(z)}{2T'(z)}\, dz \right) \tag{4.2}$$

is bounded and bounded away from zero, as can be easily seen from the fact that T'' is bounded and T' is bounded from below by a positive constant.

The regularity condition (ii) holds, if $K_2 > 0$ is small enough, by the Denjoy estimate (A).

Lemma 4.1 implies that, for sufficiently large $n \in \mathbb{N}$, there can be at most one point $\zeta_n^* \in [-1, 0]$, such that $f_n'(\zeta_n^*) = 1$. Due to the continuity of f_n', $f_n(z) - z$ is monotone in each of the intervals $[-1, \zeta_n^*]$ and $[\zeta_n^*, 0]$. If n is odd and $c > 1$, or n is even and $c < 1$, then $c^{(n)} < 1$ and, by Lemma 4.1, $f_n''(z) < -K_4$, for some $K_4 > 0$, all $z \in [-1, 0]$, and sufficiently large $n \in \mathbb{N}$. Thus, we either have $f_n'(z) > 1$, for $z \in [-1, -\frac{1}{2})$ or $f_n'(z) < 1$, for $z \in (-\frac{1}{2}, 0]$. In either case, $f_n(-\frac{1}{2}) + \frac{1}{2}$ can be, uniformly in n, bounded from below by a positive constant. Since,

$$f_n(z) = f_n\left(-\frac{1}{2}\right) + \int_{-1/2}^{z} \left(f_n'(-1/2) + \int_{-1/2}^{\zeta} f_n''(\zeta')\, d\zeta' \right) d\zeta, \tag{4.3}$$

if $f_n'(z) < 1$, for $z \in (-\frac{1}{2}, 0]$, we obtain $f_n(-\frac{1}{2}) + \frac{1}{2} \geq f_n(0) + \frac{K_4}{8} \geq \frac{K_4}{8}$. In the other case, the proof is similar. This ensures that point $-\frac{1}{2}$ does not belong to any of the gates and the regularity condition (iii) holds, if $K_3 > 0$ is small enough.

The regularity conditions (iv) follows immediately from Lemma 4.1, for small enough $K_3 > 0$ and $K_4 > 0$. Notice that by choosing $K_3 > 0$ small enough, we can always achieve that, finitely many renormalizations f_n, for which Lemma 4.1 does not guarantee concavity, have no gates, i.e. satisfy $f_n(z) - z \geq K_3$, for all $z \in [-1, 0]$.

If n is even and $c > 1$, or n is odd and $c < 1$, then $c^{(n)} > 1$ and, by Lemma 4.1, $f_n''(z) > K_6$, for some $K_6 > 0$, and sufficiently large $n \in \mathbb{N}$. Together with the fact that, by the Denjoy estimate (A), $f_n(-1) + 1 = |\tau_n(\Delta_{q_{n-1}}^{(n)})|$ is of the same order as $f_n(0) = |\tau_n(\Delta_0^{(n)})|$, this implies that the renormalizations f_n satisfy the regularity condition (v) as well, if $K_5 > 0$ is small enough. To see this, notice that, if $c^{(n)} > 1$, we either have $f_n'(z) < 1$, for $z \in [-1, -\frac{1}{2})$ or $f_n'(z) > 1$, for $z \in (-\frac{1}{2}, 0]$. In either case, both $f_n(-1) + 1$ and $f_n(0)$ are, uniformly in n, bounded away from zero. It follows from (4.3) that in the second case, $f_n(0) > \frac{K_6}{8}$; in the other case, the proof is similar. We can choose the constant $K_5 > 0$ small enough such that $f_n(-1) \geq K_5 - 1$ and $f_n(0) \geq K_5$, for sufficiently large n.

The regularity condition (vi) follows immediately, for small enough $K_5 > 0$ and $K_6 > 0$, from Lemma 4.1. We choose $K_5 > 0$ small enough such that finitely many renormalizations f_n, for which Lemma 4.1 does not guarantee convexity, have no tunnels. **QED**

Proof of Theorem 1.2. To prove Theorem 1.2, we need to verify that the conditions of Theorem 2.2 hold true in the case of circle diffeomorphisms with a break point. Condition (a) is an assumption of Theorem 1.2. To verify condition (b), we will show that the renormalization sequences f_n and $\widetilde{f}_n$ of $C^{2+\alpha}$-smooth circle diffeomorphisms with breaks T and $\widetilde{T}$ are K-regular, for some $K \in \mathbb{R}^6_+$, with respect to the sequence $\mathfrak{n}$ consisting of all odd numbers in $\mathbb{N}_0$, if $c > 1$, and all even numbers in $\mathbb{N}_0$, if $0 < c < 1$. This follows from Lemma 4.2. Condition (c) follows from the Denjoy estimate (A) (see Lemma 2 in [18]). Condition (d) follows from Theorem 1.1, proven in [10]. QED

Acknowledgements

We would like to thank the referee for the careful reading of the manuscript and numerous suggestions.

References

[1] V.I. Arnol'd, Small denominators I: On the mapping of a circle into itself, *Izv. Akad. Nauk. Math. Serie* **25** (1961), 21–86; *Transl. A.M.S. Serie 2* **46** (1965).

[2] K. Cunha and D. Smania, Renormalization for piecewise smooth homeomorphisms on the circle, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** 3 (2013), 441–462.

[3] K. Cunha and D. Smania, Rigidity for piecewise smooth homeomorphisms on the circle, *Adv. Math.* **250** (2014), 193–226.

[4] E. de Faria, W. de Melo, Rigidity of critical circle maps I, *J. Eur. Math. Soc.* **1** 4 (1999), 339–392.

[5] E. de Faria, W. de Melo, Rigidity of critical circle maps II, *J. Am. Math. Soc.* **13** 2 (2000), 343–370.

[6] M.R. Herman, Sur la conjugasion différentiable des difféomorphismes du cercle a des rotations, *Publ. Math. Inst. Hautes Etudes Sci.* **49** (1979), 5–234.

[7] Y. Katznelson, D. Orstein, The differentiability of conjugation of certain diffeomorphisms of the circle, *Ergod. Th. & Dynam. Sys.* **9** (1989), 643–680.

[8] K. Khanin, D. Khmelev, Renormalizations and rigidity theory for circle homeomorphisms with singularities of break type, *Commun. Math. Phys.* **235** 1 (2003), 69–124.

[9] K. Khanin, S. Kocić, Abscence of robust rigidity for circle diffeomorphisms with breaks, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30** 3 (2013), 385–399.

[10] K. Khanin, S. Kocić, Renormalization conjecture and rigidity theory for circle diffeomorphisms with breaks, *Geom. Funct. Anal.* **24** 6 (2014), 2002–2028.

[11] K.M. Khanin, Ya.G. Sinai, A new proof of M. Herman's theorem, *Commun. Math. Phys.* **112** 1 (1987), 89–101.

[12] K. Khanin, A. Teplinsky, Robust rigidity for circle diffeomorphisms with singularities, *Invent. Math.* **169** (2007), 193–218.

[13] K. Khanin, A. Teplinsky, Herman's theory revisited, *Invent. Math.* **178** (2009), 333–344.

[14] K. Khanin, A. Teplinsky, Renormalization horseshoe and rigidity theory for circle diffeomorphisms with breaks, *Comm. Math. Phys.* **320** 2 (2013), 347–377.

[15] K.M. Khanin, E.B. Vul, Circle Homeomorphisms with Weak Discontinuities, *Adv. Sov. Math.* **3** (1991), 57–98.

[16] S. Kocić, Generic rigidity for circle diffeomorphisms with breaks, *Commun. Math. Phys.* **344** 2 (2016), 427–445.

[17] S. Marmi, P. Moussa, J.-C. Yoccoz, Linearization of generalized interval exchange maps, *Ann. Math* **176** 3 (2012), 1583–1646.

[18] Ya. G. Sinai, K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44** 1 (1989), 57–82.

[19] M. Yampolsky, Hyperbolicity of renormalization of critical circle maps, *Publ. Math. Inst. Hautes Etudes Sci.* **96** (2002), 1–41.

[20] J.-C. Yoccoz, Conjugaison differentiable des difféomorphismes du cercle donc le nombre de rotation vérifie une condition Diophantienne, *Ann. Sci. École Norm. Sup.* **17** (1984), 333–361.