Renormalization conjecture and rigidity theory for circle diffeomorphisms with breaks

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August 24, 2014

Abstract

We prove the renormalization conjecture for circle diffeomorphisms with breaks, i.e., that the renormalizations of any two $C^{2+\alpha}$-smooth ($\alpha \in (0,1)$) circle diffeomorphisms with a break point, with the same irrational rotation number and the same size of the break, approach each other exponentially fast in the C^2-topology. As was shown in [18], this result implies the following strong rigidity statement: for almost all irrational numbers ρ, any two circle diffeomorphisms with a break, with the same rotation number ρ and the same size of the break, are C^1-smoothly conjugate to each other. As we proved in [17], the latter claim cannot be extended to all irrational rotation numbers. These results can be considered an extension of Herman's theory on the linearization of circle diffeomorphisms.

1 Introduction and statement of the results

Rigidity theory for circle diffeomorphisms is the subject of the classical theory of Herman [14], further developed by Yoccoz [38]. It states that any sufficiently smooth circle diffeomorphism with a Diophantine rotation number ρ is smoothly conjugate to the rigid rotation $R_\rho : x \mapsto x+\rho \pmod 1$, i.e., there is a smooth circle homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ such that $T \circ \varphi = \varphi \circ R_\rho$. The crucial step in the whole theory is to establish C^1 smoothness of φ, from which one can derive higher smoothness results using Hadamard convexity

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inequalities and bootstrap techniques. It turns out that for almost all Diophantine rotation numbers, the conjugacy is arbitrarily smooth if the diffeomorphism is smooth enough. On the other hand, Arnold showed that the conjugacy can be not even absolutely continuous for some Liouville (non-Diophantine) irrational rotation numbers, even in the case of analytic circle diffeomorphisms. A natural approach to Herman's theory is based on renormalization [34]. In this approach, the rigidity statement can be obtained using the convergence of renormalizations of sufficiently smooth circle diffeomorphisms with the same irrational rotation number. In fact, the renormalizations of circle diffeomorphisms approach a family of linear maps with derivative equal to 1. Such a convergence implies rigidity and smooth linearization if the rotation number satisfies a Diophantine condition.

This paper presents the renormalization and rigidity theory for circle diffeomorphisms with a single singular point where the derivative has a jump discontinuity. We call such maps *circle diffeomorphisms with breaks* and these points the *break points*. Our main result is the theorem on the exponential convergence of renormalizations for circle maps with breaks provided that they have the same irrational rotation number and the same *size of the break*, i.e., the square root of the ratio of the left and right derivatives of a map at the break point. More precisely, we prove the following.

Theorem 1.1 *Let $\alpha \in (0,1)$ and let $c \in \mathbb{R}^+\backslash\{1\}$. There exists $\mu \in (0,1)$, such that for every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same irrational rotation number $\rho \in (0,1)$, and the same size of the break c, there exists $C > 0$, such that the renormalizations f_n and $\widetilde{f}_n$ of T and $\widetilde{T}$, respectively, satisfy $\|f_n - \widetilde{f}_n\|_{C^2} \leq C\mu^n$, for all $n \in \mathbb{N}$.*

We emphasize that the convergence result holds for all irrational rotation numbers and that the exponential rate of convergence is *universal*, i.e., it is independent of the maps and, in particular, their rotation numbers, as long as the size of their breaks is the same. As we have previously shown in [18], Theorem 1.1 implies a strong rigidity statement for such maps.

Theorem 1.2 *For almost all irrational $\rho \in (0,1)$, any two $C^{2+\alpha}$-smooth circle maps with a break T and $\widetilde{T}$, with the same rotation number ρ, and the same size of the break c, are C^1-smoothly conjugate to each other.*

We have earlier proved in [17], answering a question of [16] (question II therein), that this claim cannot be extended to all irrational rotation numbers.

To explain how maps with breaks appear naturally in the context of one-dimensional dynamics, we start with a rigid rotation by an angle ρ on a unit circle $\mathbb{T}^1$, i.e., a linear map R_ρ. It is well known that such a map can be regarded as an interval exchange transformation of two intervals. While the intervals are transformed by isometries, in the case of an interval exchange transformation, it is quite natural to consider nonlinear interval exchange transformations, where the maps acting on the intervals are assumed to be smooth and strictly monotone. Such generalized interval exchange transformations were recently introduced by Marmi, Moussa and Yoccoz [30]. In the case of two intervals there are just two branches $h_1 : [0, \xi] \to [h_1(0), 1]$ and $h_2 : [\xi, 1] \to [0, h_2(1)]$, $\xi \in (0, 1)$, which satisfy a matching condition $h_1(0) = h_2(1) \in (0, 1)$. By matching the derivatives $(h_1)'_+(0) = (h_2)'_-(1)$, $(h_1)'_-(\xi) = (h_2)'_+(\xi)$, we obtain a circle diffeomorphism T with a lift whose restrictions to $[0, \xi]$ and $[\xi, 1]$ are given by h_1 and $h_2 + 1$, respectively. Such a matching condition is rather artificial in the setting of nonlinear interval exchange transformations. Without the derivative matching one obtains a circle diffeomorphism T with two break points at $x_{br}^{(1)} = 0$ and $x_{br}^{(2)} = \xi$. Since the two break points belong to the same orbit of T, i.e., $Tx_{br}^{(2)} = x_{br}^{(1)}$, one can piecewise-smoothly conjugate T to a circle map with a single break point. A natural question to ask is when two maps of this type are smoothly, or piecewise-smoothly, conjugate to each other. The main and only missing piece in answering the latter question has been settled by the theory presented in this paper. In a sense, this theory is a one-parameter extension of Herman's theory where the break size c plays the role of the parameter. While in the case of circle diffeomorphisms, corresponding to $c = 1$, renormalizations converge to a one-dimensional space of linear maps with derivative 1, in the case $c \neq 1$, the renormalizations converge to a two-dimensional space of fractional linear transformations with very non-trivial dynamics on the limiting attractor [21]. Theorem 1.1 and Theorem 1.2 correspond to the non-linearizable case of nonlinear interval exchange transformations of two intervals. The linearizable case of more general interval exchange transformations has recently been considered by Marmi, Moussa and Yoccoz in [30]. The special case of cyclic permutations, which corresponds to circle maps with more than one point of break and with the product of the sizes of breaks being equal to 1, was considered by Cunha and Smania [5]. In their case, the renormalizations converge to piecewise-affine (linear) maps, rather than fractional linear ones. Theorem 1.1 and Theorem 1.2 are, so far, the only results in the non-affine case, for generic rotation numbers.

In circle dynamics, the behavior of renormalizations plays a crucial role in proving global rigidity results. *Rigidity*, in this context, is the phenomenon of smooth conjugacy between any two maps within a given topological equivalence class. For sufficiently smooth circle diffeomorphisms, the topological equivalence classes are defined uniquely by the irrational *rotation numbers*. Denjoy proved [8] that every C^2-smooth (or even C^1-smooth with a derivative of bounded variation) circle diffeomorphism with an irrational rotation number ρ is topologically conjugate to the rigid rotation R_ρ. Almost 30 years later, Arnold proved [2] that every analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rigid rotation R_ρ, is analytically conjugate to it. This local linearization result is essentially a KAM (Kolmogorov-Arnold-Moser) type problem, and Arnold proved this result using the perturbative tools of KAM theory [3,27,32]. In one-dimensional setting, however, one can expect stronger rigidity results, and Arnold conjectured that the claim holds true if the assumption of closeness to the rotation is removed. This *global* rigidity result was proved by Herman [14], and improved by Yoccoz [38]. To make a precise statement, we define the Diophantine class $\mathcal{D}(\mathfrak{b})$, for some $\mathfrak{b} \geq 0$, as the set of all irrational $\rho \in (0,1)$, for which there exists $\mathfrak{C} > 0$ such that $|\rho - p/q| > \mathfrak{C}/q^{2+\mathfrak{b}}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ (ρ is Diophantine if it belongs to a class $\mathcal{D}(\mathfrak{b})$ for some $\mathfrak{b} \geq 0$). For $C^{2+\alpha}$-smooth circle diffeomorphisms, $\alpha \in (0,1)$, and Diophantine rotation numbers of class $\mathcal{D}(\mathfrak{b})$, the conjugacy is $C^{1+\alpha-\mathfrak{b}}$-smooth, provided $\alpha > \mathfrak{b}$ [15,20]. An important step in the proof of this result is the proof of C^1-smoothness of conjugacy, which follows from the exponential convergence of renormalizations of any two $C^{2+\alpha}$-smooth circle diffeomorphisms with the same irrational rotation number [34].

The action of the renormalization operator can be naturally extended to a larger class, involving not only circle diffeomorphisms but also circle diffeomorphisms with a singular point where the derivative vanishes (*critical circle maps*) or has jump discontinuity. It has been expected that the renormalizations of any two $C^{2+\alpha}$-smooth circle maps in these classes, with the same irrational rotation number and the same type of singularity, approach each other exponentially fast in the C^2-topology. The type of singularity is characterized by the order of the critical point $\beta > 1$ (the derivative of the map T near the critical point x_c behaves as $|x - x_c|^{\beta-1}$), in the case of critical circle maps, and by the size of the break $c \neq 1$, in the case of circle maps with a break. In the case of non-analytic critical circle maps, this conjecture is still open. For analytic critical circle maps the conjecture is true, as was proved by de Faria and de Melo [12] and Yampolsky [37]. The methods they used to prove this result are based on holomorphic dynamics and cannot be extended to the non-analytic case. It is expected, however, that the renormalization conjecture is extremely general and holds for all orders $\beta > 1$ of the critical point. It is interesting to mention that, contrary to the case of circle diffeomorphisms, for critical circle maps, C^1-rigidity should holds without additional Diophantine-type conditions on the rotation numbers. For $C^{2+\alpha}$-smooth critical circle maps, it was shown by Khanin and Teplinsky [19] that a proof of the renormalization conjecture would imply *robust rigidity*, i.e., C^1-rigidity for all irrational rotation numbers. De Faria and de Melo [11] previously proved that the proof of the renormalization conjecture would imply even stronger $C^{1+\epsilon}$-rigidity of C^3-smooth critical circle maps, for some $\epsilon > 0$, for almost all irrational rotation numbers. They also proved that this statement cannot be extended to all irrational rotation numbers.

In this paper, we prove the renormalization conjecture for circle maps with a break. It has been known for more than two decades that the renormalizations of circle maps with a break approach a family of fractional linear transformations [22]. This makes certain aspects of the renormalization analysis of maps with breaks simpler than in the critical case. On the other hand, circle maps with breaks are characterized by strongly *unbounded geometry*, i.e., the ratio of lengths of nearby elements of dynamical partitions can be arbitrarily small (exponentially small in the corresponding partial quotient k_{n+1} of the rotation number). This makes other aspects of the renormalization analysis of these maps significantly more difficult than in the case of critical circle maps, for which the geometry is bounded. Previous rigidity results for maps with breaks [16, 21], beyond the family of fractional linear transformations, have been restricted to a small (zero measure) set of irrational rotation numbers for which one has *bounded geometry*, i.e., when the neighboring intervals of dynamical partitions are of comparable size. For circle maps with a break, the intervals of the dynamical partitions of the circle can decrease at an arbitrary rate. This is essentially a new phenomenon in the renormalization theory. It creates major difficulties that we overcome in this paper. To deal with this problem, we introduce a new notion of renormalization strings and analyze their asymptotic properties. This allows us to prove the final result: Any two sufficiently smooth circle maps with a break, with the same irrational rotation number and the same size of the break, belong to the same universality class (i.e., their renormalizations approach each other). The decomposition of the sequence of renormalizations into strings and the exploitation of two different mechanisms of contraction play the key role in our proof.

In spite of full universality, the unbounded geometry prevents robust rigidity in the case of circle maps with a break. The same is true in the case of circle diffeomorphisms, for which the geometry is much less unbounded (the ratio of the lengths of neighboring intervals of the n-th dynamical partition is at most of the order of k_{n+1}). In fact, if the lengths of the smallest elements of the dynamical partitions decrease sufficiently rapidly, one can even find examples of analytic circle diffeomorphisms and analytic (outside the break point) circle diffeomorphisms with breaks of the same size, which are topologically but not C^1-smoothly conjugate to each other. In [17], we constructed such examples of analytic circle maps with breaks of the same size, in the same topological conjugacy class, for which no conjugacy is even Lipschitz continuous. Nevertheless, Theorem 1.1 implies C^1-rigidity for almost all irrational rotation numbers (Theorem 1.2). For those rotation numbers, the geometry is super-exponentially bounded, i.e., the logarithms of the ratios of the nearby elements of dynamical partitions are bounded by an exponential function of the renormalization step [18]. An explicit condition under which the rigidity was established in [18] is the exponential bound $k_{n+1} \leq C_0\lambda_0^{-n}$, for some $C_0 > 0$ and $\lambda_0 \in (\mu, 1)$ (with μ as in Theorem 1.1), on the growth of the partial quotients k_{n+1} of the rotation numbers for the subsequence of odd n if the break size $c < 1$ or the subsequence of even n if $c > 1$. This condition is different from the Diophantine condition on rotation numbers that guarantees C^1-rigidity of circle diffeomorphisms for almost all irrational rotation numbers.

At the end of this introduction, let us place our results in a larger context of renormalization in dynamics. The idea of renormalization originated in quantum field theory and is due to Stueckelberg and Petermann [35]. In statistical mechanics, renormalization methods provided an explanation for critical phenomena, by classifying systems into different universality classes, according to their scaling limits and corresponding critical exponents. The renormalization methods in dynamics were introduced by Feigenbaum [9, 10] and Coullet and Tresser [7], to explain the metric universality of period-doubling bifurcations in one-parameter families of one-dimensional maps. The universal properties of the dynamics of one-dimensional systems can be understood by studying an associated infinite-dimensional dynamical system: a *renormalization operator* $\mathcal{R}$ acting on a functional space of the original systems. Typically, the action of the renormalization operator separates the systems into different *universality classes*, according to their approach to different attractors. As shown by continuous efforts of Sullivan [36], McMullen [31] and Lyubich [28], the Feigenbaum-Coullet-Tresser universality follows from the existence of a hyperbolic fixed point on a space of such maps, with one unstable direction. The theory was extended to infinitely renormalizable unimodal maps of other combinatorial types. In addition to providing the proof for the universality of infinitely renormalizable unimodal maps, and applications to rigidity theory of circle maps discussed above [5, 6, 16–22, 34], renormalization also led to advances in several other areas of dynamics including complex dynamics [31, 36], KAM theory [13, 23–25], the break-up of invariant tori [1, 29], and the reducibility of cocycles and skew-product flows [4, 26]. This list of topics and references is by no means complete.

The paper is organized as follows. In Section 2, we define the renormalizations of circle maps and provide the basic definitions and earlier results that we use. In Section 3, we prove general estimates of the renormalization parameters, including the parameter a_n (a ratio of the lengths of successive renormalization segments). In Section 4, we define the strings of renormalizations with large a_n tails, and obtain a result on the closeness of renormalizations in the tail to fractional linear maps with the same (associated) rotation number. In Section 5, we show that renormalizations with small parameters a_n are also close to fractional linear maps with the same rotation number. In Section 6, we prove an almost commuting property of the renormalization operator and a projection operator onto the space of fractional linear transformations. Finally, in Section 7, we develop a method to combine the two different mechanisms of closeness of renormalizations established in Section 4 and Section 5 and prove Theorem 1.1.

2 Preliminaries

2.1 Renormalization of commuting pairs

For every orientation-preserving homeomorphism T of the circle $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$ there is a unique rotation number $\rho := \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is a lift of T to $\mathbb{R}$. If $\rho \in (0,1)$ is irrational, it can be expressed uniquely as an infinite *continued fraction* *expansion*

$$\rho = [k_1, k_2, k_3, \dots] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}, \tag{2.1}$$

where $k_n \in \mathbb{N}$. Conversely, every infinite sequence of *partial quotients* k_n defines uniquely an irrational number ρ as the limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, \dots, k_n]$. The denominators satisfy the recursion relation $q_{n+1} = k_{n+1}q_n + q_{n-1}$, with $q_0 = 1$ and $q_{-1} = 0$.

To define the renormalizations, we start with a *marked point* $x_0 \in \mathbb{T}^1$, and consider the *marked trajectory* $x_i = T^i x_0$, with $i \geq 0$. The subsequence $(x_{q_n})_{n\geq 0}$ indexed by the denominators of the sequence of rational convergents of the rotation number ρ, will be called the sequence of *dynamical convergents*. We define $x_{q_{-1}} := x_0 - 1$. The combinatorial equivalence of all circle homeomorphisms with the same irrational rotation number implies that the order of the dynamical convergents of T is the same as the order of the dynamical convergents for the rigid rotation R_ρ. The well-known arithmetic properties of the rational convergents now imply that dynamical convergents alternate their order such that

$$x_{q_{-1}} < x_{q_1} < x_{q_3} < \dots < x_0 < \dots < x_{q_2} < x_{q_0}. \tag{2.2}$$

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$, and called the n-th *renormalization segments* associated to the marked point x_0. The n-th renormalization segment associated to the marked point x_i will be denoted by $\Delta_i^{(n)}$. The intervals $\Delta_i^{(n-1)} := T^i(\Delta_0^{(n-1)})$, for $i = 0, \dots, q_n - 1$ and $\Delta_i^{(n)} := T^i(\Delta_0^{(n)})$, for $i = 0, \dots, q_{n-1} - 1$, cover the whole circle without overlapping except at end points and, thus, form the n-th *dynamical partition* $\mathcal{P}_n$ of the circle. The first return map on the interval $\Delta_0^{(n-1)} \cup \Delta_0^{(n)}$ is given by T^{q_n} restricted to $\Delta_0^{(n-1)}$ and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n)}$. The n-th *renormalization* of an orientation-preserving homeomorphism T of the circle $\mathbb{T}^1$, with a rotation number $\rho = [k_1, k_2, k_3, \dots]$, with respect to the marked point $x_0 \in \mathbb{T}^1$, is given by a pair of functions (f_n, g_n), $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, obtained by rescaling the first return map, i.e.

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}, \qquad g_n := \tau_n \circ T^{q_{n-1}} \circ \tau_n^{-1}. \tag{2.3}$$

Here, τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0. Thus, $f_n : [-1, 0] \to \mathbb{R}$, and $g_n : [0, a_n] \to \mathbb{R}$, where $a_n := \tau_n(x_{q_n})$. The sequence of renormalizations (f_n, g_n) can also be generated by the action of a renormalization operator $\mathcal{R}$ on a space of commuting pairs. Renormalization of commuting pairs was first introduced in [33]. A *commuting pair* is a pair (f, g) of two real-valued, continuous and strictly-increasing functions f and g, with $f(0) \geq 0$ and $g(0) \leq 0$, defined on $[g(0), 0]$ and $[0, f(0)]$, respectively, satisfying $f(g(0)) = g(f(0))$. If $g(0) = -1$, the commuting pair is called *normalized*. If (f, g) is a commuting pair with $g(0) < 0$, then $\overline{(f, g)} := (\tau \circ f \circ \tau^{-1}, \tau \circ g \circ \tau^{-1})$

with $\tau(z) = -z/g(0)$ is a normalized commuting pair. A commuting pair is *non-degenerate* if $f(0) > 0$. For a normalized, non-degenerate pair (f, g), we define the *height* $k \in \mathbb{N}_0$ by the condition $f^k(-1) \leq 0 < f^{k+1}(-1)$. On a set of *renormalizable* commuting pairs, i.e. commuting pairs with finite and nonzero height, we define a renormalization operator as $\mathcal{R}(f, g) := \overline{(f^k \circ g, f)}$. Pairs which are not renormalizable are called non-renormalizable.

A pair (f, g) is called *infinitely-renormalizable* if $\mathcal{R}^n(f, g)$ is renormalizable for all $n \in \mathbb{N}_0$. Clearly, if the rotation number of T is irrational, then $\mathcal{R}(f_n, g_n) = (f_{n+1}, g_{n+1})$, for all $n \in \mathbb{N}_0$. Here, $f_0 = \mathcal{T}|_{[-1,0]}$ is the restriction of a lift $\mathcal{T}$ of T to $[-1, 0]$ satisfying $\mathcal{T}(0) \in (0, 1]$ and $g_0 : x \mapsto x - 1$ defined at $[0, \mathcal{T}(0)]$.

For normalized pairs (f, g) such that $f(-1) < 0$, we define a rotation number $\rho(f, g) \in [0, 1]$, by substituting its consecutive heights for partial quotients in the continued fraction expansion $\rho(f, g) = [k_1, k_2, \dots]$, where k_n is the height of $\mathcal{R}^{n-1}(f, g)$ (the symbol "∞" is the terminator of the sequence). On the set of rotation numbers, the renormalization operator acts as Gauss map: $\mathcal{G}[k_1, k_2, \dots] = [k_2, \dots]$, i.e. $\rho(\mathcal{R}(f, g)) = \mathcal{G}\rho(f, g)$.

2.2 Circle diffeomorphisms with breaks

In this paper, we consider renormalizations of $C^{2+\alpha}$-smooth circle *diffeomorphisms with breaks*, for $\alpha > 0$, i.e. homeomorphisms $T : \mathbb{T}^1 \to \mathbb{T}^1$ for which there exists a point $x_{br} \in \mathbb{T}^1$ such that: (i) T is $C^{2+\alpha}$ smooth on $[x_{br}, x_{br} + 1]$; (ii) $\inf_{x \neq x_{br}} T'(x) > 0$; and (iii) there exist one-sided derivatives $T'_-(x_{br}) \neq T'_+(x_{br})$. We refer to x_{br} as the *break point* and to

$$c = \sqrt{\frac{T'_-(x_{br})}{T'_+(x_{br})}} \neq 1, \tag{2.4}$$

as the *size of the break*. We will use the break point x_{br} as the marked point x_0. One can verify that renormalizations of circle diffeomorphisms with a break of size c satisfy the condition $c_n^2 = \frac{f_n'(0)g_n'(f_n(0))}{g_n'(0)f_n'(-1)}$, where $c_n = c$ if n is even, $c_n = c^{-1}$ if n is odd.

We refer to commuting pairs (f, g) satisfying $c = \sqrt{\frac{f'(0)g'(f(0))}{g'(0)f'(-1)}} \in \mathbb{R}_+ \backslash \{1\}$ as the commuting pairs with a break of size c. For the renormalization operator acting on commuting pairs (f, g) with breaks of size c, we sometimes write $\mathcal{R}_c$ instead of $\mathcal{R}$. Notice that the renormalization operator maps renormalizable commuting pairs with a break of size c to commuting pairs with a break of size c^{-1}.

It is well known [22] that renormalization maps f_n and g_n for circle diffeomorphisms with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$ approach, exponentially fast, two particular families of fractional linear transformations

$$F_{a_n,b_n,M_n,c_n}(z) := \frac{a_n + (a_n + b_n M_n)z}{1 - (M_n - 1)z}, \qquad G_{a_n,b_n,M_n,c_n}(z) := \frac{-c_n + \frac{c_n - b_n M_n}{a_n} z}{c_n + \frac{M_n - c_n}{a_n} z}, \tag{2.5}$$

where

$$a_n := \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad b_n := \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}, \tag{2.6}$$

and

$$\begin{aligned} M_n &:= \exp\left((-1)^n \int_{\Delta_0^{(n-1)}} \frac{(T^{q_n})''(z)}{2(T^{q_n})'(z)}\, dz \right) = \exp\left((-1)^n \sum_{i=0}^{q_n-1} \int_{\Delta_i^{(n-1)}} \frac{T''(z)}{2T'(z)}\, dz \right) \\ &= \exp\left(\frac{1}{2} \int_{x_{q_{n-1}}}^{x_0} (\ln(T^{q_n})'(z))'\, dz \right) = \sqrt{\frac{(T^{q_n})'(0)}{(T^{q_n})'(-1)}} = \sqrt{\frac{f_n'(0)}{f_n'(-1)}}. \end{aligned} \tag{2.7}$$

We will sometimes abbreviate the notation by writing $\mathcal{F}_n := F_{a_n,b_n,M_n,c_n}$ and $\mathcal{G}_n := G_{a_n,b_n,M_n,c_n}$.

It is easy to see that $V = \mathrm{Var}_{\mathbb{T}^1} \ln T' < \infty$. It follows that the map T satisfies the Denjoy's lemma [22], which implies that $|\ln(T^{q_n})'(x)| \leq V$, for all $x \in \mathbb{T}^1$. In particular, we have

(A) $|\ln f_n'(x)| \leq V$, for all $x \in [-1, 0]$ (at the end points, both the left and right derivatives are considered).

The following estimates have been proved in [22]. For every $C^{2+\alpha}$-smooth, $\alpha > 0$, circle diffeomorphism T with a break of size c, there exist constants $\mathcal{C} > 0$ and $\lambda \in (0, 1)$, such that, for all $n \in \mathbb{N}$, we have

(B) $\|f_n - F_{a_n,b_n,M_n,c_n}\|_{C^2} \leq \mathcal{C}\lambda^n$, $\|g_n - G_{a_n,b_n,M_n,c_n}\|_{C^1} \leq \mathcal{C}\lambda^n$,

(C) $|a_n + b_n M_n - c_n| \leq \mathcal{C} a_n \lambda^n$, and

(D) $|M_{n+1} - c_{n+1}(1 + a_{n+1} a_n (M_n - 1))| \leq \mathcal{C} a_{n+1} a_n \lambda^n$,

We show here (Proposition 3.7 below) that λ is universal, i.e., that it can be chosen independently of T, depending on c and α only.

Property (B) is a statement about the approach of renormalization maps to fractional linear transformations. Property (C) is a consequence of a commutation-type relation of the maps f_n and g_n [22]. Let us define the *total nonlinearity* of f_n as $\mathcal{N}(f_n) = \int_{-1}^{0} \frac{f_n''(x)}{f_n'(x)}\, dx$. Property (D) then provides a relation between the total nonlinearities of the maps f_n and f_{n+1}, taking into account that $M_n = \exp(\frac{1}{2}\mathcal{N}(f_n))$. We note that F_{a_n,b_n,M_n,c_n} is the unique fractional linear map that satisfies $F_{a_n,b_n,M_n,c_n}(0) = f_n(0)$, $F_{a_n,b_n,M_n,c_n}(-1) = f_n(-1)$ and $\mathcal{N}(F_{a_n,b_n,M_n,c_n}) = \mathcal{N}(f_n)$.

We also define

$$N_n := \exp\left((-1)^n \sum_{i=0}^{q_{n-1}-1} \int\limits_{\Delta_i^{(n)}} \frac{T''(z)}{2T'(z)}\, dz \right). \tag{2.8}$$

Clearly, $M_n N_n = \exp\left((-1)^n \int_{S^1} \frac{T''(z)}{2T'(z)}\, dz \right) = c^{(-1)^n} = c_n$.

For a normalized commuting pair (f, g) with a positive height, we define, the *canonical lift* $H_{f,g}(w) : \mathbb{R} \mapsto \mathbb{R}$, satisfying $H_{f,g}(w+1) = H_{f,g}(w) + 1$, and

$$H_{f,g}(w) := \begin{cases} H^{(1)}_{f,g}(w), & w \in \left[-1, \phi\left(f^{-1}(0)\right)\right], \\ 1 + H^{(2)}_{f,g}(w), & w \in \left[\phi\left(f^{-1}(0)\right), 0\right], \end{cases} \tag{2.9}$$

where

$$H^{(1)}_{f,g}(w) := \phi \circ f \circ \phi^{-1}, \qquad H^{(2)}_{f,g}(w) := \phi \circ g \circ f \circ \phi^{-1}, \tag{2.10}$$

and $\phi : [-1, f(0)] \to \mathbb{R}$ is the fractional linear transformation that maps $(-1, 0, f(0))$ into $(-1, 0, 1)$, i.e.

$$\phi(z) := \frac{(f(0)+1)z}{2f(0) + (f(0)-1)z}. \tag{2.11}$$

The derivative of the latter coordinate transformation is given by

$$\phi'(z) = \frac{2f(0)(f(0)+1)}{(2f(0) + (f(0)-1)z)^2}. \tag{2.12}$$

Notice that $H^{(1)}_{f,g}(\phi\left(f^{-1}(0)\right)) = 1 + H^{(2)}_{f,g}(\phi\left(f^{-1}(0)\right))$ and $H^{(1)}_{f,g}(-1) = H^{(2)}_{f,g}(0)$ and, thus, $H_{f,g}$ is continuous on $\mathbb{R}$. In fact, it is a lift of an orientation-preserving circle homeomorphism. We will denote the circle map generated by it by the same symbol. Notice that the rotation number $\rho(H_{f,g}) = \rho(f, g)$. Notice further that $H_{f,g}$ for a commuting pair (f, g) with a break of size c has, in general, two break points. Nevertheless, they belong to the same orbit and the product of the sizes of their breaks is equal to c.

It is sometimes convenient to consider the following two parameter families of fractional linear maps

$$F_{a_n,v_n,c_n}(z) := \frac{a_n + c_n z}{1 - v_n z}, \qquad G_{a_n,v_n,c_n}(z) := \frac{-c_n + z}{c_n - \frac{c_n - 1 - v_n}{a_n} z}, \tag{2.13}$$

where

$$v_n := M_n - 1. \tag{2.14}$$

The derivatives of these maps are given by

$$F'_{a,v,c}(z) = \frac{c + av}{(1 - vz)^2}, \qquad G'_{a,v,c}(z) = \frac{c(1 - \frac{c-1-v}{a})}{(c - \frac{c-1-v}{a} z)^2}. \tag{2.15}$$

We define $F_n := F_{a_n,v_n,c_n}$ and $G_n := G_{a_n,v_n,c_n}$, if $a_n < c_n$; otherwise we set $F_n := F_{c_n,v_n,c_n}$ and $G_n := G_{c_n,v_n,c_n}$.

The estimate (B) gives that, for some $\mathscr{C} > 0$,

(B') $\|f_n - F_n\|_{C^2} \leq \mathscr{C}\lambda^n$, $\|g_n - G_n\|_{C^1} \leq \mathscr{C}\lambda^n$.

We will identify each point $(a, v) \in \mathbb{R}^2$ with the corresponding pair of fractional linear maps $(F_{a,v,c}, G_{a,v,c})$. The following sets play an important role in the renormalization of commuting pairs of fractional linear transformations. We define

$$\mathcal{D}_c := \left\{(a, v) : 1/2 \leq \frac{v}{c-1} < 1, \frac{c(c-v-1)}{v} \leq a \leq c\right\}, \tag{2.16}$$

and $\check{\mathcal{D}}_c := \mathcal{D}_c \cap \{(a, v) : a > (c-1)^2/4v\}$, if $c > 1$, and $\check{\mathcal{D}}_c := \mathcal{D}_c \cap \{(a, v) : v > a(c-1)^2/4c + c - 1\}$, if $c < 1$. It was shown in [21] that the renormalization operator maps all infinitely renormalizable pairs in $\check{\mathcal{D}}_c$ into $\check{\mathcal{D}}_{1/c}$. Moreover, these sets are absorbing areas for the dynamics of the renormalization operator on a space of commuting pairs of fractional linear maps, i.e. each infinitely renormalizable commuting pair of fractional linear maps eventually falls inside these sets, under the action of the renormalization operator $\mathcal{R}$. The set of points in $\{(a, v) : 0 < a \leq c, a + v - c + 1 > 0\} \supset \check{\mathcal{D}}_c$ with the same irrational rotation number $\rho \in (0, 1)$ is a continuous curve $a = \gamma_{\rho,c}(v)$, $v > -1$, such that the slope of any secant line, in the (v, a) coordinate system, belongs to the interval $(-1, 0)$. As was also shown in [21], the slopes of the of the curve $\gamma_{\rho,c}$ also lie in this interval. Furthermore, for $c > 1$ and all irrational $\rho \in (0, 1)$, all curves $\gamma_{\rho,c}$ lie above the hyperbola $a = \frac{(c-1)^2}{4v}$.

We end this section with some comments about the notation. We write $A_n = \Theta(B_n)$, if there exits a constant $K_1 > 0$, such that $K_1^{-1}B_n \leq A_n \leq K_1 B_n$, for all n. We write $A_n = \mathcal{O}(B_n)$, if there exists a constant $K_2 \in \mathbb{R}$, such that $-K_2 B_n \leq A_n \leq K_2 B_n$, for all n.

3 A priori estimates of the renormalization parameters

In this section, we give some general estimates of the renormalization parameters of a $C^{2+\alpha}$-smooth ($\alpha > 0$) circle map T with a break of size $c \in \mathbb{R}_+ \backslash \{1\}$.

Proposition 3.1 $\ln M_n = \mathcal{O}(1)$*, for all* $n \in \mathbb{N}$.

Proof. Since the interiors of the intervals $\Delta_i^{(n-1)}$, for $i = 0, \ldots, q_n - 1$ do not overlap, the claim follows from the definition of M_n and the facts that T'' is bounded and that T' is bounded from below by a positive constant. **QED**

Proposition 3.2 *$M_n = c_n + \mathcal{O}(a_n)$, for all $n \in \mathbb{N}$.*

Proof. Since by Denjoy estimate (A), $b_n = 1 - \Theta(a_n)$, the claim follows directly from estimate (C). **QED**

Proposition 3.3 *There exists $\epsilon > 0$ such that, for n sufficiently large, if $c_n > 1$, then $a_n \in (\epsilon, c_n - \epsilon b_n + \mathcal{O}(\lambda^n))$; if $c_n < 1$, then $a_n \in (0, c_n - \epsilon)$;*

Proof. Consider first the case $c_n > 1$. Assume that $a_n \leq \epsilon$ for some small $\epsilon > 0$. It follows from Proposition 3.2 and the Denjoy estimate (A) that $\mathcal{F}_n'(0) = (a_n + b_n)M_n = (1+\mathcal{O}(a_n))M_n$ is close to c_n. Due to (B), for sufficiently large n, $f_n'(0) = c_n+\mathcal{O}(\lambda^n)+\mathcal{O}(a_n)$ is close to $c_n > 1$ as well. By Proposition 3.1, $\mathcal{F}_n''(z) = 2(M_n - 1)\frac{(a_n+b_n)M_n}{(1-(M_n-1)z)^3}$ is bounded and, by (B), so is $f_n''(z)$. If a_n is small enough, $f_n(z) = z$ at some point $z \in [-1, 0)$, close to 0, which contradicts the fact that the rotation number of T is irrational. This gives a lower bound on a_n. The estimate (C) implies $a_n \leq c_n - b_n M_n + \mathcal{O}(\lambda_n) \leq c_n - \Theta(b_n) + \mathcal{O}(\lambda^n)$, which gives an upper bound.

In the case $c_n < 1$, it follows from (C) and Proposition 3.1 that $c_n - a_n = \Theta(b_n) + \mathcal{O}(\lambda^n) \geq \Theta(a_n) - \Theta(\lambda^n)$. Here, we have also used that $b_n \geq a_{n+1}a_n$ and that $a_{n+1} > \epsilon$, for sufficiently large n. In order for $c_n - a_n$ to be very small, a_n must be very small, which is impossible. **QED**

Corollary 3.4 *There exists $\delta \in (0,1)$ such that, for sufficiently large n, $a_{n+1}a_n < 1 - \delta$.*

Proof. It follows directly from Proposition 3.3 and the fact that $c_n c_{n+1} = 1$. **QED**

Proposition 3.5 *For every $\epsilon_0 > 0$, we have $\frac{M_n-1}{c_n-1} \in (\delta - \epsilon_0, 1 + \epsilon_0)$, for sufficiently large n.*

Proof. It follows from (D) that

$$M_{n+1} - 1 = c_{n+1}(1 + (M_n - 1)a_n a_{n+1}) - 1 + \mathcal{O}(\lambda^n). \tag{3.1}$$

Applying this estimate recursively, first for M_{n+2} and then for M_{n+1}, we obtain

$$\frac{M_{n+2}-1}{c_{n+2}-1} = a_{n+2}a_{n+1}a_{n+1}a_n \frac{M_n - 1}{c_n - 1} + (1 - a_{n+2}a_{n+1}) + \mathcal{O}(\lambda^n), \tag{3.2}$$

since $c_{n+2} = c_n$. Since, by Corollary 3.4, $a_{n+1}a_n \leq 1 - \delta$, by iterating the latter equality we obtain that $\frac{M_n-1}{c_n-1}$ is bounded from below by a positive constant, that can be chosen arbitrarily close to δ, for sufficiently large n. The previous equality can be put in the form

$$1 - \frac{M_{n+2}-1}{c_{n+2}-1} = a_{n+2}a_{n+1}\left[a_{n+1}a_n\left(1 - \frac{M_n-1}{c_n-1}\right) + (1 - a_{n+1}a_n)\right] + \mathcal{O}(\lambda^n). \tag{3.3}$$

By iterating this identity, we find $1 - \frac{M_{n+2}-1}{c_{n+2}-1} > -\epsilon_0$, for sufficiently large n. **QED**

Proposition 3.6 *For sufficiently large n, $f_n''(z)$ is bounded away from zero and positive if $c_n > 1$ and negative if $c_n < 1$.*

Proof. It follows from the Denjoy estimate (A) that $a_n = \Theta(1 - b_n)$ and thus $a_n + b_n = \Theta(1)$. The claim follows from Proposition 3.1 and Proposition 3.5, using (B) and the explicit form of $\mathcal{F}_n''(z) = 2(M_n - 1)\frac{(a_n+b_n)M_n}{(1-(M_n-1)z)^3}$. **QED**

Proposition 3.7 *Let $c \in \mathbb{R}_+ \backslash \{1\}$ and $\alpha \in (0, 1]$. There exist a universal constant $\lambda \in (0, 1)$, such that the estimates (B), (C) and (D) hold true for every $C^{2+\alpha}$-smooth circle map T with a break of size c, and every $n \in \mathbb{N}$.*

Proof. There exist a universal constant $\bar{V} > 0$, such that for every $C^{2+\alpha}$-smooth circle map T with a break of size c the following holds. There exists $N_0 \in \mathbb{N}$, such that for all $n \geq N_0$, we have $|\ln f_n'(z)| \leq \bar{V}$, for all $z \in [-1, 0]$. This follows from the estimate (B), estimate (C) and Proposition 3.5. Namely, estimate (C) and Proposition 3.5 imply that for any $\epsilon_0 > 0$ (independent of T) and sufficiently larger n, $\mathcal{F}_n'(z)$ is bounded by $c_n/(c_n+(c_n-1)\epsilon_0)^2$ from one side and by $c_n^2+c_n(c_n-1)\epsilon_0$ from the other. The estimate (B) then implies that for any $C^{2+\alpha}$ smooth circle map T with a break of size $c \in \mathbb{R}_+\backslash\{1\}$ and for any $\bar{\epsilon} > 0$, for sufficiently large n, we have $\min\{c^2, c^{-2}\} - \bar{\epsilon} < f_n'(z) < \max\{c^2, c^{-2}\} + \bar{\epsilon}$, and the initial claim holds for any $\bar{V} > 2|\ln c|$. It is easy to show (see Lemma 2 in [34]), that there is a universal constant $\bar{\lambda} = (1 + e^{-\bar{V}})^{-1/2}$, such that $\frac{|\Delta_i^{(n+2)}|}{|\Delta_i^{(n)}|} \leq \bar{\lambda}^2$, for $n \geq N_0$, provided that $\Delta_i^{(n)} \subset \Delta_0^{(N_0-1)} \cup \Delta_0^{(N_0)}$. Using the standard small distortion argument, for any $\bar{\bar{\lambda}} > \bar{\lambda}$, we obtain $\frac{|\Delta_j^{(n+2)}|}{|\Delta_j^{(n)}|} \leq \bar{\bar{\lambda}}^2$, for all $j = 0, \ldots, q_{n+1} - 1$ and n sufficiently large. Following [22], this implies that the estimates (B), (C) and (D) are valid with the same $\lambda = \bar{\bar{\lambda}}^\alpha$, for every such map T, with some $\mathcal{C} > 0$ (depending on T), for every $n \in \mathbb{N}$. **QED**

4 Strings of renormalizations with large a_n tails

We define a *string of renormalizations* (or simply a *string*) to be a (finite or infinite) sequence f_n with $n \in [n_1, n_2 - 1]$, $n_1 \in \mathbb{N} \cup \{0\}$ and $n_2 \in \mathbb{N} \cup \{\infty\}$. We call $n_2 - n_1$, the *length* of the string. It can be finite or infinite. We choose some positive $\sigma < \epsilon$ from Proposition 3.3, and assume that the strings have *tails* with (relatively) *large* a_n, i.e. for all $n \in [n_1 + 1, n_2 - 1]$, we have $a_n > \sigma\lambda_{1^-}^n$, for some $\lambda_{1^-} \in (\lambda, 1)$. Each finite string ends with $c_{n_2} < 1$ and $a_{n_2} \leq \sigma\lambda_{1^+}^{n_2}$, for some $\lambda_{1^+} \in [\lambda_{1^-}, 1)$. We consider two *types of strings*: (i) an *initial string*, starting at some $n_1 = n_0 \in \mathbb{N}_0$, and (ii) an *ordinary string*, starting at some $n_1 \in \mathbb{N}$ with $a_{n_1} \le \sigma\lambda_{1+}^{n_1}$. If $\lambda_{1+} > \lambda_{1-}$, there is certain freedom in the choice of strings within the renormalization sequence f_n. We will use this freedom later on. We assume that the initial string is sufficiently long so that the estimates of the previous section are already valid. Notice that the initial string can be made arbitrary long by taking σ sufficiently small and that for an ordinary string we have $a_{n_1} \le \sigma < \epsilon$ and thus $c_{n_1} < 1$.

The objective of this section is to show that, with an exponentially small (in n) change of the parameters a_n and v_n of the pair of fractional linear maps $(F_{a_n,v_n,c_n}, G_{a_n,v_n,c_n})$, one can obtain a pair of fractional linear maps with the same rotation number as (f_n, g_n), for sufficiently large n in the initial string and for all integer $n \in [n_1+1, n_2-1]$ in an ordinary string.

We emphasize that this section deals with single strings only, and that the constants hidden in $\mathcal{O}(\cdot)$ and $\Theta(\cdot)$ notations are independent of n_1 and n_2.

Proposition 4.1 *There exists $\epsilon_1 > 0$, such that $\frac{v_n}{c_n-1} \in (\epsilon_1, 1-\Theta(\lambda_{1-}^n))$, for sufficiently large n, in the initial string, and for all $n \in [n_1+2, n_2-1]$ in an ordinary string. In an ordinary string, we also have $\frac{v_{n_1}}{c_{n_1}-1} \in (1-\Theta(\lambda_{1+}^n), 1+\Theta(\lambda_{1+}^{n_1}))$ and $\frac{v_{n_1+1}}{c_{n_1+1}-1} \in (1-\Theta(\lambda_{1+}^n), 1+\Theta(\lambda^{n_1}))$.*

Proof. Part of the argument here is similar to that of Proposition 3.5. We repeat it for the benefit of the reader. It follows from (D) and definition (2.14) that

$$v_{n+1} = c_{n+1}(1 + v_n a_n a_{n+1}) - 1 + \mathcal{O}(\lambda^n). \tag{4.1}$$

Applying this estimate recursively, first for v_{n+2} and then for v_{n+1}, we obtain

$$\frac{v_{n+2}}{c_{n+2}-1} = a_{n+2}a_{n+1}a_{n+1}a_n\frac{v_n}{c_n-1} + (1 - a_{n+2}a_{n+1}) + \mathcal{O}(\lambda^n), \tag{4.2}$$

since $c_{n+2} = c_n$. It is now easy to see from Corollary 3.4 that if $\frac{v_n}{c_n-1}$ is negative, in two steps it will increases by a positive constant; once it becomes positive, it will stay larger than a positive constant that can be chosen arbitrarily close to δ. This proves the desired lower bound for sufficiently large n in the initial string, if the string is long enough. A similar argument has been previously used in [21]. For an ordinary string, it follows directly from the definition (2.14) and Proposition 3.2 that $\frac{v_{n_1}}{c_{n_1}-1} = 1 + \mathcal{O}(\lambda_{1+}^{n_1})$. Identity (4.1) implies $\frac{v_{n_1+1}}{c_{n_1+1}-1} = 1 - \Theta(a_{n_1}) + \mathcal{O}(\lambda^{n_1})$. The recursion relation (4.2) leads to the desired lower bound for the remaining n in the string.

By rewriting the equality (4.2) as

$$1 - \frac{v_{n+2}}{c_{n+2}-1} = a_{n+2}a_{n+1}\left[a_{n+1}a_n\left(1 - \frac{v_n}{c_n-1}\right) + (1 - a_{n+1}a_n)\right] + \mathcal{O}(\lambda^n), \tag{4.3}$$

we see that, for sufficiently large n in the initial string, we have $1-\frac{v_n}{c_n-1} > \Theta(\lambda_{1-}^n)$. If for some $n\in\mathbb{N}_0$, $1-\frac{v_n}{c_n-1} < -\delta$, it will increase in two steps by an amount larger than a positive constant and, thus, in a finite number of steps, it will become positive. Once it is positive, it will remain positive for all larger n belonging to the same even or odd subsequence. This proves the desired upper bound for sufficiently large n in the initial string. For an ordinary string, the above estimates on $\frac{v_n}{c_n-1}$, for $n=n_1$ and $n=n_1+1$, and the recursive relation (4.3) imply the desired upper bounds. **QED**

Proposition 4.2 *For $n\in[n_1+1,n_2-1]$ in either the initial or an ordinary string, we have*

$$a_n \geq \frac{c_n(c_n-v_n-1)}{v_n} - \Theta(\lambda^n). \tag{4.4}$$

Proof. It follows directly from (4.1) that

$$\frac{v_n+1-c_n}{a_n} = c_n v_{n-1} a_{n-1} + \frac{1}{a_n}\mathcal{O}(\lambda^n), \tag{4.5}$$

and, thus,

$$\frac{c_n-v_n-1}{a_n} - \frac{v_n}{c_n} = c_{n-1}-1-v_{n-1}a_{n-1}(a_n+c_n) + \frac{1}{a_n}\mathcal{O}(\lambda^n). \tag{4.6}$$

We further have

$$|v_{n-1}|a_{n-1}(c_n+a_n) \leq \left| v_{n-1}b_{n-1} + v_{n-1}\frac{a_{n-1}}{c_{n-1}}\right| = \left| v_{n-1}\frac{c_{n-1}-a_{n-1}}{1+v_{n-1}} + v_{n-1}\frac{a_{n-1}}{c_{n-1}}\right| + \mathcal{O}(\lambda^n). \tag{4.7}$$

For $\frac{v_{n-1}}{c_{n-1}-1}\in(0,1]$, the right-hand side of (4.7) is smaller or equal to $|c_{n-1}-1|+\Theta(\lambda^n)$, since it is the absolute value of an increasing function of v_{n-1}, which takes the value $c_{n-1}-1$ at $v_{n-1}=c_{n-1}-1$. Thus, if $c_{n-1}>1$, we have

$$\frac{c_n-v_n-1}{a_n} - \frac{v_n}{c_n} \geq \frac{1}{a_n}\mathcal{O}(\lambda^n), \tag{4.8}$$

and, since v_n is bounded from above by a negative constant, by Proposition 4.1, we have (4.4). If $c_{n-1}<1$, then one gets the same inequality (4.4) for sufficiently large n in the initial string and for $n\in[n_1+2,n_2-1]$ in an ordinary string. For $n=n_1+1$, in an ordinary string, we obtain (4.4) (without the error term) directly from (4.6), using the fact that v_n and a_n are bounded, as follows from Proposition 3.1 and Proposition 3.3. **QED**

Proposition 4.3 *For sufficiently large n in the initial string and for every $n\in[n_1+1,n_2-1]$ in an ordinary string, the following holds. The point (a_n,v_n) belongs to the $\mathcal{O}(\lambda^n)$-neighborhood of $\check{\mathcal{D}}_{c_n}$. If $c_n>1$, then $\frac{v_n}{c_n-1}\in(\frac{1}{2}-\Theta(\lambda^n),1+\Theta(\lambda^n))$. If $c_n<1$, then $\frac{v_n}{c_n-1}\in(\frac{1}{2}+\epsilon_2,1-\Theta(\lambda_{1-}^n))$, for some $\epsilon_2>0$.*

Proof. It follows from Proposition 3.3, Proposition 4.1 and Proposition 4.2 that, if $c_n > 1$, then we have $\frac{v_n}{c_n-1} \in (\frac{1}{2} - \Theta(\lambda^n), 1+\Theta(\lambda^n))$. If $c_n < 1$, then $\frac{v_n}{c_n-1} \in (\frac{1}{2}+\epsilon_2, 1-\Theta(\lambda_{1^-}^n))$, for some $\epsilon_2 > 0$. Together with Proposition 3.3 and Proposition 4.2, these estimates show that (a_n, v_n) belongs to $\mathcal{O}(\lambda^n)$-neighborhood of $\mathcal{D}_{c_n}$ (defined by (2.16)), for all $n \in [n_1+1, n_2-1]$ either in the initial or in an ordinary string.

Let $c_n > 1$. Since the rotation number is irrational, using (B'), for all $z \in [-1, 0]$, we have

$$z < f_n(z) \leq F_n(z) + \mathcal{O}(\lambda^n). \tag{4.9}$$

In particular, for some $z_0 = -\frac{c_n-1}{2v_n} + \mathcal{O}(\lambda^n) \in (-1, 0)$, we obtain

$$F_n(z_0) - z_0 = \frac{2}{c_n+1}\left(a_n - \frac{(c_n-1)^2}{4v_n}\right) + \mathcal{O}(\lambda^n) \geq \mathcal{O}(\lambda^n), \tag{4.10}$$

and, thus,

$$a_n - \frac{(c_n-1)^2}{4v_n} \geq \mathcal{O}(\lambda^n). \tag{4.11}$$

This inequality, together with (4.1), implies

$$\frac{v_{n+1}+1-c_{n+1}}{c_{n+1}a_{n+1}} \geq \frac{(c_n-1)^2}{4} + \frac{1}{a_{n+1}}\mathcal{O}(\lambda^n) = \frac{(c_{n+1}-1)^2}{4(c_{n+1})^2} + \frac{1}{a_{n+1}}\mathcal{O}(\lambda^n). \tag{4.12}$$

The estimates (4.11) and (4.12) show that, for all $n \in [n_1+1, n_2-1]$ either in the initial or in an ordinary string, (a_n, v_n), in fact, belongs to $\mathcal{O}(\lambda^n)$-neighborhood of $\check{\mathcal{D}}_{c_n}$. **QED**

Proposition 4.4 *For sufficiently large n in the initial string and for all $n \in [n_1+1, n_2-1]$ in an ordinary string, we have $a_n + v_n + 1 - c_n > \Theta(a_n)$.*

Proof. If $c_n > 1$ then it follows from Proposition 3.3, Proposition 4.2 and Proposition 4.3 that, for sufficiently large n in the initial string and for all $n \in [n_1+1, n_2-1]$ in an ordinary string,

$$a_n + v_n + 1 - c_n \geq a_n - a_n\frac{v_n}{c_n} - \Theta(\lambda^n) > \Theta(a_n). \tag{4.13}$$

If $c_n < 1$ then it follows from Proposition 4.1 that, for sufficiently large n in the initial string and for all $n \in [n_1+1, n_2-1]$ in an ordinary string,

$$a_n + v_n + 1 - c_n > a_n + \Theta(\lambda_{1^-}^n) > a_n. \tag{4.14}$$

This proves the claim. **QED**

Proposition 4.5 *There exists a constant $C_1 > 1$ such that, for sufficiently large n, and for every z in the domain of the corresponding function, we have*

$$C_1^{-1} < F_n'(z), G_n'(z) < C_1, \qquad C_1^{-1} a_n < \phi_n'(z) < \frac{C_1}{a_n}. \tag{4.15}$$

Proof. The bounds on F_n' and G_n' follow from the definition (2.14) and Proposition 3.1, together with the estimates of Proposition 3.3 or estimates (A) and (C), respectively. The bound on ϕ_n' is obvious, taking into account Proposition 3.3. **QED**

Proposition 4.6 *There exists $C_2 > 0$ such that $|H_{f_n,g_n}(w) - H_{F_n,G_n}(w)| \le \frac{C_2}{a_n}\lambda^n$, for sufficiently large n and for all $w \in \mathbb{R}$.*

Proof. Since f_n and F_n are monotonically increasing, $f_n^{-1}(0)$ and $F_n^{-1}(0)$ are defined uniquely. Furthermore, since $f_n(0) = a_n > 0$ and $f_n(-1) = -b_n < 0$, due to Proposition 4.5 and estimate (B'), for sufficiently large $n \in \mathbb{N}$, we have $f_n^{-1}(0) \in (-1, 0)$. Since $F_n(0) = a_n > 0$ and $F_n(-1) \le 0$, due to Proposition 4.5, we have $F_n^{-1}(0) \in [-1, 0)$. Notice that, since $f_n(0) = F_n(0)$, ϕ is the same for (f_n, g_n) and (F_n, G_n). On $[-1, \min\{\phi(F_n^{-1}(0)), \phi(f_n^{-1}(0))\})$, using property (B'), we have

$$\begin{aligned} |H_{f_n,g_n}(w) - H_{F_n,G_n}(w)| &= |\phi \circ f_n \circ \phi^{-1}(w) - \phi \circ F_n \circ \phi^{-1}(w)| \\ &= \phi'(\zeta)|f_n \circ \phi^{-1}(w) - F_n \circ \phi^{-1}(w)| \le \frac{C_1}{a_n}\mathscr{C}\lambda^n, \end{aligned} \tag{4.16}$$

where ζ is a point between $f_n \circ \phi^{-1}(w)$ and $F_n \circ \phi^{-1}(w)$. On $[\max\{\phi(F_n^{-1}(0)), \phi(f_n^{-1}(0))\}, 0)$, we have

$$\begin{aligned} |H_{f_n,g_n}(w) - H_{F_n,G_n}(w)| &= |\phi \circ g_n \circ f_n \circ \phi^{-1}(w) - \phi \circ G_n \circ F_n \circ \phi^{-1}(w)| \\ &= \phi'(\zeta_1)|g_n \circ f_n \circ \phi^{-1}(w) - G_n \circ F_n \circ \phi^{-1}(w)| \\ &\le \phi'(\zeta_1)\big(G_n'(\zeta_2)|f_n \circ \phi^{-1}(w) - F_n \circ \phi^{-1}(w)| \\ &+ |(g_n - G_n) \circ f_n \circ \phi^{-1}(w)|\big) \le \frac{C_1}{a_n}(C_1 + 1)\mathscr{C}\lambda^n. \end{aligned} \tag{4.17}$$

Here, ζ_1 is a point between $g_n \circ f_n \circ \phi^{-1}(w)$ and $G_n \circ F_n \circ \phi^{-1}(w)$, and ζ_2 is a point between $f_n \circ \phi^{-1}(w)$ and $F_n \circ \phi^{-1}(w)$.

Since the functions H_{f_n,g_n} and H_{F_n,G_n} are continuous and monotonically increasing, and since $H_{f_n,g_n}(\phi(f_n^{-1}(0))) = 0$ and $H_{F_n,G_n}(\phi(F_n^{-1}(0))) = 0$, we obtain a similar estimate

$$|H_{f_n,g_n}(w) - H_{F_n,G_n}(w)| \le \frac{C_1}{a_n}(C_1 + 2)\mathscr{C}\lambda^n, \tag{4.18}$$

for all w in the interval $[\min\{\phi(F_n^{-1}(0)), \phi(f_n^{-1}(0))\}, \max\{\phi(F_n^{-1}(0)), \phi(f_n^{-1}(0))\}]$. Thus, on the whole interval $[-1, 0]$, we have the desired estimate, and the claim follows. **QED**

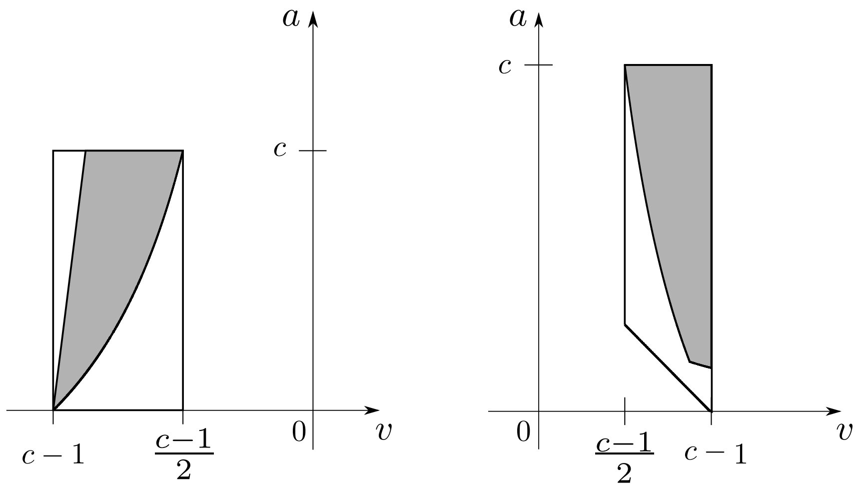

Figure 1: Regions $\check{\mathcal{D}}_c$ (shaded) and $\Phi^0_{c,n}$ (trapezoidal) for $0 < c < 1$ (left) and $c > 1$ (right), and some large $n \in \mathbb{N}$.

Let $\varepsilon \geq 0$ and $\varsigma > 0$. If $c > 1$, we define

$$\Phi^\varepsilon_{c,n} := \left\{ (a, v) : \varepsilon < a \leq c, \frac{1}{2} - \varsigma\lambda^n < \frac{v}{c-1} < 1 + \varsigma\lambda^n, a + v + 1 - c > \varepsilon \right\}. \tag{4.19}$$

If $c < 1$, we define

$$\Phi^\varepsilon_{c,n} := \left\{ (a, v) : \varepsilon\lambda^n_{1^-} < a < c - \varepsilon, \frac{1}{2} + \varepsilon < \frac{v}{c-1} < 1 - \varepsilon\lambda^n_{1^-} \right\}. \tag{4.20}$$

Proposition 3.3, Proposition 4.3, and Proposition 4.4 show that there exist $\varepsilon > 0$ and $\varsigma > 0$ such that, for sufficiently large n in the initial string and for all $n \in [n_1 + 1, n_2 - 1]$ in any ordinary string, (a_n, v_n) belongs to $\Phi^\varepsilon_{c_n,n}$. In the following, we assume that ε and ς have been chosen such that this is the case. Also, we abbreviate the notation by setting $H_n := H_{F_n,G_n}$. The value that H_n takes at point w will be denoted by $H_n(w; a_n, v_n)$, when necessary to specify the parameters a_n and v_n of F_n and G_n.

Proposition 4.7 *There exists* $\mu_{c_n,\varepsilon} > 0$ *such that, for any* $(a_n, v_n) \in \Phi^\varepsilon_{c_n,n}$, *we have* $\frac{\partial H^{(i)}_n(w;a_n,v_n)}{\partial a_n} \geq \mu_{c_n,\varepsilon} a_n^5$, *for* $i = 1, 2$.

Proof. The proof that we give here is an improvement of the method developed in [21]. To abbreviate the notation, let us write a, v, c instead of a_n, v_n, c_n, respectively. A direct calculation gives us that $\frac{\partial H^{(1)}_n}{\partial a} = -4wP_1(w)/Q_1^2(w)$, for $w \in [-1, -\frac{a+1}{2c+1-a}]$, where $P_1(w) = -(2vc+2ac+v+va^2-2c^2)w+2ac+a^2v-v+2c$ and $Q_1(w) = -(a+1)^2+w(-1+4va+a^2-2ac+2c)$. Clearly, the denominator $Q_1^2(w)$ is bounded and, thus, $-4w/Q_1^2(w)$ is bounded from below be a positive constant. Since $P_1(-1) = 2a(c+av)+2c(v+a-c+1) > \Theta(a)$, $P_1(-\frac{a+1}{2c+1-a}) = 2(c+1)(a+1)(c+av)/(2c+1-a)$ is bounded from below by a positive constant, and P_1 is linear in w, we can conclude that on the whole interval $P_1(w) > \Theta(a)$ and, thus, the claim is proved for $i = 1$.

Similarly, we can find $\frac{\partial H_n^{(2)}}{\partial a} = 2P_2(w)/Q_2^2(w)$, for $w \in [-\frac{a+1}{2c+1-a}, 0]$, where $P_2(w) = 4a^2c(c+1)v^2w^2 + (2c^2a^2 + 2c^2 + 2a^4c - 4c^3a^2 + a^2 + a^4 + c - 2^3 + 7a^2c - 2ca)vw^2 + c(4a^3c + c - 6a^2c^2 - 2c^2 - 3ca^2 + 1 - 3a^2 + 2a^3 + 6ac)w^2 - 2(c+1)(a^2-1)(a^2+c)vw - 2c(c+1)(a^2 - 1)(2a-c+1)w + (a+1)^2(a^2+c)v + c(a+1)^2(2a-c+1)$, and $Q_2(w) = -(a+1)(a^2 - ac + a + c) + (3a^2c - a + a^3 + 4c^2a - c)w - 2a(a+1)v + 2a(1+c)(a-1)vw$. As the denominator $Q_2^2(w)$ is bounded, the term $2/Q_1^2(w)$ is bounded from below be a positive constant. We also have $P_2(-\frac{a+1}{2c+1-a}) = 4ca(1+c)(1+a)^2(v+a+1-c)(c+av)/(2c+1-a)^2 > \Theta(a^2)$ and $P_2(0) = (a+1)^2(a(c+av)+c(v+a+1-c)) > \Theta(a)$. Furthermore, since the derivative $\frac{\partial P_2(w)}{\partial w}$ is bounded, there exists $\epsilon_3 > 0$ such that, for every $w \in [-\epsilon_3 a^{3/2}, 0]$, we have $P_2(w) > \Theta(a)$. In order to provide a lower bound on $P_2(w)$ in the interval $[-\frac{a+1}{2c+1-a}, -\epsilon_3 a^{3/2})$, notice first that $P_2(w)/w^2$ is a quadratic polynomial in $1/w$, which has minimum at point $1/w = (1+c)(a-1)/(a+1)$, outside our interval of interest $(-\infty, -\frac{2c+1-a}{a+1}]$. Therefore, within this interval, it reaches the global minimum at point $1/w_{min} = -\frac{2c+1-a}{a+1}$. Finally, for $w \in [-\frac{a+1}{2c+1-a}, -\epsilon_3 a^{3/2})$, we have $P_2(w) \geq \epsilon_3^2 a^3 P_2(w_{min})/w_{min}^2 \geq \Theta(a^5)$, since $P_2(w_{min}) > \Theta(a^2)$.
QED

The rotation number $\rho(a_n, v_n)$ of the fractional linear pair (a_n, v_n) does not necessarily equal $\rho_n = \rho(f_n, g_n)$. For that reason, we define the projection operator $\mathcal{P}$ from the space of all commuting pairs (f_n, g_n) with well-defined rotation number to $\check{\mathcal{D}}_{c_n}$, as $\mathcal{P}(f_n, g_n) = (a_n^*, v_n^*)$, where $(a_n^*, v_n^*) = (\gamma_{\rho_n,c_n}(v_n), v_n)$ if $(\gamma_{\rho_n,c_n}(v_n), v_n) \in \check{\mathcal{D}}_{c_n}$, or let (a_n^*, v_n^*) be the closest to (a_n, v_n) intersection point of the curve $a = \gamma_{\rho_n,c_n}(v)$ with the boundary of $\check{\mathcal{D}}_{c_n}$. Since this projection is determined uniquely by f_n, and the rotation number ρ_n, and since we will always have in mind the n-th renormalizations of a particular circle map T, we will write $\mathcal{P}f_n = (a_n^*, v_n^*)$.

Proposition 4.8 *Let* $\lambda_{1^-} > \sqrt[6]{\lambda}$. *Then,* $\gamma_{\rho_n,c_n}(v_n) - a_n = \mathcal{O}((\lambda/\lambda_{1^-}^6)^n)$, *for all* $n \in [n_1+1, n_2-1]$ *either in the initial or in an ordinary string.*

Proof. We assume that $a_n > \gamma_{\rho_n,c_n}(v_n)$. In the opposite case, the proof is similar and even simpler. It follows from Proposition 4.6 that, for all $w \in [-1, 0]$, we have

$$H_{f_n,g_n}(w) \geq H_n(w; a_n, v_n) - \frac{C_2}{a_n}\lambda^n. \tag{4.21}$$

Since $(a_n, v_n) \in \Phi_{c_n,n}^{\varepsilon}$, for large enough n in the initial string and for $n_1 < n < n_2$ in an ordinary string, at least half of the segment $[\gamma_{\rho_n,c_n}(v_n), a_n] \times \{v_n\}$ lies inside $\Phi_{c_n,n}^{\varepsilon/2}$. Moreover, that is the segment $[\hat{a}_n, a_n] \times \{v_n\}$, for some $\hat{a}_n \in [\gamma_{\rho_n,c_n}(v_n), a_n]$. To see this, notice that $(\gamma_{\rho_n,c_n}(v_n), v_n)$ belongs to $\Phi^0_{c_n,n}$ and that the distance between the bottom boundaries of the regions $\Phi^\varepsilon_{c_n,n}$ and $\Phi^0_{c_n,n} \supset \Phi^\varepsilon_{c_n,n}$, in the direction of coordinate a, is proportional to ε.

Using Proposition 4.7, we find

$$H_{f_n,g_n}(w) \geq H_n(w; \gamma_{\rho_n,c_n}(v_n), v_n) + \mu_{c_n,\varepsilon/2}\frac{a_n^6 - \hat{a}_n^6}{6} - \frac{C_2}{a_n}\lambda^n. \tag{4.22}$$

If $\mu_{c_n,\varepsilon/2}\frac{a_n^6-\hat{a}_n^6}{6} - \frac{C_2}{a_n}\lambda^n > 0$, then $H_{f_n,g_n}(w) > H_n(w; \gamma_{\rho_n,c_n}(v_n), v_n)$ for all $w \in [-1, 0]$. This gives that the rotation number $\rho(f_n, g_n) > \rho(\gamma_{\rho_n,c_n}(v_n), v_n)$, while, in fact, they are equal. Thus,

$$a_n^6 - \hat{a}_n^6 \leq \frac{6C_2\lambda^n}{a_n\mu_{c_n,\varepsilon/2}} \leq \frac{6C_2\lambda^n}{\sigma\lambda_{1^-}^n \min\{\mu_{c,\varepsilon/2}, \mu_{c^{-1},\varepsilon/2}\}}. \tag{4.23}$$

The claim follows since $a_n - \gamma_{\rho_n,c_n}(v_n) \leq 2(a_n - \hat{a}_n) \leq 2(a_n^6 - \hat{a}_n^6)/a_n^5$. **QED**

Lemma 4.9 *If $\lambda_{1^-} > \sqrt[6]{\lambda}$, then $a_n^* - a_n = \mathcal{O}((\lambda/\lambda_{1^-}^6)^n)$ and $v_n^* - v_n = \mathcal{O}((\lambda/\lambda_{1^-}^6)^n)$, for all $n \in [n_1 + 1, n_2 - 1]$ either in the initial or in an ordinary string.*

Proof. The claim follows from the definition of the projection operator $\mathcal{P}$, Proposition 4.3 and Proposition 4.8. Just notice that the slopes of the curves $a = \frac{c_n(c_n-v-1)}{v}$ and $a = \frac{4c_n(v+1-c_n)}{(c_n-1)^2}$ inside of $\check{\mathcal{D}}_{c_n}$ are bounded away from the interval $(-1, 0)$ and, thus, their intersection with γ_{ρ_n,c_n} is transversal. **QED**

On the set of commuting pairs (F, G) in $\mathcal{D}_c$, we consider two sets of coordinates (a, v) and (x, y), where

$$(x, y) = \mathcal{U}(a, v) = \left(av, \frac{v + 1 - c}{ca}\right). \tag{4.24}$$

The coordinates x and y, introduced in [21], can be viewed as independent indicators of nonlinearity of F and G. To see this, notice that if we perform a linear scaling $z = at$, the map $F_{a,v,c}(z)$ is transformed into $F_c(t) = \frac{1+ct}{1-xt}$. Similarly, by simple inversion $z = -t$, $G_{a,v,c}(z)$ is transformed into $G_c(t) = \frac{1+t/c}{1-yt}$. The following corollary follows directly from Lemma 4.9.

Corollary 4.10 *If $\lambda_{1^-} > \sqrt[8]{\lambda}$, then $y_n^* - y_n = \mathcal{O}((\lambda/\lambda_{1^-}^8)^n)$, for all $n \in [n_1 + 1, n_2 - 1]$ either in the initial or in an ordinary string.*

Proposition 4.11 ([21]) *For any $(x, y) \in \check{\mathcal{D}}_c$, $\mathcal{R}_c(x, y) = (x', y')$ with $y' = x$.*

5 Renormalizations with small a_n

In this section, we consider renormalizations at the beginning of ordinary strings, satisfying $a_n \le \sigma\lambda_{1+}^n$. In particular, $c_n < 1$, since in the opposite case a_n is bounded from below by a positive constant due to Proposition 3.3.

Lemma 5.1 *Let $\lambda_2 \in (\lambda_{1+}, 1)$. For sufficiently small $\sigma > 0$ and all $n \in \mathbb{N}$, if $a_n \le \sigma\lambda_{1+}^n$, then $a_n^* - a_n = \mathcal{O}(\lambda_2^n)$ and $v_n^* - v_n = \mathcal{O}(\lambda_2^n)$. Also, $\mathcal{P}f_n = (a_n^*, v_n^*)$ and $\mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1} = (\bar{a}_n, \bar{v}_n)$ are $\mathcal{O}(\lambda_2^n)$-close, as points in $\check{\mathcal{D}}_{c_n}$.*

Proof. It follows from [17] (see Proposition 3.2 and Proposition 3.4 therein) that for every $\varkappa > 0$ there exists a constant $C_3 > 0$ such that for sufficiently large $n \in \mathbb{N}$ and sufficiently large k_{n+1} ($\sigma > 0$ sufficiently small), we have $\gamma_1^{-(\chi+\varkappa)k_{n+1}} \le C_3 a_n \le C_3\sigma\lambda_{1+}^n$, where $\gamma_1 := (f_n)'_+(-1)$, $\gamma_2 := (f_n)'_-(0)$, $\chi := \frac{\ln\gamma_2}{\ln\gamma_2 + \ln\gamma_1^{-1}}$ and, thus, $k_{n+1} > \frac{n\ln\lambda_{1+}^{-1} - \ln(C_3\sigma)}{(\chi+\varkappa)\ln\gamma_1}$. Since the map f_n is exponentially close to the fractional linear map F_{a_n,v_n,c_n}, in the C^2-topology, due to (B'), and since $v_n = c_n - 1 + \mathcal{O}(a_n)$ is, by Proposition 3.2 and our assumption on a_n, exponentially close to $c_n - 1$, it follows that γ_1 and γ_2 are exponentially close to c_n^{-1} and c_n, respectively, i.e., $\gamma_1 - c_n^{-1} = \mathcal{O}(\lambda_{1+}^n)$ and $\gamma_2 - c_n = \mathcal{O}(\lambda_{1+}^n)$. Consider now an arbitrary fractional linear map F_{a_n',v_n',c_n} with $(a_n', v_n') \in \check{\mathcal{D}}_{c_n}$ and with the same height k_{n+1} as f_n. Denote its corresponding derivatives at -1 and 0 by γ_1' and γ_2', respectively. Notice that the estimates of Proposition 3.4 of [17] still hold for such a map, since the second derivative of F_{a_n',v_n',c_n} is negative and uniformly bounded. Let $\chi' := \frac{\ln\gamma_2'}{\ln\gamma_2' + \ln(\gamma_1')^{-1}}$. Using once again Proposition 3.2 and Proposition 3.4 of [17], we find that for every $\varkappa' > 0$ and $\sigma > 0$ sufficiently small

$$a_n' \le C_4(\gamma_1')^{-(\chi'-\varkappa')k_{n+1}} \le C_4(\gamma_1')^{-\frac{1}{\ln\gamma_1}\frac{\chi'-\varkappa'}{\chi+\varkappa}\left(n\ln\lambda_{1+}^{-1} - \ln(C_3\sigma)\right)} = C_4 e^{\frac{\ln\gamma_1'}{\ln\gamma_1}\frac{\chi'-\varkappa'}{\chi+\varkappa}\ln\left(C_3\sigma\lambda_{1+}^n\right)}, \quad (5.1)$$

for some $C_4 > 0$. By choosing $\sigma > 0$ small enough, we can make γ_1' and γ_2' arbitrarily close to c_n^{-1} and c_n, respectively, and thus $\frac{\ln\gamma_1'}{\ln\gamma_1}\frac{\chi'-\varkappa'}{\chi+\varkappa}$ can be made arbitrarily close to 1. Therefore, for every $\lambda_2 > \lambda_{1+}$, there exists $C_5 > 0$, such that

$$a_n' \le C_4\left(C_3\sigma\lambda_{1+}^n\right)^{\frac{\ln\gamma_1'}{\ln\gamma_1}\frac{\chi'-\varkappa'}{\chi+\varkappa}} \le C_5\lambda_2^n. \quad (5.2)$$

In particular, this is true for $a_n' = a_n^*$, i.e. the first component of $\mathcal{P}f_n$, and the first component of $\mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}$. The estimates on the second components follow from the fact that these two points belong to $\check{\mathcal{D}}_{c_n}$. **QED**

6 Renormalization and projection operators

On the set of commuting pairs in $\mathcal{D}_c$, we consider two metrics: the standard metric

$$d((a,v),(\widetilde{a},\widetilde{v})) = |a-\widetilde{a}| + |v-\widetilde{v}|. \qquad (6.1)$$

and the metric

$$d_c((x,y),(\widetilde{x},\widetilde{y})) = |x-\widetilde{x}| + |y-\widetilde{y}|. \qquad (6.2)$$

Here, the parameters (a,v) and (x,y) correspond to a pair (F,G), and $(\widetilde{a},\widetilde{v})$ and $(\widetilde{x},\widetilde{y})$ correspond to $(\widetilde{F},\widetilde{G})$.

Proposition 6.1 *Let $\lambda_{1^-} > \sqrt[8]{\lambda}$ and $\lambda_2 > \lambda/\lambda_{1^-}^8$. For all $n \in [n_1+1, n_2-1]$ either in the initial or in an ordinary sting, we have $d_{c_n}(\mathcal{P}f_n, \mathcal{R}_{c_n}\mathcal{P}f_{n-1}) = \mathcal{O}(\lambda_2^n)$.*

Proof. Let $\mathcal{R}_{c_n}(a_n^*, v_n^*) = (\bar{a}_{n+1}, \bar{v}_{n+1})$. It follows directly from (4.1) that $y_{n+1} = x_n + \mathcal{O}((\lambda/\lambda_{1^-})^n)$, for $n_1 \le n < n_2 - 1$. It follows from Proposition 4.11 and Lemma 4.9 that $\bar{y}_{n+1} = x_n^* = x_n + \mathcal{O}((\lambda/\lambda_{1^-}^6)^n)$, for $n_1 < n \le n_2 - 1$. Lemma 5.1 gives us $\bar{y}_{n_1+1} = x_{n_1}^* = x_{n_1} + \mathcal{O}(\lambda_2^{n_1})$. Therefore, we obtain $\bar{y}_{n+1} - y_{n+1} = \mathcal{O}(\lambda_2^n)$, for $n_1 \le n < n_2 - 1$. Using Corollary 4.10, we find $\bar{y}_{n+1} - y_{n+1}^* = \mathcal{O}(\lambda_2^n)$, also for $n_1 \le n < n_2 - 1$.

Recall that the slopes of the curve $\gamma_{\rho_{n+1},c_{n+1}}$ lie in the interval $(-1,0)$. On the other hand, the slopes of the straight lines $y = \bar{y}_{n+1}$ and $y = y_{n+1}^*$, in the (v,a) plane, lie outside of this interval. In what follows, we will show this claim for the first of these lines only, as the same argument applies to the second line. Notice first that the slopes of these straight lines, whose equations take the forms $a = \frac{v+1-c_{n+1}}{c_{n+1}\bar{y}_{n+1}}$ and $a = \frac{v+1-c_{n+1}}{c_{n+1}y_{n+1}^*}$, are $\frac{da}{dv} = (c_{n+1}\bar{y}_{n+1})^{-1}$ and $\frac{da}{dv} = (c_{n+1}y_{n+1}^*)^{-1}$, respectively. Since $(\bar{a}_{n+1}, \bar{v}_{n+1}) \in \check{\mathcal{D}}_{c_{n+1}}$, it follows directly from the definition of $\check{\mathcal{D}}_{c_{n+1}}$ that

$$\left|\frac{da}{dv}\right| = \left|\frac{\bar{a}_{n+1}}{\bar{v}_{n+1}+1-c_{n+1}}\right| \ge \left|\frac{c_{n+1}}{\bar{v}_{n+1}}\right| \ge \frac{1}{|c_n-1|}. \qquad (6.3)$$

If $c_n > 1$, then $\frac{da}{dv} > 0$ and, thus, the slope of the line $y = \bar{y}_{n+1}$ lies in the interval $[\frac{1}{c_n-1}, \infty)$ (in fact, it is in $[\frac{1}{c_n-1}, \frac{4c_n}{(c_n-1)^2}))$. If $c_n < 1$, then $\frac{da}{dv} < 0$ and, thus, the slope of this line belongs to the interval $(-\infty, \frac{1}{c_n-1}]$. In both cases, the slope of the line $y = \bar{y}_{n+1}$ is bounded away from the interval $(-1,0)$. The same can be said about the line $y = y_{n+1}^*$. Hence, the intersections of these lines with the curve $\gamma_{\rho_{n+1},c_{n+1}}$ are uniformly transversal, i.e., the angles between these lines and the curve at the intersection points, are greater than some positive constant. Therefore, the distance of the intersection points of these lines with the curve $\gamma_{\rho_{n+1},c_{n+1}}$, i.e. $d((\bar{a}_{n+1}, \bar{v}_{n+1}), (a_{n+1}^*, v_{n+1}^*))$, is of the order of the angle between these lines, and, thus, $d((\bar{a}_{n+1}, \bar{v}_{n+1}), (a_{n+1}^*, v_{n+1}^*)) < \Theta(|\bar{y}_{n+1} - y_{n+1}^*|)$. The latter estimate follows from the formula $\tan\bar{\theta} - \tan\theta^* = \frac{\sin(\bar{\theta}-\theta^*)}{\cos\bar{\theta}\cos\theta^*}$, where $\bar{\theta}$ and θ^* are the angles between the v axis the lines perpendicular to $y = \bar{y}_{n+1}$ and $y = y_{n+1}^*$, respectively. Finally, for $n_1 \le n < n_2 - 1$, we obtain $\bar{x}_{n+1} - x_{n+1}^* = \mathcal{O}(\lambda_2^n)$. **QED**

7 Convergence of renormalizations

On the set of renormalizable commuting pairs in $\check{\mathcal{D}}_c$, with the same irrational rotation number, the renormalization operator is Lipschitz and the two-step renormalization operator is a contraction, in the metric d_c.

Proposition 7.1 ([21]) *For every positive $c \neq 1$, there exist constants $B > 0$ and $\beta \in (0,1)$ such that for any two points (a,v) and $(\widetilde{a},\widetilde{v})$ in $\check{\mathcal{D}}_c \backslash \{(0, c-1)\}$, with the same irrational rotation number, and corresponding coordinates (x,y) and $(\widetilde{x},\widetilde{y})$, respectively, we have*

$$d_{\frac{1}{c}}(\mathcal{R}_c(\widetilde{x},\widetilde{y}), \mathcal{R}_c(x,y)) \leq B d_c((\widetilde{x},\widetilde{y}),(x,y)) \tag{7.1}$$

and

$$d_c(\mathcal{R}_{\frac{1}{c}} \circ \mathcal{R}_c(\widetilde{x},\widetilde{y}), \mathcal{R}_{\frac{1}{c}} \circ \mathcal{R}_c(x,y)) \leq \beta d_c((\widetilde{x},\widetilde{y}),(x,y)). \tag{7.2}$$

On the other hand, the C^2-norm of the distance of fractional linear maps is easily controlled by their distance in the d metric.

Proposition 7.2 *There exists $C_6 > 0$ such that if $(a,v),(\widetilde{a},\widetilde{v}) \in \mathcal{D}_c$ then we have*

$$\|F_{\widetilde{a},\widetilde{v},c} - F_{a,v,c}\|_{C^2} \leq C_6(|\widetilde{a} - a| + |\widetilde{v} - v|). \tag{7.3}$$

Proof. The claim follows from

$$F_{\widetilde{a},\widetilde{v},c}(z) - F_{a,v,c}(z) = \frac{\widetilde{a} - a}{1 - \widetilde{v}z} + \frac{(a+cz)(\widetilde{v} - v)z}{(1-vz)(1-\widetilde{v}z)}, \tag{7.4}$$

and analogous expressions for the derivatives and the second derivatives, since $(1 - vz)$ and $(1 - \widetilde{v}z)$ are bounded away from zero on $[-1, 0]$, and all other variables are bounded. **QED**

The following proposition provides a relation between the metrics.

Proposition 7.3 *There exists $K > 1$, such that for any two commuting pairs (a,v) and $(\widetilde{a},\widetilde{v})$ in $\check{\mathcal{D}}_c \backslash \{(0, c-1)\}$, on the same curve $\gamma_{\rho,c}$, and with corresponding coordinates (x,y) and $(\widetilde{x},\widetilde{y})$, we have*

$$K^{-1} d((\widetilde{a},\widetilde{v}),(a,v)) \leq d_c((\widetilde{x},\widetilde{y}),(x,y)) \leq \frac{K}{a\widetilde{a}} d((\widetilde{a},\widetilde{v}),(a,v)). \tag{7.5}$$

Proof. The first inequality follows from the second of the inverse relations

$$a = \frac{c-1}{2cy}\left(-1 + \sqrt{1 + \frac{4cxy}{(c-1)^2}}\right), \qquad v = \frac{c-1}{2}\left(1 + \sqrt{1 + \frac{4cxy}{(c-1)^2}}\right). \tag{7.6}$$

The only situation when we use the fact that both (a,v) and $(\widetilde{a},\widetilde{v})$ belong to the same curve $\gamma_{\rho,c}$ is when $\frac{4cxy}{(c-1)^2}$ is close to -1. In $\check{\mathcal{D}}_c$, this is only the case when a is close to c, v is close to $\frac{c-1}{2}$ and, thus, y is close to $\frac{1-c}{2c^2}$. The level sets of y in the (v,a) plane are straight lines with slope $\frac{1}{cy}$, which is close to $\frac{2c}{1-c}$. These slopes are bounded away from the interval $(-1,0)$ and, thus, the intersection with $\gamma_{\rho,c}(v)$ is transversal. Therefore, $|a-\widetilde{a}|$ and $|v-\widetilde{v}|$ (and, thus, $|x-\widetilde{x}|$) are at most of the same order as $|y-\widetilde{y}|$. The second inequality follows from the direct relations (4.24). **QED**

Remark 1 Without the assumption that the pairs belong to the same curve $\gamma_{\rho,c}$ in Proposition 7.3, one would have a weaker inequality $d^2((\widetilde{a},\widetilde{v}),(a,v)) \le K d_c((\widetilde{x},\widetilde{y}),(x,y))$.

Proposition 7.4 *Let $\widetilde{\lambda}_{1^-} < \lambda_1 < \widetilde{\lambda}_{1^+}$. Consider the sequences of renormalizations (f_n,g_n) and $(\widetilde{f}_n,\widetilde{g}_n)$, $n\in\mathbb{N}_0$, of any two $C^{2+\alpha}$-smooth circle maps T and $\widetilde{T}$ with a break of size c and the same irrational rotation number ρ, with corresponding parameters (a_n,v_n) and $(\widetilde{a}_n,\widetilde{v}_n)$, respectively. For any $\epsilon_4>0$, there exist $C_7,C_8>0$ such that for $n\in\mathbb{N}_0$, if $a_n\le\sigma\lambda_1^n$, for some $\sigma>0$ sufficiently small, then $\widetilde{a}_n\le C_7\sigma^{1-\epsilon_4}\widetilde{\lambda}_{1^+}^n$. If $a_n>\sigma\lambda_1^n$, then $\widetilde{a}_n>C_8\sigma^{1+\epsilon_4}\widetilde{\lambda}_{1^-}^n$.*

Proof. The proof of the first claim is similar to the proof of Lemma 5.1. Notice that similar reasoning can be applied to obtain the estimates on the parameters $\widetilde{a}_n$ associated to the renormalizations $\widetilde{f}_n$ on any $C^{2+\alpha}$-smooth circle map with a break $\widetilde{T}$ with the same rotation number as T and the same size of the break c. One obtains that, for any $\epsilon_4>0$, there exist $C_7>0$ such that, if $a_n\le\sigma\lambda_1^n$, then $\widetilde{a}_n\le C_7\sigma^{1-\epsilon_4}\widetilde{\lambda}_{1^+}^n$. The proof of the second claim is by contraposition, exchanging first the roles of T and $\widetilde{T}$. **QED**

Proposition 7.4 allows us to partition the two sequences of renormalizations for two maps T and $\widetilde{T}$, with the same irrational rotation number, into finite or infinite sequence strings $S_i := (f_{n_1(i)},\ldots,f_{n_2(i)-1})$ and $\widetilde{S}_i := (\widetilde{f}_{n_1(i)},\ldots,\widetilde{f}_{n_2(i)-1})$, with $1\le i<N$, $N\in\mathbb{N}_0\cup\{\infty\}$ and $n_2(i)=n_1(i+1)$, in such a way that, for each i, the lengths of the i-th strings are the same. More precisely, starting with some $n_0\in\mathbb{N}_0$, we can partition the sequence of renormalizations for T indexed by $n\ge n_0$ into strings S_i of lengths $n_2(i)-n_1(i)$ uniquely, by choosing $\sigma>0$ and $\lambda_{1^-}=\lambda_{1^+}=\lambda_1\in(\sqrt[8]{\lambda},1)$. We choose $\lambda_2\in(\max\{\lambda_1,\lambda/\lambda_1^8\},1)$. The sequence of renormalizations for $\widetilde{T}$ can then be partitioned into strings $\widetilde{S}_i$ of the same lengths $n_2(i)-n_1(i)$, with some $\widetilde{\sigma}>0$, $\widetilde{\lambda}_{1^-}\in(\sqrt[8]{\lambda/\lambda_2},\lambda_1)$ and $\widetilde{\lambda}_{1^+}\in(\lambda_1,\lambda_2)$.

Lemma 7.5 *There exists $C_9>0$ such that, for $\lambda_3\in(\max\{\beta^{1/2},\lambda_2\},1)$ and for every $n\in[n_1+1,n_2-1]$ either in the initial or in an ordinary string, we have*

$$d_{c_n}(\mathcal{P}\widetilde{f}_n,\mathcal{P}f_n)\le C_9\lambda_3^n. \tag{7.7}$$

Proof. Using the triangle inequality, we find

$$\begin{aligned} d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n) \le d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}) + d_{c_n}(\mathcal{P}f_n, \mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}) \\ + d_{c_n}(\mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}, \mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}). \end{aligned} \tag{7.8}$$

It follows from Proposition 6.1 that, for every n either in the initial or in an ordinary string such that $n_1 < n < n_2$, we have $d_{c_n}(\mathcal{P}f_n, \mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}) \le C_{10}\lambda_2^n$ and $d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}) \le C_{10}\lambda_2^n$, for some $C_{10} > 0$, assuming that we have chosen $\widetilde{\lambda}_{1^-} > \sqrt[8]{\lambda/\lambda_2}$ and $\widetilde{\lambda}_{1^+} < \lambda_2$. Applying (7.8) recursively, in a string of more than two renormalizations, and using Proposition 7.1, we obtain

$$\begin{aligned} d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n) \le\ & 2C_{10}\lambda_2^n + d_{c_n}(\mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}\widetilde{f}_{n-2}, \mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}f_{n-2}) \\ + d_{c_n}(\mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}, \mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}\widetilde{f}_{n-2}) & + d_{c_n}(\mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}, \mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}f_{n-2}) \\ \le\ & 2(1 + B\lambda_2^{-1})C_{10}\lambda_2^n + d_{c_n}(\mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}\widetilde{f}_{n-2}, \mathcal{R}_{c_{n-1}} \circ \mathcal{R}_{c_{n-2}}\mathcal{P}f_{n-2}) \\ \le\ & 2(1 + B\lambda_2^{-1})C_{10}\lambda_2^n + \beta d_{c_n}(\mathcal{P}\widetilde{f}_{n-2}, \mathcal{P}f_{n-2}). \end{aligned} \tag{7.9}$$

By iterating the resulting inequality, we obtain

$$\begin{aligned} d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n) \le 2(1 + B\lambda_2^{-1})C_{10}\sum_{i=0}^{k-1}\lambda_2^{n-2i}\beta^i + \beta^k d_{c_n}(\mathcal{P}\widetilde{f}_{n-2k}, \mathcal{P}f_{n-2k}) \\ \le C_{11}\lambda_4^n + \beta^k d_{c_n}(\mathcal{P}\widetilde{f}_{n-2k}, \mathcal{P}f_{n-2k}), \end{aligned} \tag{7.10}$$

for some $\lambda_4 > \max\{\beta^{1/2}, \lambda_2\}$ and $C_{11} > 0$. If $d_{c_n}(\mathcal{P}\widetilde{f}_{n-2k}, \mathcal{P}f_{n-2k}) \le C_9\lambda_3^{n-2k}$, $n_1 < n - 2k < n < n_2$, for some $\lambda_3 > \lambda_4$ and $C_9 > 0$, then $d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n) \le \lambda_3^n(C_{11}(\lambda_4/\lambda_3)^n + C_9(\sqrt{\beta}/\lambda_3)^{2k}) \le C_9\lambda_3^n$, if C_9 is large enough.

To complete the proof by induction, we need to verify that the estimates are also true for $n = n_1 + 1$ and $n = n_1 + 2$, in an ordinary string. In the initial string, the initial estimates are certainly satisfied, for some large n_1' and $n_1' + 1$, if C_9 is chosen sufficiently large. For an ordinary string and $n = n_1 + 1$ (if smaller than n_2), we have from (7.8) and Proposition 6.1 that $d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n) \le 2C_{10}\lambda_2^n + d_{c_n}(\mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}, \mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1})$. Using similar reasoning as in the proof of Proposition 6.1, we find $d_{c_n}(\mathcal{R}_{c_{n-1}}\mathcal{P}\widetilde{f}_{n-1}, \mathcal{R}_{c_{n-1}}\mathcal{P}f_{n-1}) \le C_{12}|\bar{\widetilde{y}}_n - \bar{y}_n| = C_{12}|\widetilde{x}_{n-1}^* - x_{n-1}^*| \le C_{13}\lambda_2^n$, for some $C_{12}, C_{13} > 0$. The equality follows from Proposition 4.11. In the last inequality, we have used Lemma 5.1 and Proposition 7.4. For $n = n_1 + 2$ (if smaller than n_2), the claim follows from (7.8), using the estimate on $d_{c_{n_1+1}}(\mathcal{P}\widetilde{f}_{n_1+1}, \mathcal{P}f_{n_1+1})$, Proposition 6.1 and Proposition 7.1. **QED**

Proof of Theorem 1.1. Using the triangle inequality and Proposition 7.2, we find

$$\|\widetilde{f}_n - f_n\|_{C^2} \le \|\widetilde{f}_n - \mathcal{P}\widetilde{f}_n\|_{C^2} + \|f_n - \mathcal{P}f_n\|_{C^2} + C_6 d(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n). \tag{7.11}$$

For n belonging either to the initial or to an ordinary string such that $n_1 < n < n_2$, we have $\|f_n - \mathcal{P}f_n\|_{C^2} \le C_{14}(\lambda/\lambda_1^6)^n$, for some $C_{14} > 0$, as follows from property (B') and Lemma 4.9. Therefore, for some $C_{15} > 0$, we have

$$\|\widetilde{f}_n - f_n\|_{C^2} \le C_{15}(\lambda/\lambda_1^6)^n + C_{15}(\lambda/\widetilde{\lambda}_{1-}^6)^n + KC_6 d_{c_n}(\mathcal{P}\widetilde{f}_n, \mathcal{P}f_n). \tag{7.12}$$

Using Lemma 7.5, we obtain, from this estimate, for some $C_{16} > 0$,

$$\|\widetilde{f}_n - f_n\|_{C^2} \le C_{16}\lambda_3^n. \tag{7.13}$$

It remains to prove the same estimate for $n = n_1$ in every ordinary sting. For $n = n_1$, this estimate follows directly from

$$\|\widetilde{f}_n - f_n\|_{C^2} \le \|\widetilde{f}_n - \widetilde{F}_n\|_{C^2} + \|f_n - F_n\|_{C^2} + C_6 d((\widetilde{a}_n, \widetilde{v}_n), (a_n, v_n)), \tag{7.14}$$

using property (B'), Proposition 7.4 and Proposition 3.2 (together with the definition (2.14)).

It follows from Proposition 3.7 and the estimates above that the constant $\mu = \lambda_3$ can be chosen uniformly. It depends only on the size of the break, c, and does not depend on the rotation number of the maps. **QED**

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