DISJOINT PATHS IN PLANAR GRAPHS

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS . . . . . **iii**

LIST OF FIGURES . . . . . **vi**

SUMMARY . . . . . **ix**

I INTRODUCTION **1**

1.1 Background 1

1.1.1 Graph definitions 1

1.1.2 Algorithmic complexity 4

1.2 Applications 5

1.2.1 Multicommodity flows 5

1.2.2 General linkages 7

1.2.3 Graph minors 9

1.3 (2,3)-linkages 10

II OBSTRUCTIONS AND REDUCTIONS **13**

2.1 Terminology and Notation 13

2.2 Obstructions 15

2.3 Reductions 22

2.4 Main Theorem 26

III MAXIMIZING PATHS **27**

3.1 Defining a maximizing path 27

3.2 Examining path bridges 33

3.3 Location of X relative to a 35

3.4 Location of X relative to path 43

3.5 Separating X from a 61

3.5.1 Defining common paths 62

3.5.2 The case $t' = x_1$. 65

3.5.3 The case that T falls to the left of P 71

IV SPECIAL CASE $|X| = 2$ 92

4.1 Obstructions 92

4.2 Theorem 2 93

4.3 A special case 94

4.4 Proof of Theorem 2 96

4.5 1-connected case 97

V PROOF OF GENERAL CASE 100

5.1 Independent inductive cases 100

5.2 Vertex of X on outer face 106

5.3 Remaining inductive cases 117

5.4 Proof of Theorem 1 139

REFERENCES 145

LIST OF FIGURES

1 Complete and complete bipartite examples 2
2 Obstructions in $\mathcal{O}_1$ 11
3 Example of 3-separation (G_1, G_2) 14
4 Obstructions in $\mathcal{O}_1$ 16
5 Obstructions in $\mathcal{O}_2$ and $\mathcal{O}_3$ 16
6 Obstructions in $\mathcal{O}_4$ 19
7 Obstructions in $\mathcal{O}_5$ 20
8 Obstruction in $\mathcal{O}_6$ 21
9 Obstructions in $\mathcal{O}_7$ 22
10 Graphs satisfying property (R1) 23
11 Graphs satisfying property (R2) 25
12 The subgraph L_P (shaded), where P is from a_1 to x 28
13 Bridges without L_P attachment 30
14 Where $b_1, b_2 \subseteq V(U_P)$ 32
15 Where $X \setminus \{x_1\}$ is not contained in $V(L_P)$ 34
16 Lemma 3.7, Claim 1 36
17 Lemma 3.7, Claim 2 37
18 Lemma 3.7, choice of path W 38
19 Lemma 3.7, Claim 4 39
20 Lemma 3.7, Claim 5 41
21 Lemma 3.7 final cases 42
22 Establishing separations when special paths exist 44
23 x_2 cofacial with a_1 46
24 Separations when $b_1 \in V(D)$ 50
25 Separations when s' not cofacial with b_2 56
26 Combinations of separations, Case (5.1) 58
27 Case (5.2) 59
28 Combinations of separations, Case (5.2) 61
29 Initial path definitions 63

30 Defining path R 64
31 Where $t_0 = b_2$ 66
32 Where path O_x does not exist 67
33 Forcing crossed 3-separations when $t' = x_1$ 68
34 Cases of crossed 3-separations 70
35 Where a_2 is a cutvertex of L_Q 71
36 Defining $Z'(U)$ 72
37 Where x_1 not in L_Q 74
38 Where $x_1 \in M_Q$ 76
39 Defining separation (H_z, M_z) 77
40 Example of $z_2 \neq b_1$, not cofacial s or s' 79
41 2-cut in L_Q corresponding to 3-cut in L_P 81
42 Where $\{w_2, s'\} \subseteq V(C)$. 84
43 Where S_0 and T_0 both exist 88
44 Where $s' = a_1$ 89
45 $x_1 \in V(C), \quad (\Gamma, \{x_1, x_2\}) \in \Theta_4$ 95
46 Condition (4) 98
47 Subgraph definitions for $(\Gamma, X') \in \mathcal{O}_5$ 101
48 Case 1, $x_1 \in V(H)$ 102
49 $(\Gamma, X') \in \mathcal{O}_6$ 105
50 $x_1 \in V(C)$, $X' \subset \overline{Int}(D)$. 107
51 $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, separation (O_k, N_k). 111
52 $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, leading to $(\Gamma, X) \in \mathcal{O}_5$. 112
53 $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, $(\Gamma, X') \in \mathcal{O}_3$. 113
54 $X_2 = \emptyset$, $(\Gamma, X') \in \mathcal{O}_4$. 116
55 Where $X_2 = \emptyset$, $(\Gamma, X') \in \mathcal{O}_7$. 117
56 Separation with $X' \nsubseteq V(L_P)$ 118
57 Example of $(\Gamma, X') \in \mathcal{O}_1$ with $X' \nsubseteq V(L_P)$ 120
58 Example of $(\Gamma, X') \in \mathcal{O}_1$ with $X' \subseteq V(M_P)$ 122
59 $(\Gamma, X') \in \mathcal{O}_1$, $X' \subseteq V(M_P)$, $(\Gamma, \{x_1, x_2\})$ reducible 125
60 $(\Gamma, X') \in \mathcal{O}_2$ 129

61 Where $(\Gamma, X') \in \mathcal{O}_3$ 132
62 Examples of $(\Gamma, X') \in \mathcal{O}_7$ 135
63 Examples of $(\Gamma, X') \in \mathcal{O}_7$ 137
64 $(\Gamma, X') \in \mathcal{O}_7$ with (Γ, X_3) reducible 138
65 (Γ, X') satisfies (R1), $a'_1 = a_1$ 141
66 Disjoint paths in H when $\{a_1, a_2\} \cap V(K) = \emptyset$ 143
67 When $x_1 \in V(P_i)$ 144
68 When $x_1 \in V(Q_1)$ 144

SUMMARY

Disjoint paths in graphs, and disjoint subgraphs, have applications in such areas as circuit design and multicommodity flows. They also lie at the heart of graph minor questions. Here we consider a question of finding disjoint paths connecting specific subsets of vertices in a planar graph. The specific form of that question is motivated by a problem of Robertson and Seymour on $(2,3)$-linkages.

Consider a plane graph G, an ordered set $\{a_1, b_1, a_2, b_2\}$ of vertices lying on the outer boundary of G, and a vertex set X disjoint from $\{a_1, b_1, a_2, b_2\}$. We ask when there exist three disjoint paths: one between the vertices b_1 and b_2, one from a_1 to the set X, and one from a_2 to X. We establish a good characterization of those graphs which do not contain such paths, called obstructions. The proof is constructive, implying an algorithm which either produces the desired paths or identifies G as an obstruction.

The introductory chapter provides the basic definitions needed to understand the main result. It also briefly discusses some applications giving rise to disjoint path questions. We then describe a related graph structure question, regarding linkages, to which this result contributes.

The remainder of the document is devoted to the details of the main result. We first define obstructions, those graphs which do not have the desired paths. We then describe graph reductions, which reduce the disjoint path question on a graph to the same question on some smaller subgraph. Several technical tools and lemmas are then established. A special case of the main result is proven, where the special set X includes just two vertices. Finally, this special case is applied together with the technical lemmas to prove the general case.

CHAPTER I

INTRODUCTION

1.1 Background

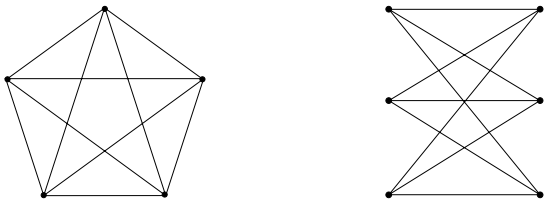
This section provides the basic background in graph theory needed to understand our main result. We define graphs, and paths in graphs. We briefly consider the ideas of connectivity and cutsets as they relate to the existence of disjoint paths. We also review some basic concepts of algorithmic complexity, which will be helpful when describing previous results in the field.

1.1.1 Graph definitions

A *graph* consists of a set of vertices and a set of edges, where each edge is an unordered pair of vertices. Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. All graphs considered here will have finite vertex and edge sets. Unless otherwise specified, they will be *simple*, meaning that each pair of vertices appears at most once in the edge set, and that the two vertices of each pair are distinct. Where $u, v \in V(G)$, we say u is *adjacent* to v if there is an edge $uv \in E(G)$. We may also call v a *neighbor* of u. The edge uv is *incident* with each of the vertices u and v.

We represent graphs visually by drawing them on some surface. The vertex set is represented by points, and an edge is represented by a line connecting the two vertices with which it is incident. We call a graph *planar* if it can be drawn in the plane in such a way that no edges cross one another. A *plane* graph is one which is so drawn.

Two specific classes of graphs which will be mentioned in our discussion are complete graphs and bipartite graphs. A graph is *complete* if every distinct pair of its vertices is included in its edge set. We denote by K_n the complete graph on n vertices. A graph G is *bipartite* if its vertices can be partitioned into two disjoint sets A and B such that $A \cup B = V(G)$, and for any edge uv we have either $u \in A, v \in B$ or $u \in B, v \in A$. A



(a) K_5 (b) $K_{3,3}$

Figure 1: Complete and complete bipartite examples

complete bipartite graph G is a bipartite graph with vertex partition $A \cup B = V(G)$ such that for every vertex pair $a \in A$, $b \in B$, we have the edge $ab \in E(G)$. The graph $K_{m,n}$ is the complete bipartite graph with partition sets of size m and n respectively. Figure 1 shows the examples K_5 and $K_{3,3}$.

A *path* in a graph can be thought of as a way of traveling from one vertex to another by traversing edges of the graph. We require that no vertex be revisited in the course of this traversal. Rigorously, a path is an ordered list of alternating vertices and edges, beginning and ending with vertices, $v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, e_n, v_n$, where each edge e_i is incident with v_{i-1} and v_i, and no vertex appears more than once in the list. Since we are dealing with simple graphs only, a path can be determined completely by an ordered list of vertices. Where v_0, v_1 appear consecutively in the ordered list, we know the edge v_0v_1 is included. Where v_0 and v_n are the first and last vertices of the path, we say the path is *from* v_0 *to* v_n. Note that since we are dealing with undirected graphs, a path from v_0 to v_n is associated with a path from v_n to v_0 in the obvious way. In most cases, we will make no distinction between the two.

Paths may also be considered as special cases of *subgraphs*. We say a graph H is a subgraph of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Suppose we have two different subgraphs, $H_1, H_2 \subseteq G$. We call these *disjoint* if they have no vertices in common. That is, $V(H_1) \cap V(H_2) = \emptyset$. This definition of disjoint is sometimes called *vertex-disjoint*. One could also consider *edge-disjoint* subgraphs, requiring $E(H_1) \cap E(H_2) = \emptyset$. Notice that if two subgraphs are vertex-disjoint, they are necessarily edge-disjoint.

Thinking of a path as a way "to get there from here" in a graph, it is natural to ask when a path exists between two vertices. We call a graph G *connected* if for each pair of vertices $u, v \in V(G)$, there is some path from u to v in G. We may further ask whether there are multiple paths from u to v. In order to make sense of this question, we need to be clear about what we mean by multiple paths. Suppose we have many paths from u to v, but every one of them passes through vertex x. Then removal of vertex x destroys all paths from u to v, and $G-x$ is disconnected. (Here $G-x$ is the graph formed from G by removal of vertex x, and all edges incident with x.) We might think of paths as subgraphs, and say that two paths are independent if they are disjoint. If we want their end vertices to be the same, however, we need a modification. Given a path P from u to v, let us call all vertices in $V(P)$ except u and v *internal* vertices of P. Two paths P and Q will be called *internally disjoint* if they share no internal vertices. (Some authors call this *openly disjoint*.)

We now extend the definition of connectivity, and consider its implications for disjoint paths. A *cutset* of G is a set of vertices $T \subseteq V$ whose removal produces a disconnected graph. If T is a cutset with $|T| = k$, it is called a *k-cut* in G. A graph G is called *n-connected* if $|V(G)| \geq k+1$, and G has no k-cut with $k < n$. A classic result of Menger shows that G is k-connected if and only if it contains k internally disjoint paths between any two vertices. (This formulation is due to Whitney (1932).)

Menger's theorem actually tells us more than this. Given vertex sets $A, B \subseteq V(G)$, we call any path from $a \in A$ to $b \in B$ an A-B path. If $T \subseteq V$ is such that every A-B path includes some vertex of T, then T is said to *separate* A from B. Menger's theorem tells us that the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint paths from A to B.

The existence of multiple disjoint paths in a graph is certainly intimately related to high connectivity. If we ask for paths in some specific configuration, however, the exact relationship is not obvious. Suppose, for example, we have disjoint set $A, B \subseteq V(G)$, each of size k. If no set of less than k vertices separates A from B in G, then there are k disjoint paths from A to B in G. This gives us no control, however, over which vertices are connected. Do we have a path from $a_1 \in A$ to $b_1 \in B$? Or can we get k disjoint paths

only if we include a path from a_1 to b_2? The question of finding paths connecting specific subsets or pairs of vertices is what we will consider here.

1.1.2 Algorithmic complexity

When considering applications, we will be concerned with not only knowing whether disjoint paths exist, but actually finding them. Where algorithms are available to do this, we will want some way to measure their efficiency. To this end, we consider several ideas from complexity theory. A complete treatment of the definitions presented here can be found in [2].

The running time of an algorithm is measured by counting the number of operations performed. This assumes that we have a consistent definition of "operation". We generally consider worst-case running times. We use the notation $f(n) = \mathcal{O}(g(n))$ to mean that the asymptotic growth of $f(n)$ is bounded above by that of $g(n)$. That is, there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. We say an algorithm is *polynomial time* if the worst case running time is $\mathcal{O}(n^r)$ for some fixed constant r, where n is the size of the input.

We say a problem is polynomially solvable if there is some polynomial time algorithm which will solve any instance of that problem. The complexity class P is the set of all problems which are polynomially solvable.

The complexity class NP (nondeterministic polynomial time) is a bit more difficult to describe concisely. This applies to problems for which a proposed solution can be verified in polynomial time, but no algorithm is known to find a solution in polynomial time. Suppose for example that we are given vertex lists in a graph, with the claim that they represent disjoint paths in that graph. We can verify the claim quite easily. This is distinct from being able to find sets of disjoint paths when given a graph.

An *NP-complete* problem is in some sense a representative of the class NP. The complexity class of a problem is often determined by a technique called *reduction*. If we can reduce problem Y to problem X, then any algorithm which will solve X can be modified to solve problem Y. A problem X is NP-complete if every other problem in NP can be

reduced to X. If we were to find a polynomial time algorithm to solve any NP-complete problem, we would be able to solve all NP-complete problems in polynomial time. This would show $P = NP$, a conclusion there is extremely strong evidence against.

In most cases, we show a problem is in P by exhibiting a polynomial time algorithm to solve it. However, results in the field of graph minors (specifically, the excluded minor results of Robertson and Seymour [13]) sometimes allow us to state that a polynomial time algorithm does exist, without specifying what that algorithm may be.

1.2 Applications

We now consider some of the applications of disjoint paths and disjoint subgraphs. Our discussion of physical applications is limited to multicommodity network flows, but this is certainly not the only example. Disjoint paths are highly studied in the field of VLSI (very large systems integration), for instance, as they have clear application in the design of circuit layouts. Most physical examples of disjoint path problems, however, are conceptually similar to the flows we describe.

Our main result, somewhat technical in nature, has direct applications within graph theory. We discuss these in the form of linkages and graph minors.

1.2.1 Multicommodity flows

Network flows are a well-studied topic in the field of optimization, and have become a fairly standard example of a physical application of graph theory concepts. Here a graph is used to model a physical network, where something is being transported from node to node (vertex to vertex), along pathways defined by the graph's edges. This may be used to model such systems as electrical networks, telecommunications, or the transportation of physical goods, whether within a single production plant or across a national distribution system.

Networks are generally modeled by *directed* graphs. In a directed graph, each edge is an ordered pair of vertices. An edge (u, v) is thought of as an edge from u to v. A *network* is a directed, connected graph $G = (V, E)$ with two special nodes s and t, which we will call the source and the sink respectively. Each edge is assigned a non-negative capacity, $c(u, v)$.

The source node generally has only outgoing edges, and the sink only incoming edges. A *flow* on a network is a function $f : E \to \mathbb{R}^+$ satisfying two sets of conditions:

1. For each $(u, v) \in E$, $f(u, v) \leq c(u, v)$.

2. For each fixed $v \in V \setminus \{s, t\}$, $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,w)\in E} f(v, w)$.

The first condition simply requires that the flow on each edge not exceed the capacity of that edge. The second ensures conservation of flow at each node. If we think of the flow as representing the volume of material moving along a transportation network, we may see material as being produced at the source, consumed at the sink, and moving without loss through all internal nodes. The *value* of a flow is simply $\sum_{(s,v)\in E} f(s, v)$, or equivalently $\sum_{(v,t)\in E} f(v, t)$, the net quantity being transported from source to sink.

The usual question asked in the context of network flows is, given a particular network, what is the flow of maximum value? Or, generally more useful in practice, what is the maximum integer-valued flow? (In distributing a product, for example, where we have a some predetermined shipping unit.) When asking for integer-valued flows, one generally restricts capacities to integer values as well. There are now fairly standard algorithms which can determine the maximum flow on a network in $O(|V|^3)$ time.

A *multicommodity* flow extends the standard network flow formulation to a network with multiple sources and sinks. We again have a directed graph $G = (V, E)$, now with $\{s_1, \ldots, s_k, t_1, \ldots t_k\} \subseteq V$, constituting a source s_i and sink t_i for each of k commodities. The edges of E are considered *capacity* edges. A *demand edge* (t_i, s_i) is added between each sink/source pair, with a demand (or request) value $r(s_i, t_i)$. Call the set of demand edges D. In principal, we might think of a flow as a multivalued function $F : E \cup D \to \mathbb{R}^k$, where each coordinate F_i represents the flow of commodity i, again subject to capacity and conservation conditions. The technical formulation used is actually a bit different, as we now describe.

We wish to find a collection $\mathcal{C}$ of circuits in G, and a function $F : \mathcal{C} \to \mathbb{R}^+$, satisfying the following conditions:

1. Each $C \in \mathcal{C}$ contains exactly one demand edge.

2. For each $d \in D$, $\sum_{d \in C \in \mathcal{C}} F(C) = r(d)$.

3. For each $e \in E$, $\sum_{e \in C \in \mathcal{C}} F(C) \leq c(e)$.

We can think of each circuit as representing the flow of a single commodity. This is established by the first condition above. The second condition ensures that the demand for each commodity is met, and the third that no edge capacities are exceeded.

If we consider an integer-valued multicommodity flow problem where each edge capacity and each demand is 1, we have the *edge-disjoint path* problem. Since each edge can carry only one commodity, any flow will specify a collection of k paths, each from s_i to t_i, where no pair of paths has any edge in common. Conversely, any collection of such paths defines a flow. If we further require that at most one commodity may pass through each node, we have the *vertex-disjoint path* problem. This is equivalent to the *k-linkage* problem, which will be discussed in Section 1.2.2.

Sebo [16] gives a polynomial time algorithm which determines the integer solvability (i.e., whether a flow exists) when the graph representing capacity and demand is planar, and the number of commodities is bounded by a predetermined integer. While polynomial time algorithms are available under some restrictions, no general characterization for solvability of multicommodity flows is known.

1.2.2 General linkages

Given a graph G, consider an ordered set of distinct vertices $X = \{s_1, \ldots, s_k, t_1, \ldots t_k\} \subseteq V(G)$, which we will call *terminals*. A collection of disjoint paths $P_1, \ldots, P_k$ in G, with each P_i from s_i to t_i, is called an *X-linkage*, or a *k-linkage*. Suppose G is a graph with at least $2k$ vertices, and for every set X of $2k$ terminals, G has an X-linkage. Then G is called k-linked.

Here we recall our earlier discussion on graph connectivity. Suppose we are given a graph G and sets $S = \{s_1, \ldots, s_k\}, T = \{t_1, \ldots t_k\} \subseteq V(G)$. If G is k-connected, then by Menger's theorem we have k disjoint paths from S to T. This does not ensure that there is an X-linkage. High enough connectivity, though, does ensure that a graph is k-linked. Results by Jung [6] and Larman [8], for example, show that $2k$-connected graphs

with certain subgraphs are indeed k-linked. A result of Mader [10] shows that the necessary subgraph can be guaranteed by some minimal edge density. High edge density can in turn be guaranteed by high connectivity. So there is some function $f(k) : \mathbb{N} \to \mathbb{N}$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason give the first linear bound on this $f(k)$ in [1]. They show that any graph which is $22k$-connected is k-linked.

Can we easily determine, given a specific instance of a graph G, whether G is k-linked? Karp showed in 1975 [7] that this question is NP-complete. Lynch [9] showed that the question remains NP-complete even if we restrict our input to graphs which are planar. Middendorf and Pfeiffer [11] extended this to a further restricted class, where the graph formed from G by the addition of an edge between s_i and t_i for each terminal pair is planar. These results all consider k as part of the input.

When k is fixed, the algorithmic outlook is more hopeful. In [14], Robertson and Seymour describe an algorithm which for fixed k has running time $O(|V(G)|^3)$, and determines whether a graph G has an X-linkage for specified $X = \{s_1, \ldots, s_k, t_1, \ldots t_k\} \subseteq V(G)$. Unfortunately, this algorithm is "not practically feasible, since it involves the manipulation of enormous constants". In [12], Reed et al. show that there is a linear time algorithm answering this question for planar graphs, where again k is fixed. This result can be extended to any fixed compact surface.

The linear time algorithm of Reed et al. actually addresses a more general question. It determines whether, given vertex sets $X_1, X_2, \ldots X_k \subset V(G)$, there exist disjoint trees $T_1, \ldots T_k$ in G, with each $X_i \subseteq V(T_i)$. A *tree* here is a connected graph for which each vertex is a cutvertex. (An equivalent definition of a tree is a connected graph which contains no cycles.) Disjoint paths are special cases of disjoint trees, where $|X_i| = 2$ for each $i = 1, \ldots, k$.

Polynomial time algorithms do not extend to directed graphs, where the question of the existence of a k-linkage remains NP-complete even when k is fixed at 2. In the remainder of our discussions, then, we will restrict ourselves to undirected graphs. In the case of multicommodity flows, this corresponds to an assumption that edges appear in pairs (u, v) and (v, u), so that there may be flow in either direction between two adjacent vertices.

We have seen that there are some known sufficient conditions for a graph to be k-linked. No full characterization of k-linked graphs is known. We have also seen that there are algorithms for determining whether specific linkages exist in a given graph. These algorithms generally construct the desired linkage, or show that it does not exist. The lemma below (Lemma 1.1) demonstrates the type of characterization we might hope for, in this case for 2-linkages. This result, now well-known, is proven by Thomassen in [18]. It was also obtained (but stated without proof) by Seymour [17]. The conditions described in (2) can be thought of as planar embedding "modulo 3-cuts".

Lemma 1.1. *Given a 2-connected graph G and vertices $s_1, s_2, t_1, t_2 \in V(G)$, then either*

1. *there are disjoint paths from s_1 to t_1 and from s_2 to t_2, or*
2. *for some $n \geq 0$ there are pairwise disjoint sets $A_1, \ldots, A_n \subseteq V(G) \setminus \{s_1, s_2, t_1, t_2\}$ such that*
   (a) *for $i \neq j$, no vertex of A_i has a neighbor in A_j,*
   (b) *each A_i has at most three neighbors in $V(G) \setminus A_i$, and*
   (c) *where G' is the graph formed from G by, for each $1 \leq i \leq n$, removing the vertices of A_i and adding edges between each pair of neighbors of A_i, G' can be embedded in a disk with s_1, s_2, t_1, t_2 appearing in order along the boundary of the disk.*

Here we are not concerned with exhibiting a specific linkage. We do have a very simple description of the structure which obstructs the existence of a linkage. This has proven useful in examining other graph structure properties. Our approach to $(2,3)$-linkages, for example, will rely on this lemma as a basic tool.

1.2.3 Graph minors

Given a graph G, a graph H is called a *minor* of G if it can be obtained from a subgraph of G by contracting edges. Contraction of an edge $uv \in E(G)$ is performed by replacing the two vertices u and v with a single new vertex, adjacent to each neighbor of u or v.

Perhaps the most widely known result on graph minors is Kuratowski's Theorem: A graph is planar if, and only if, it does not contain K_5 or $K_{3,3}$ (Figure 1) as a minor. (The

statement in terms of minors is actually by Wagner (1937). Kuratowski's version (1930) is in terms of subdivisions.) Graph minors in general are closely related to questions of surface embeddings. For example, within their extensive series of papers on graph minors [13], Robertson and Seymour have shown that for any surface, there is a finite collection of graphs such that every graph not embeddable in the surface contains one member of this collection as a minor. Graph minors also have applications to graph coloring and to many other structural graph properties.

Where H is a minor of the graph G, vertices of H are realized as disjoint connected subgraphs of G, and edges of H as internally disjoint paths between those subgraphs. So when we ask questions about graph minors, we are really asking about the existence of particular disjoint subgraphs and paths.

1.3 (2,3)-linkages

The specific disjoint path problem considered in this thesis is motivated by the work of Robertson, Seymour, and Thomas addressing Hadwiger's conjecture. The conjecture, made in 1943 in [5], concerns graph minors and colorings. A graph is said to be t-colorable if one can color its vertices in such a way that no two vertices of the same color are adjacent, using at most t colors.

Hadwiger Conjecture. For every integer $t \geq 0$, every loopless graph with no K_{t+1} minor is t-colorable.

This is a generalization of the famous four-color theorem, stating that all planar graphs are 4-colorable, since no planar graph contains a K_5 minor. The Hadwiger conjecture is known to be true for $t \leq 3$, and is equivalent to the four-color theorem for $t = 4$. In [15], Robertson et al. show that every minimal counterexample to the Hadwiger conjecture for $t = 5$ is an apex graph. That is, removal of a single vertex (the apex) leaves a planar graph. Hence the Hadwiger conjecture for $t = 5$ is also equivalent to the four-color theorem. The details of their proof, carefully examining the structure any minimal counterexample must have, led them to consider the following question: Given a graph G and vertices $a_0, a_1, a_2, b_1, b_2 \in V(G)$, when does G have disjoint connected subgraphs A and B with

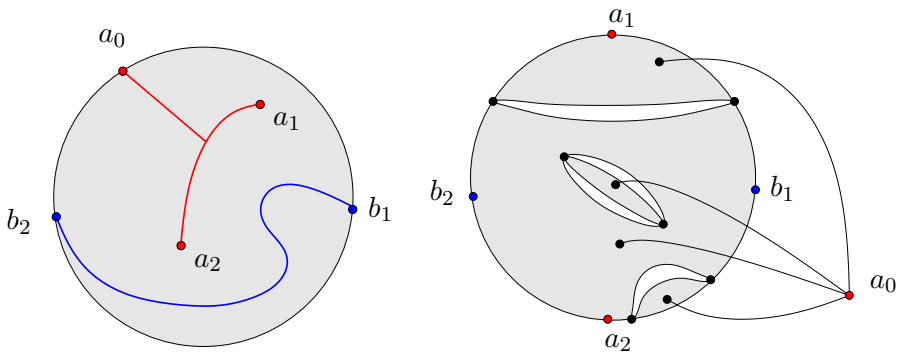


Figure 2: Obstructions in $\mathcal{O}_1$

$\{a_0, a_1, a_2\} \subseteq V(A)$ and $\{b_1, b_2\} \subseteq V(B)$? We will call such a configuration, if it exists, a $(2,3)$-*linkage* (see Figure 2(a)).

The results of this thesis can be implemented in characterizing when a $(2,3)$-linkage exists. We briefly outline an approach to that proof here. Given G and $a_0, a_1, a_2, b_1, b_2 \in V(G)$, consider the graph $G' = G - a_0$. We ask first whether G' has disjoint paths P from a_1 to a_2 and Q from b_1 to b_2. If these paths do exist, contract path P in G to a single vertex, forming the graph $\tilde{G}$. Then ask for disjoint paths from a_0 to P and from b_1 to b_2 in $\tilde{G}$. In both of these steps, we are asking for 2-linkages. From Lemma 1.1 we have a complete characterization of those graphs which do not have the desired 2-linkages. This can be extended to a characterization of those graphs which do not have a $(2,3)$-linkage.

If the paths P and Q do not exist in G', then by Lemma 1.1, G' can be embedded in the plane, modulo 3-cuts, with a_1, b_1, a_2, b_2 appearing in order on its outer cycle. If we have a plane embedding of G' then, with the addition of the vertex a_0, we see that G is an apex graph (see Figure 2(b)). Let X be the set of neighbors of a_0. We want disjoint paths in G': from a_1 to X, from a_2 to X, and from b_1 to b_2. So if we have a characterization of when these paths exist in G', we can determine when a $(2,3)$-linkage exists in G.

Notice that if the graph G is 3-connected, the graph G' is 2-connected. Suppose there is a 2-cut Z in G'. Each component of $G' - Z$ must contain some vertex of X. Keeping in mind that we want a path from a_i to X, we see that if a component of $G' - Z$ includes none of a_1, b_1, a_2, b_2, we need not consider the details of the graph structure in that component. If

we can find a path from a_i to one of the vertices of Z, then we can find a path from a_i to any vertex of X contained in that component. We know from Lemma 1.1 that we will not make use of more than one vertex of X in that component, since this would not allow for a path from b_1 to b_2. These observations will motivate our use of $(3, \{a_1, b_1, a_2, b_2\})$-connectivity in the definition of disk graphs, Definition 2.2.

CHAPTER II

OBSTRUCTIONS AND REDUCTIONS

2.1 Terminology and Notation

All graphs considered here will be finite and simple. Where there is no confusion, we may use $x \in G$ to mean $x \in V(G)$ or $x \in E(G)$ for any graph G. Given two graphs G_1 and G_2, we define $G := G_1 \cup G_2$ to be the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

When G is a connected graph which is not 2-connected, we can describe the structure of G using Tutte's *block graph* construction [19]. A *cut edge* is any edge $e \in E(G)$ whose removal disconnects G. A *block* of G is a maximal 2-connected subgraph, or a subgraph induced by a cut edge. We construct a graph $Blk(G)$ such that $V(Blk(G)) = \{\text{cutvertices } u_i \text{ of } G\} \cup \{\text{blocks } B_j \text{ of } G\}$, and u_i is adjacent to B_j in $Blk(G)$ if $u_i \in V(B_j)$ in G. We call a block B of G an *endblock* of G if B contains at most one cutvertex of G. If the graph $Blk(G)$ is a simple path, we say that G is a *chain* of blocks. Note that for a connected graph G this is equivalent to the condition that each block of G contains at most two cutvertices of G, and each cutvertex of G is contained in at most two blocks of G.

Let P be a path in a graph G, and let $u, v \in V(P)$; then $P[u, v]$ denotes the subpath of P between u and v, inclusive. $P(u, v)$ denotes the same subpath with its endpoints excluded, and $P(u, v], P[u, v)$ denote the subpath with the obvious endpoint excluded. Furthermore, if G is a plane graph and C is a cycle in G, then for any distinct $u, v \in V(C)$, we use $C[u, v]$ to indicate the subpath of C from u to v, inclusive, which follows the clockwise orientation of C. Define $C(u, v),\ C(u, v],\ C[u, v)$ in the obvious way.

Given vertex sets X, Y in a graph G, we say a path P is a path from X to Y in G if P has endpoints $x \in X$ and $y \in Y$, and is internally disjoint from $X \cup Y$. That is, $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$.

Given any plane graph G, we let ∂G be the subgraph of G consisting of vertices and

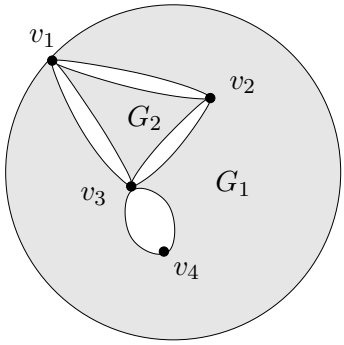


Figure 3: Example of 3-separation (G_1, G_2)

edges incident with the infinite face of G. When G is 2-connected, ∂G is a cycle, and we call this the *outer cycle* of G.

We say a collection of vertices in a plane graph G are *cofacial* if they are incident with a common face of G.

Recall that if G is a connected graph, an *k-cut* of G is a set of vertices $T \subseteq V(G)$ with $|T| = k$ such that $G - T$ has at least two components. G is called *n-connected* if it has no k-cut with $k < n$. We now define something similar to a k-cut, called a k-separation.

Definition 2.1. Let G_1 and G_2 be edge-disjoint subgraphs of a graph G. We say (G_1, G_2) is a *separation* of G if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $E(G_1) \neq \emptyset$, and $E(G_2) \neq \emptyset$. If $|V(G_1 \cap G_2)| = k$, we say (G_1, G_2) is a *k-separation* of G, or a separation *of order k*.

Figure 3 illustrates a 3-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{v_1, v_2, v_3\}$. Notice that the illustrated graph also has a 4-separation (G_1', G_2') with $V(G_1' \cap G_2') = \{v_1, v_2, v_3, v_4\}$. In the case of separation (G_1, G_2), we can find a simple closed curve Ω in the plane whose intersection with G is exactly $V(G_1 \cap G_2)$, and which separates $V(G_1) \setminus V(G_2)$ from $V(G_2) \setminus V(G_1)$ in the disk in which G is embedded. In the case of separation (G_1', G_2'), no such closed curve exists. In several later definitions, we will desire the existence of such a closed curve. We call the separation (G_1, G_2) *non-degenerate* if $|V(G_1 \cap G_2)| = k$, and $V(G_1 \cap G_2)$ can be labeled as $\{g_1, g_2, \ldots, g_k\}$ in such a way that g_i is cofacial with g_{i+1} for each $i \in \{1, \ldots, k-1\}$, and g_k is cofacial with g_1.

Definition 2.2. Let G be a graph and let a_1, b_1, a_2, b_2 be distinct vertices of G. If for any $T \subseteq V(G)$ with $|T| \leq 2$, each component of $G - T$ contains at least one element of $\{a_1, b_1, a_2, b_2\}$, then G is said to be $(3, \{a_1, b_1, a_2, b_2\})$-*connected*.

We call $\Gamma = (G, a_1, b_1, a_2, b_2)$ a *disk graph* if G is a 2-connected graph drawn in a disk in the plane with no pair of edges crossing such that a_1, b_1, a_2, b_2 appear in clockwise order along the boundary of the disk, and G is $(3, \{a_1, b_1, a_2, b_2\})$-connected.

Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We ask whether three disjoint paths exist in G: from a_1 to X, from a_2 to X, and from b_1 to b_2 respectively. If these paths exist, we say X is *feasible* in Γ. Otherwise, X is *infeasible* in Γ. We will show that if X is infeasible in Γ, then either Γ can be "reduced" (to be described in Section 2.3), or Γ falls into one of several families of graphs (to be described in Section 2.2).

2.2 Obstructions

Here we describe seven classes of disk graphs $\Gamma = (G, a_1, b_1, a_2, b_2)$ and vertex sets $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ for which X is infeasible in Γ. In all figures, shaded areas represent possible locations for vertices of X.

Definition 2.3. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_1$ if all $x \in X$ are incident with a common finite face, or $X \subseteq V(C[b_1, b_2])$, or $X \subseteq V(C[b_2, b_1])$. See Figure 4.

Proposition 2.1. *If $(\Gamma, X) \in \mathcal{O}_1$, then X is infeasible in Γ.*

Proof. Suppose there are disjoint paths P_1 from a_1 to $x_1 \in X$ and P_2 from a_2 to $x_2 \in X$. Since x_1 and x_2 are cofacial, $V(P_1) \cup V(P_2)$ separates b_1 from b_2 in G. Since this is true for any choice of P_1 and P_2, X is infeasible in Γ. □

Definition 2.4. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_2$ if G has a separation (H, K) of order 3 such that $X \subseteq V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$. See Figure 5(a).

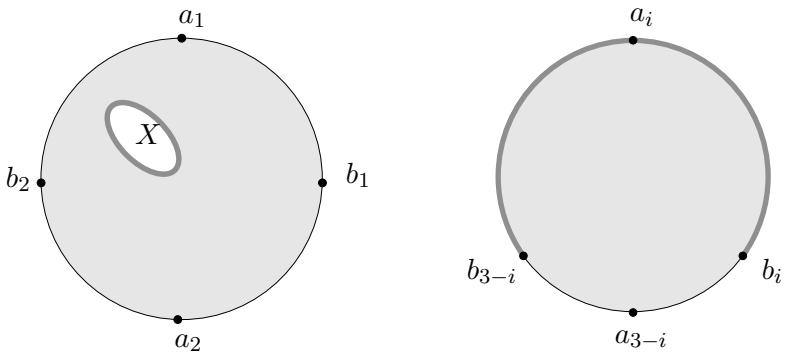
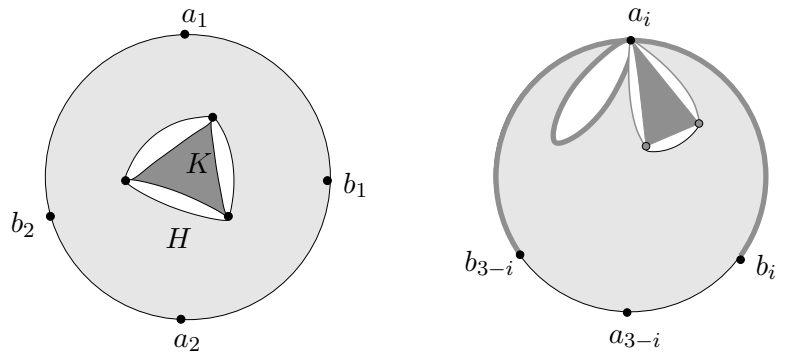


Figure 4: Obstructions in $\mathcal{O}_1$



(a) Class $\mathcal{O}_2$ (b) Class $\mathcal{O}_3$

Figure 5: Obstructions in $\mathcal{O}_2$ and $\mathcal{O}_3$

Proposition 2.2. *If $(\Gamma, X) \in \mathcal{O}_2$, then X is infeasible in Γ.*

Proof. Label $V(H \cap K) = \{h_1, h_2, h_3\}$. Suppose there are disjoint paths P_1 from a_1 to X and P_2 from a_2 to X. Then $P_i \cap V(H \cap K) \neq \emptyset$ for $i \in \{1, 2\}$. Assume by symmetry that $h_1 \in V(P_1)$ and $h_2 \in V(P_2)$. Now h_1 and h_2 are cofacial, so $V(P_1) \cup V(P_2)$ separates b_1 from b_2 in G. Since this is true for any choice of P_1 and P_2, X is infeasible in Γ. □

Definition 2.5. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_3$ (Figure 5(b)) if there is some $i \in \{1, 2\}$ such that for each $x \in X$, one of the following holds:

1. $x \in C[b_{3-i}, b_i]$, or

2. there is a finite face of G incident with both a_i and x, or

3. G has a separation (H, K) of order 3 such that $a_i \in V(H \cap K)$, $x \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$

Proposition 2.3. *If $(\Gamma, X) \in \mathcal{O}_3$, then X is infeasible in Γ.*

Proof. Assume by symmetry that the definition above is satisfied with $i = 1$. Let P_2 be any path from a_2 to X. Then some vertex of P_2 is cofacial with a_1, and $V(P_2) \cup \{a_1\}$ separates b_1 from b_2 in G. Hence X is infeasible in Γ. □

For the following definition, refer to Figure 6

Definition 2.6. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Suppose there is some $i \in \{1, 2\}$ such that $X \cap C[b_{3-i}, b_i] \neq \emptyset$. Partition X into X_1, X_2, X_3 with $X_1 = X \cap V(C[b_{3-i}, a_i])$, $X_2 = X \cap V(C[a_i, b_i])$, $X_3 = X \setminus (X_1 \cup X_2)$. We say $(\Gamma, X) \in \mathcal{O}_4$ if there is some $j \in \{1, 2\}$, $x' \in X_j$, and $C_a = C[a_i, x']$ if $x' \in C[a_i, b_i]$, $C_a = C[x', a_i]$ if $x' \in C[b_{3-i}, a_1]$, such that $X_j \cap V(C_a) = \{x'\}$ and one of the following holds:

1. $X_{3-j} = \emptyset$, and

   (a) there is a finite face of G incident with some vertex in $V(C_a)$ and all vertices in X_3 (see Figure 6(a)), or

   (b) G has a separation (H, K) of order 3 with $X_j \cup \{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K) \setminus V(C_a)| \leq 2$ (Figure 6(b) and 6(c)), or

   (c) $|X_3| = 1$, and G has a non-degenerate separation (H, K) of order 4 with $X_j \cup \{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K) \setminus V(C_a)| \leq 2$ (Figure 6(d)), or

2. $X_{3-j} \neq \emptyset$, and

   (a) G has a 2-separation (H, K) with $X_1 \cup X_2 \cup \{a_{3-i}, b_1, b_2\} \subseteq V(H)$, $X_3 \cup \{a_i\} \subseteq V(K)$, (Figure 6(e)), or

(b) $|X_3| = 1$, and G has a non-degenerate separation (H, K) of order 4 with $X_1 \cup X_2 \cup \{a_{3-i}, b_1, b_2\} \subseteq V(H)$, $X_3 \cup \{a_i\} \subseteq V(K)$ (Figure 6(f)).

Proposition 2.4. *If $(\Gamma, X) \in \mathcal{O}_4$, then X is infeasible in Γ.*

Proof. Assume by symmetry that the definition above is satisfied with $i = 1$, and $C_a = C[b_2, a_1]$. Let P be a path from a_2 to $x \in X$. If $V(P) \cap V(C[b_2, b_1]) \neq \emptyset$, then $V(P)$ separates b_1 from b_2 in G. We may assume, then, that $V(P) \cap V(C[b_2, b_1]) = \emptyset$, and $x \in X_3$.

Suppose $X_2 = \emptyset$ and there is a finite face of G incident with some vertex in $V(C_a)$ and all vertices in X_3. Then there is a vertex $v \in V(C_a)$ cofacial with x, and $V(P) \cup \{v\}$ separates $X_1 \cup \{b_2\}$ from $\{a_1, b_1\}$ in G. Since this is true for any choice of path P, X is infeasible in Γ.

Now suppose G has a separation (H, K) of order at most 4 with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $x' \subseteq V(K)$, and $|V(H \cap K) - V(C_a)| \leq 2$. Now $V(P) \cap V(H \cap K) \neq \emptyset$, so we have some vertex $vinV(C_a)$ cofacial with some vertex $p \in V(P)$, and $V(P) \cup \{v\}$ separates $X_1 \cup \{b_2\}$ from $\{a_1, b_1\}$. If $X_2 \neq \emptyset$, we find that p is also cofacial with some vertex $v' \in C[a_1, b_1]$. Hence $V(P) \cup \{v'\}$ separates $X_2 \cup \{b_1\}$ from $\{a_1, b_2\}$. Since this is true for any choice of path P, X is infeasible in Γ.

□

The following definition is illustrated in Figure 7.

Definition 2.7. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_5$ if G has separations $(H_1, M_1), (H_2, M_2)$, and (H_3, M_3), and a cycle D bounding a finite face of G such that the following hold for some $i \in \{1, 2\}$:

1. $V(H_1 \cap M_1) = \{u_1, v_1, w_1\}$, $w_1 \in V(C)$, $\{u_1, v_1\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,
2. $V(H_2 \cap M_2) = \{u_2, v_2, w_2\}$, $w_2 \in V(C)$, $\{u_2, v_2\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$,
3. $V(H_3 \cap M_3) = \{u_1, u_2\} \cup (\{w_1, w_2\} \cap \{b_1, b_2\})$, $a_i \in V(M_3)$, $\{a_{3-i}, b_1, b_2\} \in V(H_3)$,
4. if $w_1 \notin \{b_1, b_2\}$, then $u_1 \in V(C)$, and if $w_2 \notin \{b_1, b_2\}$, then $u_2 \in V(C)$, and

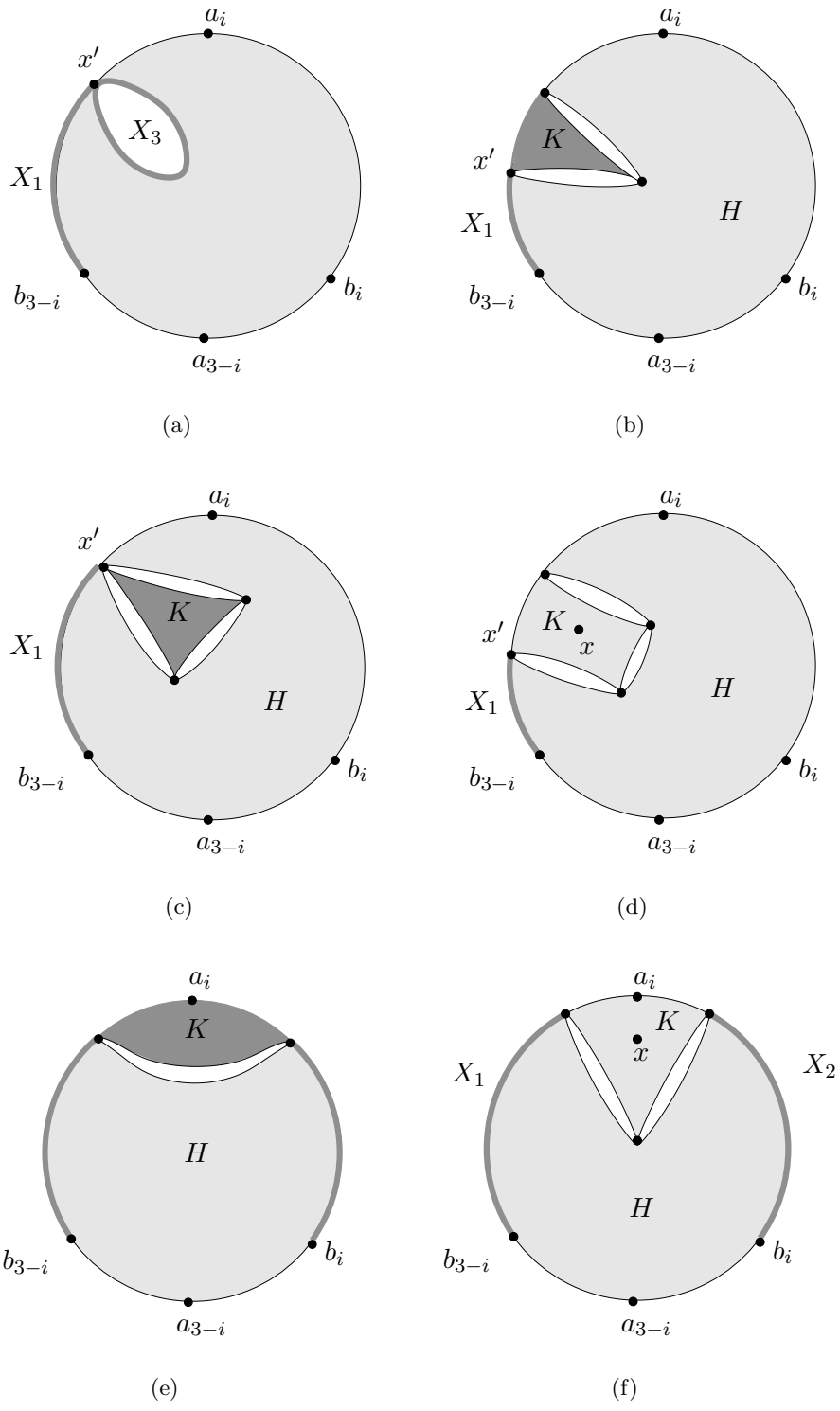


Figure 6: Obstructions in $\mathcal{O}_4$

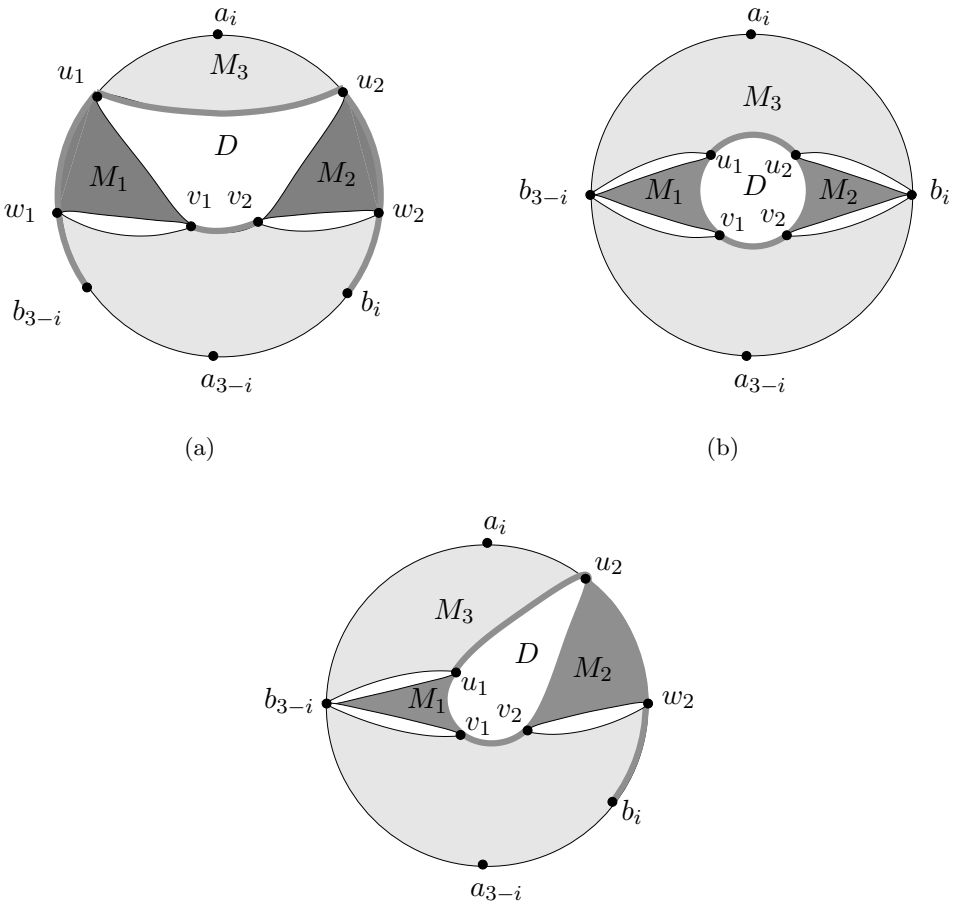


Figure 7: Obstructions in $\mathcal{O}_5$

5. $X \subseteq V(M_1 \cup M_2 \cup D \cup C[b_{3-i}, w_1] \cup C[w_2, b_i])$,

Proposition 2.5. *If $(\Gamma, X) \in \mathcal{O}_5$, then X is infeasible in Γ.*

Proof. Assume by symmetry that the above definition is satisfied for $i = 1$, so $a_1 \in V(M_3)$. Suppose there are disjoint paths P_1 from a_1 to X and P_2 from a_2 to X, each disjoint from $\{b_1, b_2\}$. Note that $V(P_1) \cap V(D) \neq \emptyset$. If $V(P_2) \cap V(C[b_2, w_1] \cup C[w_2, b_1]) \neq \emptyset$, then $V(P_2)$ separates b_1 from b_2 in G. Hence we may assume $V(P_2) \cap V(C[b_2, w_1] \cup C[w_2, b_1]) = \emptyset$, so $V(P_2) \cap V(D) \neq \emptyset$. But then $V(P_1) \cup V(P_2)$ separates b_1 from b_2 in G, and X is infeasible in Γ. □

Definition 2.8. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let

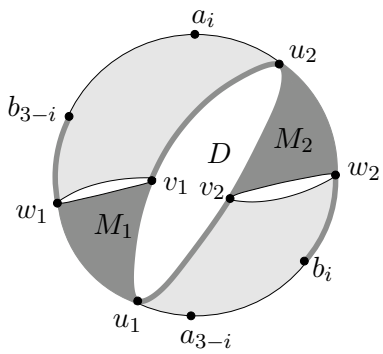


Figure 8: Obstruction in $\mathcal{O}_6$

$X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_6$ if G has separations (H_1, M_1) and (H_2, M_2) and a cycle D bounding a finite face of G such that the following hold for some $i \in \{1, 2\}$ (Figure 8):

1. $V(D \cap C) = \{u_1, u_2\}$
2. For each $j \in \{1, 2\}$ there is a path $C_j \subseteq C$ from u_j to $\{b_1, b_2\}$ with $V(C_j) \cap \{a_1, a_2\} = \emptyset$
3. $V(H_j \cap M_j) = \{u_j, v_j, w_j\}$, $v_j \in V(D)$, $w_j \in V(C_j)$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H_j)$, and
4. $X \subseteq V(M_1 \cup M_2 \cup D \cup C_1 \cup C_2)$

Proposition 2.6. *If $(\Gamma, X) \in \mathcal{O}_6$, then X is infeasible in Γ.*

Proof. Assume by symmetry that a_1, u_2, w_2, b_1 appear in clockwise order on C, as in Figure 8. Suppose there are disjoint paths P_1 from a_1 to X and P_2 from a_2 to X, each disjoint from $\{b_1, b_2\}$. If $V(P_1) \cap V(C[w_1, b_2]) \neq \emptyset$, we have $V(P_1)$ separating b_1 from b_2 in G. Similarly, if $V(P_2) \cap V(C[w_2, b_1]) \neq \emptyset$, then $V(P_2)$ separates b_1 from b_2 in G. So by planarity we may assume $V(P_1) \cap V(D) \neq \emptyset$ and $V(P_2) \cap V(D) \neq \emptyset$, and hence $V(P_1) \cap V(P_2)$ separates b_1 from b_2 in G, and X is infeasible in Γ. □

Definition 2.9. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \mathcal{O}_7$ if G has a 3-separation (H, K) and a cycle D bounding a finite face of G such that for some $i \in \{1, 2\}$, the following hold

1. $V(H \cap K) = \{u, v, w\}$, $w \in C(b_{3-i}, a_i) \cup C(a_i, b_i)$, and $u, v \in V(D)$

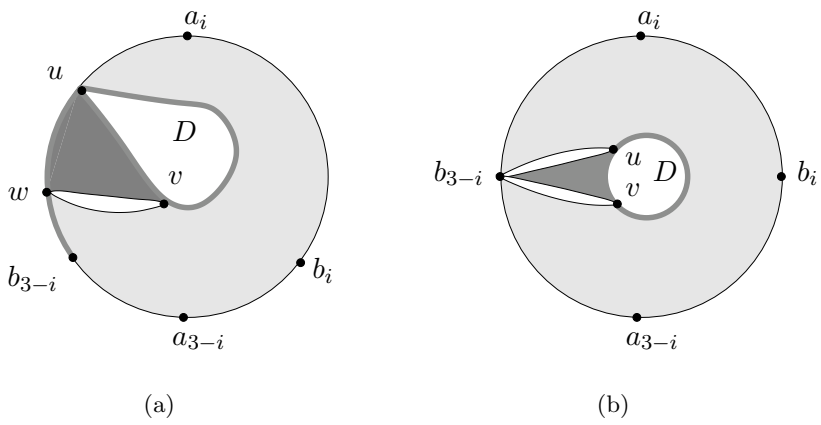


Figure 9: Obstructions in $\mathcal{O}_7$

2. $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, $X \cap V(K) \nsubseteq V(H)$,

3. if $w \in C(b_{3-i}, a_i)$, then $u \in C(w, a_i]$ and $X \subseteq V(K \cup D \cup C(b_{3-i}, w])$ (Figure 9(a)), if $w \in C(a_i, b_i)$, then $u \in C[a_i, w)$ and $X \subseteq V(K \cup D \cup C[w, b_i))$, and

4. if $w \in \{b_{3-i}, b_i\}$, then $X \subseteq V(K \cup D)$ (Figure 9(b)).

Proposition 2.7. *If $(\Gamma, X) \in \mathcal{O}_7$, then X is infeasible in Γ.*

Proof. Assume by symmetry that $w \in C[b_2, a_1]$. Suppose there are disjoint paths P_1 from a_1 to X and P_2 from a_2 to X, each disjoint from $\{b_1, b_2\}$. If $V(P_2) \cap V(C[b_2, w]) \neq \emptyset$, then $V(P_2)$ separates b_1 from b_2 in G. Hence we may assume $V(P_2) \cap V(C[b_2, w]) = \emptyset$, so $V(P_2) \cap V(D) \neq \emptyset$. By planarity, we find $V(P_1) \cap V(D) \neq \emptyset$, so $V(P_1) \cup V(P_2)$ separates b_1 from b_2 in G, and X is infeasible in Γ. □

2.3 Reductions

In some cases, we can show that the desired paths exist in the graph G if and only if certain disjoint paths exist in some subgraph of G. We define these cases as reductions. Refer to Figure 10 for the following definition.

Definition 2.10. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \backslash \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say that (Γ, X) satisfies property (R1) if G has some separation (H, K) of order 4 such that

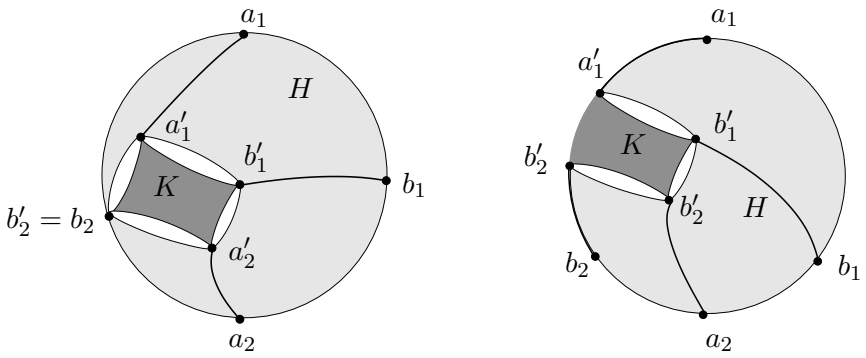


Figure 10: Graphs satisfying property (R1)

(i) $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X \subseteq V(K)$,

(ii) the vertices in $V(H \cap K)$ can be labeled as a'_1, b'_1, a'_2, b'_2 such that

  (a) a'_i is cofacial with b'_j for each $i, j \in \{1, 2\}$,

  (b) any four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $\{a'_1, b'_1, a'_2, b'_2\}$ join a_i to a'_i and b_i to b'_i respectively $(i = 1, 2)$

Among graphs with 4-separations satisfying condition (i) of property (R1), we can further characterize those graphs which satisfy condition (ii) by the following lemma.

Lemma 2.8. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Suppose G has some separation (H, K) of order 4 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and the vertices in $V(H \cap K)$ can be labeled as a'_1, b'_1, a'_2, b'_2 such that a'_i is cofacial with b'_j for each $i, j \in \{1, 2\}$, and any four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $\{a'_1, b'_1, a'_2, b'_2\}$ join a_i to a'_i, and b_i to b'_i $(i = 1, 2)$ respectively. Then one of the following is true:*

(i) G has some separation (H', K') of order at most 4 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$ and $\{a'_1, b'_1, a'_2, b'_2\} \subseteq V(K')$, with $\{a_1, b_1, a_2, b_2\} \nsubseteq V(H' \cap K')$ and $\{a'_1, b'_1, a'_2, b'_2\} \nsubseteq V(H' \cap K')$; or

(ii) $a'_i = a_i$ for some $i \in \{1, 2\}$ or $b'_i = b_i$ for some $i \in \{1, 2\}$; or

(iii) *for some $i \in \{1,2\}$, either $b_{3-i}, b'_{3-i}, a'_i, a_i$ or a_i, a'_i, b'_i, b_i occur in this clockwise order on C.*

This is simply a recasting of Theorem 2.1 in [20].

Notice also that if such a graph G has some separation (H,K) of order 4 such that $\{a_1,b_1,a_2,b_2\} \subseteq V(H)$ and one of (ii) or (iii) of Lemma 2.8 is satisfied, then $V(H \cap K)$ can in fact be labeled as a'_1, b'_1, a'_2, b'_2 such that every set of four disjoint paths from A to $V(H \cap K)$ consists of paths from a_i to a'_i and from b_i to b'_i ($i \in \{1,2\}$).

Proposition 2.9. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and $X \subseteq V(G) \setminus \{a_1,b_1,a_2,b_2\}$ with $|X| \geq 2$. If there is some edge of G incident with a_i and b_j, $i,j \in \{1,2\}$, then (Γ, X) satisfies property (R1).*

Proof. Suppose there is an edge $e = a_ib_j \in E(G)$. Then we have a separation (H,K) defined by $V(H) = \{a_1,a_2,b_1,b_2\}, E(H) = e$, and $K = G - e$. □

We also consider the case where a disk graph has a separation (H,K) similar to that in property (R1), but the vertex set X need not be contained in $V(K)$. Refer to Figure 11 for the following definition.

Definition 2.11. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, let C be the outer cycle of G, and let $X \subseteq V(G) \setminus \{a_1,b_1,a_2,b_2\}$ with $|X| \geq 2$. We say (Γ, X) satisfies property (R2) if there exist some $i \in \{1,2\}$, a partition of X into $X_1 = X \cap C(b_{3-i}, a_i)$, $X_2 = X \cap C(a_i, b_i)$, and $X_3 = X \setminus (X_1 \cup X_2)$, with $|X_3| \geq 2$, and a separation (H,K) of G of order at most 4 such that one of the following holds:

(i) X_1, X_2 are nonempty, (H,K) is of order at most 3, $X_1 \cup X_2 \cup \{a_{3-i}, b_1, b_2\} \subseteq V(H)$, $X_3 \cup \{a_i\} \subseteq V(K)$, and $|V(H \cap K) - V(C[b_{3-i}, b_i])| = 1$ (Figure 11(a)); or

(ii) $X_2 = \emptyset$, $X_1 \cup \{a_1,b_1,a_2,b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K) - V(C[b_{3-i}, a_i])| = 2$ (Figure 11(b)); or

(iii) $X_1 = \emptyset$, $X_2 \cup \{a_1,b_1,a_2,b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K) - V(C[a_i, b_i])| = 2$

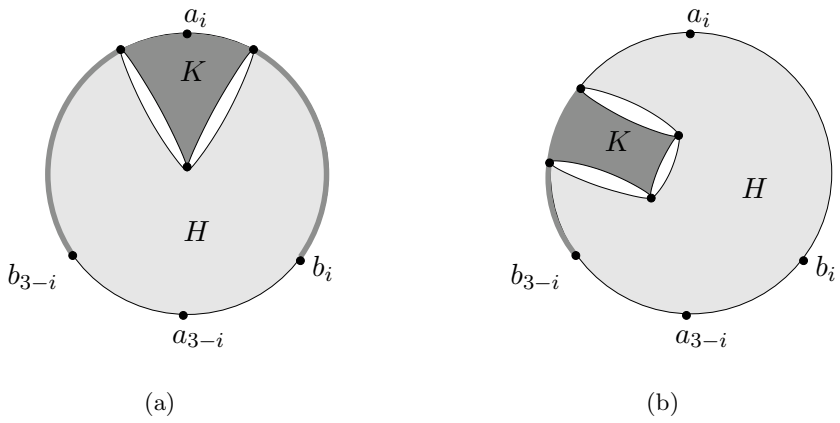


Figure 11: Graphs satisfying property (R2)

We call a disk graph and vertex set (Γ, X) *reducible* if (Γ, X) satisfies property (R1) or property (R2).

Proposition 2.10. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \backslash \{a_1, b_1, a_2, b_2\}$ such that (Γ, X) is reducible. Assume the separation (H, K) is chosen to maximize K. Then X is feasible in Γ if and only if there are three disjoint paths in K: from a_1' to X, from a_2' to X, and from b_1' to b_2'.*

In particular, if $\Gamma' = (K, a_1', b_1', a_2', b_2')$ is a disk graph with $X \cap \{a_1', b_1', a_2', b_2'\} = \emptyset$, then X is feasible in Γ if and only if X is feasible in Γ'.

Proof. Suppose there are disjoint paths P_1 from a_1 to X and P_2 from a_2 to X. We may assume there are disjoint paths Q_1 from b_1 to b_1' and Q_2 from b_2 to b_2', each disjoint from $P_1 \cup P_2$. For any such collection of paths, we must have $a_1' \in V(P_1)$ and $a_2' \in V(P_2)$. So $V(P_1) \cup V(P_2) \cup \{b_1', b_2'\}$ separates b_1 from b_2 in G. Hence X is feasible in Γ if and only if there is a path from b_1' to b_2' in K, disjoint from $P_1 \cup P_2$. □

Lemma 2.11. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Suppose G has some separation (H, K) of order 4 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X \subseteq V(K)$. Suppose further that $V(H \cap K)$ can be labeled $\{k_1, h_1, k_2, h_2\}$ in clockwise order such that there are (possibly trivial) disjoint paths in K, from k_1 to X, from k_2 to X, and from h_1 to h_2. Then one of the following holds.*

(i) X *is feasible in* Γ*, or*

(ii) (Γ, X) *is reducible, or*

(iii) $(\Gamma, X) \in \mathcal{O}_2$.

Proof. If there are not four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $V(H \cap K)$, then there must be some separation (G_1, G_2) of G of order at most three with $K \subseteq G_1$ and $\{a_1, b_1, a_2, b_2\} \subseteq V(G_2)$, and hence $(\Gamma, X) \in \mathcal{O}_2$. So we may assume there are four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $V(H \cap K)$.

Suppose there is a set of four such paths in which $\{a_1, a_2\}$ are joined to $\{k_1, k_2\}$, so $\{b_1, b_2\}$ are joined to $\{h_1, h_2\}$. Together with the paths in K, these show that X is feasible in Γ. We may assume then that the vertices in $V(H \cap K)$ can be labeled as a'_1, b'_1, a'_2, b'_2 such that any four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $\{a'_1, b'_1, a'_2, b'_2\}$ join a_i to a'_i and b_i to b'_i respectively $(i = 1, 2)$, where $a_1 = h_j,\ a_2 = h_{3-j},\ b_1 = k_j,\ b_2 = k_{3-j}$ for some $j \in \{1, 2\}$. So (Γ, X) satisfies property (R1).

□

2.4 Main Theorem

Theorem 1. *Let* $\Gamma = (G, a_1, b_1, a_2, b_2)$ *be a disk graph, and let* $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ *with* $|X| \geq 2$*. Then one of the following holds.*

(C1) X *is feasible in* Γ*,*

(C2) (Γ, X) *is reducible, or*

(C3) $(\Gamma, X) \in \bigcup_{i=1}^{7} \mathcal{O}_i$.

CHAPTER III

MAXIMIZING PATHS

3.1 Defining a maximizing path

In this section, we define the choice of a "best" path P from $\{a_1, a_2\}$ to X in a disk graph. We then show that this choice greatly restricts the location of the vertices of X within $G - V(P)$. Given any path P from $\{a_1, a_2\}$ to X, we use P to define a separation of G into "lower" and "upper" subgraphs. We use these subgraphs throughout the proofs which follow.

Given $T \subseteq V(G)$, a subgraph B of G is a *T-bridge* of G if either (i) B is induced by a single edge $xy \in E(G) \setminus E(T)$ where $x, y \in V(T)$ or (ii) $\exists$ some component H of $G - T$ such that B is the subgraph induced by $E(H) \cup \{xy \in E(G) : x \in V(T), y \in V(H)\}$. The vertices in $V(B) \cap V(T)$ are called the *attachments* of B in T.

Definition 3.1. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let P be some path from $a_i \in \{a_1, a_2\}$ to X in G which is disjoint from $C[b_i, b_{3-i}]$. We denote by L_P the minimal union of blocks of $G - V(P)$ such that $C[b_i, b_{3-i}] \subseteq L_P$, and note that L_P is a chain of blocks (see Figure 12).

By the definition of L_P, each $(L_P \cup P)$-bridge of G has at most one attachment in L_P. Since G is 2-connected, each $(L_P \cup P)$-bridge has at least one attachment in P. Hence $G - L_P$ is connected, and so there is a unique L_P-bridge of G. Let U_P be this L_P-bridge.

Let M_P denote the path induced by the edges in $E(\partial L_P) - E(C[b_i, b_{3-i}])$.

Definition 3.2. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$, and let P be some path from $a_i \in \{a_1, a_2\}$ to $x' \in X$ in G which is disjoint from $C[b_i, b_{3-i}]$.

For vertices $u, v \in M_P$ we say *u lies to the left of v* if $v \in M_P(u, b_1]$, in which case we also say *v lies to the right of u*.

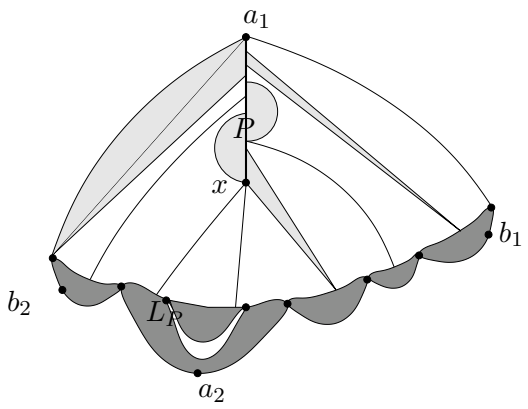


Figure 12: The subgraph L_P (shaded), where P is from a_1 to x

Given a path Q from $q' \in V(P)$ to $q \in V(M_P)$, internally disjoint from $P \cup L_P$, we say Q *falls to the left of* P if all vertices of $M_P - q$ which are cofacial with any vertex of $P(q', x']$ lie to the right of q. Symmetrically, we say Q *falls to the right of* P if all vertices of $M_P - q$ which are cofacial with any vertex of $P(q', x']$ lie to the left of q.

If D is a facial cycle of G with $D \cap P \neq \emptyset$ and $D \cap L_P \neq \emptyset$, then there are two paths from P to M_P which are subgraphs of D. We say the face bounded by D is *left of* P if both of these paths fall to the left of P, and we say it is *right of* P if both of these paths fall to the right of P.

We will use paths from P to L_P extensively, so let us take a moment to observe their character. Suppose we have such a path S from $s' \in V(P)$ to $s \in V(M_P)$. Then S is certainly contained in some $(L_P \cup P)$-bridge of G. G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, so if S has any internal vertices, the bridge must have at least two attachments on P. So a path from P to L_P is either a single edge, or it is contained in some bridge with at least two attachments in P. We sometimes refer to a bridge which is a single edge from P to L_P as a *trivial* bridge, and a bridge with at least one internal vertex as *non-trivial*.

Suppose (Γ, X) satisfies the definition of $\mathcal{O}_3$ for fixed $i \in \{1, 2\}$. That is, for each $x \in X$, one of the following holds:

(i) $x \in C[b_{3-i}, b_i]$, or

(ii) there is a finite face of G incident with both a_i and x, or

(iii) G has a separation (H, K) of order 3 such that $a_i \in V(H \cap K)$, $x \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$

We say $(\Gamma, X) \in \mathcal{O}_3$ *with respect to* a_i. This distinction allows us to state the following lemma more exactly, and will thus simplify several later proofs.

Lemma 3.1. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Then for each $i \in \{1, 2\}$, either $(\Gamma, X) \in \mathcal{O}_3$ with respect to a_{3-i}, or $G - (C[b_i, b_{3-i}])$ contains a path P from a_i to X for which a_{3-i} is not a cutvertex of L_P.*

Proof. Given $i \in \{1, 2\}$, let $H = G - a_{3-i}$. Observe that since G is 2-connected, H is connected. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, H is a chain of blocks, with b_i and b_{3-i} lying in the endblocks of H. Define D to be the path in H induced by the edges in $E(\partial H) - E(C[b_{3-i}, b_i])$.

Suppose no desired path P from a_i to X exists. Then for each $x_j \in X$, there is no path from a_i to x_j in $G - D$. Hence by planarity either $x_j \in V(D)$ or there is some separation (H_j, M_j) of H of order 2 such that $V(H_j \cap M_j) \subseteq V(D)$, $x_j \in V(M_j)$, and $\{a_i, b_1, b_2\} \subseteq V(H_j)$.

If $x_j \in V(D)$, then x_j is cofacial with a_{3-i}. Where the separation (H_j, M_j) exists, let H'_j be the subgraph of G induced by $V(H_j) \cup \{a_{3-i}\}$, and let M'_j be the union of all H'_j-bridges of G. We then have $a_{3-i} \in V(H'_j \cap M'_j)$, $x_j \in V(M'_j)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H'_j)$, and we see that $(\Gamma, X) \in \mathcal{O}_3$ with respect to a_{3-i}. □

Definition 3.3. Given $\Gamma = (G, a_1, b_1, a_2, b_2)$ a disk graph, and given $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$, we let $\mathcal{P}_{(\Gamma, X)}$ be the collection of all paths P from $a_i \in \{a_1, a_2\}$ to X in $G - C[b_i, b_{3-i}]$ such that a_{3-i} is not a cutvertex of L_P.

Note from Lemma 3.1 that if $(\Gamma, X) \notin \mathcal{O}_3$, then $\mathcal{P}_{(\Gamma, X)} \neq \emptyset$.

Definition 3.4. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. For a path $P \in \mathcal{P}_{(\Gamma, X)}$ from a_i to X in G, let

$\mathcal{R}(P) = \{v \in V(L_P) \setminus V(M_P) : \exists$ a path from a_{3-i} to v in $L_P - V(M_P)\}$,

$\mathcal{A}(P) = \{v \in V(L_P) : v$ lies in the block of L_P containing $a_{3-i}\}$, and

$\mathcal{L}(P) = \{v \in V(G) : v$ lies in the same component of $G - V(P)$ as $L_P\}$.

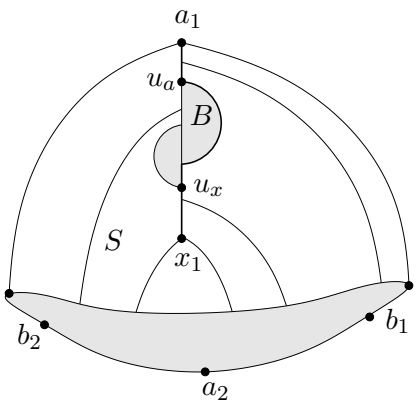


Figure 13: Bridges without L_P attachment

Suppose we have a path $P \in \mathcal{P}_{(\Gamma,X)}$ from $\{a_1, a_2\}$ to X, and there is some $x_2 \in X \cap \mathcal{R}(P)$. Let Q be some path from a_2 to x_2 in L_P disjoint from M_P. Note that M_P provides a path from b_1 to b_2 which is disjoint from $P \cup Q$, so we see that X is feasible in Γ. We can think of $\mathcal{R}(P)$ as the set of "reachable" vertices associated with the path. If there is some reachable vertex in X, we will have the desired disjoint paths. Our approach, then, is to choose a path which maximizes the set of reachable vertices, and then examine the potential locations of vertices in X within the subgraph L_P.

Definition 3.5. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \backslash \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say that a path $P \in \mathcal{P}_{(\Gamma,X)}$ is a *maximizing* path if, among all paths in $\mathcal{P}_{(\Gamma,X)}$, considered under set inclusion,

(i) $\mathcal{R}(P)$ is maximal,

(ii) subject to (i), $\mathcal{A}(P)$ is maximal,

(iii) subject to (ii), $V(L_P)$ is maximal, and

(iv) subject to (iii), $\mathcal{L}(P)$ is maximal.

Lemma 3.2. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a maximizing path. Then every $(P \cup L_P)$-bridge of G has exactly one attachment in L_P.*

Proof. Without loss of generality, assume P is from a_1 to $x_1 \in X$. (See Figure 13.) By the definition of L_P, each $(L_P \cup P)$-bridge of G has at most one attachment in L_P.

Let $B_1, \ldots, B_n$ be the $(P \cup L_P)$-bridges of G which have no attachments in L_P. For each B_k, let P_k be the shortest subpath of P including all attachments of B_k in P. Since G is 2-connected, $B_k \cup P_k$ must be 2-connected. For any $K \subseteq \{1, 2, \ldots, n\}$, define $\mathcal{B}_K = \bigcup_{k \in K}(B_k \cup P_k)$. Choose a maximal $J \subseteq \{1, 2, \ldots, n\}$ such that $\mathcal{B}_J$ is 2-connected. Let $B := \mathcal{B}_J$, and let $u_a,\ u_x$ be the vertices of $B \cap P$ which lie closest along P to a_1 and x_1, respectively.

Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, we see that $\{u_a, u_x\}$ is not a 2-cut in G, so there must be some path S from $s' \in P(u_a, u_x)$ to $s \in P[a_1, u_a) \cup P(u_x, x_1] \cup L_P$, internally disjoint from $P \cup L_P$. We claim that we may choose such a path with $s \in V(L_P)$. Otherwise, each such path S is contained in some $(P \cap L_P)$-bridge B_j of G which has no attachment in L_P. Then $B \cup B_j$ is 2-connected, and $J \cup \{j\}$ contradicts the maximality of J.

Note that either $s' \in \partial B(u_x, u_a)$ or $s' \in \partial B(u_a, u_x)$. Suppose without loss of generality that $s' \in \partial B(u_x, u_a)$ (see Figure 13). We form the new path Q from P by replacing $P[u_a, u_x]$ by $\partial B[u_a, u_x]$. Some vertices of $B - V(Q)$ now lie in a $(Q \cup L_Q)$-bridge of G with attachment s in L_Q. Note that $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, $L_P \subseteq L_Q$, and $\mathcal{L}(P) \subseteq \mathcal{L}(Q)$, but $\mathcal{L}(P) \neq \mathcal{L}(Q)$. This contradicts the choice of P as a maximizing path, and hence cannot occur. □

The following lemma applies a similar argument to limit the type of paths which may exist within $(P \cup L_P)$-bridges. Lemmas 3.3 and 3.4 are illustrated in Figure 14.

Lemma 3.3. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to $x_1 \in X$. Suppose there is a path W from $w' \in V(P) \setminus \{x_1\}$ to $w \in P(w', x_1]$ which is internally disjoint from $P \cup L_P$. Then all paths from $P(w', w)$ to M_P which are internally disjoint from $P \cup L_P \cup W$ have a common endpoint in M_P.*

Proof. Let S be a path from $s' \in P(w', w)$ to $s \in V(M_P)$ and let R be a path from $r' \in P(w', w)$ to $r \in V(M_P)$, such that S and R are each internally disjoint from $P \cup L_P \cup W$.

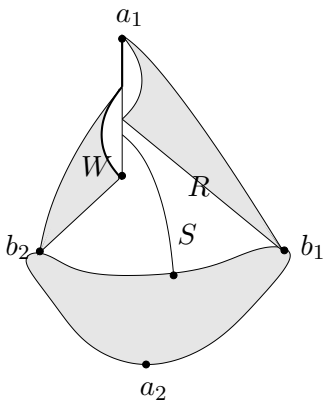


Figure 14: Where $b_1, b_2 \subseteq V(U_P)$

Suppose that $s \neq r$.

Let $Q := P[a_1, w'] \cup W \cup P[w, x_1]$. Now $(S \cup R) \subseteq L_Q$. Note that $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $V(L_P) \subseteq V(L_Q)$, but $V(L_P) \neq V(L_Q)$. This contradicts the choice of P as a maximizing path. □

Lemma 3.4. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a maximizing path, and assume P is from a_1 to x_1. If $X \setminus \{x_1\} \subseteq V(L_P)$ and $\{b_1, b_2\} \subseteq V(U_P)$, then (Γ, X) is reducible.*

Proof. Suppose $b_1, b_2 \in V(U_P)$. If any vertex $v \in P(a_1, x_1]$ is cofacial with each of b_1, b_2, we have a 4-separation (H, K) of G with $V(H \cap K) = \{v, b_1, a_2, b_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and $X \subseteq V(K)$. Hence (Γ, X) satisfies property (R1). We may assume, then, that no such vertex exists. So there is some path from $P[a_1, x_1)$ to $M_P(b_2, b_1)$. Let S be such a path, from $s' \in P[a_1, x_1)$ to $s \in M_P(b_2, b_1)$, chosen to minimize $P[a_1, s']$. Assume by symmetry that S falls to the right of P, so s' is cofacial with b_1.

Now s' is not cofacial with b_2. From our choice of S, there is no path from $P[a_1, s')$ to $M_P(b_2, b_1)$, so there is some path W from $w' \in P[a_1, s')$ to $w \in P(s', x_1]$, internally disjoint from P. Choose such a path to minimize $P[a_1, w']$, so w' is cofacial with b_2.

Now w' is not cofacial with b_1. Again considering our choice of S, we must have some path U from $u' \in P[a_1, w')$ to $u \in P(w', x_1]$, internally disjoint from P. By Lemma 3.2, this must be contained in a $(P \cup L_P)$-bridge with some attachment in M_P. Label this bridge

Y, and let $y \in M_P[s, b_1]$ be its attachment in L_P. There is some path from u' to y in Y, internally disjoint from P. By our choice of S, we must have $y = b_1$. There is also a path from u to y in Y, internally disjoint from P, but this contradicts Lemma 3.3.

□

3.2 Examining path bridges

Here we begin with a maximizing path P, and consider whether $X \subseteq L_P$. We find that all vertices of $X \setminus V(L_P)$ must lie in a specific obstructive relationship.

Lemma 3.5. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a maximizing path, and assume P is from a_1 to $x_1 \in X$. Let $Y = X \setminus V(L_P)$. Then either*

1. *$Y = \{x_1\}$, or*
2. *G has a facial cycle D and a 3-separation (H, K) with $V(H \cap K) = \{u, r, w\}$ such that*
   (a) *$u \in V(P)$, $\{r, w\} \subseteq V(D \cap M_P)$, and $D[w, r] = M_P[r, w]$,*
   (b) *$\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $Y \subseteq V(K)$, and*
   (c) *$Y \subseteq D[r, w]$.*

In case (2), $(\Gamma, Y) \in \mathcal{O}_1 \cap \mathcal{O}_2$.

Proof. Suppose $Y \neq \{x_1\}$. Label $Y = \{x_1, y_1, \ldots, y_m\}$. For each $y_i \in Y$, let B_i denote the $(P \cup L_P)$-bridge of G containing y_i. Let u_i, v_i be the attachments of B_i in P chosen to minimize each of $P[a_1, u_i]$ and $P[v_i, x_1]$. Now choose $j \in \{1, \ldots, m\}$ to minimize $P[a_1, u_j]$. (See Figure 15.)

By Lemma 3.2, B_j must have a unique attachment in L_P. Let w be this attachment. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $\{u_j, w\}$ is not a 2-cut in G. This shows $u_j \neq v_j$, and that there is some path from $P(u_j, x_1]$ to $L_P \cup P[a_1, u_j)$ in $G - \{u_j, w\}$. From Lemma 3.2 we see that G in fact has some path R from $r' \in P(u_j, x_1]$ to $r \in V(L_P)$ which is internally disjoint from $P \cup L_P \cup B_j$.

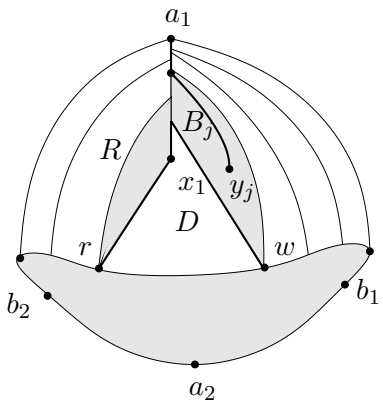


Figure 15: Where $X \setminus \{x_1\}$ is not contained in $V(L_P)$

Suppose $\{u_j, w, r\}$ is not a 3-cut in G. Then there is some path S from $P(u_j, x_1]$ to L_P which is disjoint from r and is internally disjoint from $P \cup L_P \cup B_j$. Now B_j contains a path R' from u_j to y_j which is internally disjoint from $P \cup L_P$. Let $Q = P[a_1, u_j] \cup R'$. Then $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $(R \cup S \cup L_P) \subseteq L_Q$, contradicting the choice of P as a maximizing path.

Hence we may assume $T := \{u_j, w, r\}$ is a 3-cut in G. Let K be the T-bridge of G containing y_j. Then we have a separation (H, K) of G with $V(H \cap K) = T$ and $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$. Since $P(u_j, x_1] \subseteq K$, we have $x_1 \in V(K)$. From our choice of j, and from planarity, we see that $Y \subseteq V(K)$. So condition (b) is satisfied by this separation.

Now there is some finite face of G bounded by a cycle D such that $D \cap L_P = D[w, r]$. Suppose there is some path P_1 from u_j to Y in K which is disjoint from $D[r, w]$. Letting $Q := P[a_1, u_j] \cup P_1$, we see that $D[r, w] \subset L_Q$, so we have $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $V(L_P) \subseteq V(L_Q)$, but $V(L_P) \neq V(L_Q)$. This contradicts the choice of P as a maximizing path, and hence cannot occur, so there can be no such path P_1.

From planarity, we see that for each $y \in Y$, either $y \in D[r, w]$ or there is some 2-cut in $D[r, w]$ separating y from $\{u_j, r, w\}$. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, no such 2-cut exists. Hence $Y \subseteq D[r, w]$, and conditions (a) and (c) are satisfied.

□

Let us state as a corollary the special case of this lemma where $|X| = 2$.

Corollary 3.6. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X = \{x_1, x_2\} \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a maximizing path, and assume P is from a_1 to x_1. Then either $x_2 \in V(L_P)$ or $(\Gamma, X) \in \mathcal{O}_1 \cap \mathcal{O}_2$.*

3.3 Location of X relative to a

Now let us suppose we have a maximizing path from a_1 to x_1, and $X' := X \setminus \{x_1\} \subseteq V(L_P)$. We show that X' cannot be separated from a_2 by a cutvertex in L_P. That is, the minimal chain of blocks in L_P containing X' must include the block containing a_2. It can actually be shown that under the conditions of Lemma 3.7, there is some $x_2 \in X \cap \mathcal{A}(P)$. This weaker result will be enough for our purposes, however, and the proof is more straightforward.

Lemma 3.7. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 2$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a maximizing path, and assume P is from a_1 to x_1. Let $X' = X \setminus \{x_1\}$. Suppose $X' \subseteq V(L_P)$ and there is some cutvertex of L_P separating X' from a_2. Then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.*

Proof. Label the unique block of L_P containing a_2 as B_a. Label the minimal chain of blocks containing X' as B_x. Since there is a cutvertex of L_P separating X' from a_2, $|B_a \cap B_x| \leq 1$. We may assume without loss of generality that B_x and B_a appear in clockwise order along $C[b_1, b_2]$. See Figures 16 and 17 for illustrations.

Let $\{y_1, y_2\} = V(L_P \cap U_P \cap C)$, such that $C[b_1, b_2] \subseteq C[y_1, y_2]$. If $b_1 \notin B_x$, let z_0 be the cutvertex of L_P separating X' from b_1 which appears closest to y_1 along M_P, and let z_1 be the cutvertex of L_P which separates X' from b_1 with $z_1 \in V(B_x)$ ($z_0 = z_1$ is possible). If $b_1 \in B_x$, let $z_0 = z_1 = b_1$. In either case, let z_2 be the cutvertex of L_P separating a_2 from X' with $z_2 \in V(B_x)$, and let z_3 be the cutvertex of L_P separating a_2 from X' with $z_3 \in V(B_a)$ ($z_2 = z_3$ is possible). If $y_2 \in V(B_a)$, let $z_4 = y_2$, otherwise let z_4 be the cutvertex of L_P separating a_2 from b_2 with $z_4 \in V(B_a)$. Finally, if no block of L_P includes both b_2 and z_4, let z_5 be the cutvertex of L_P separating them which appears closest to z_4 along M_P. If some block contains both b_2 and z_4, let $z_5 = b_2$.

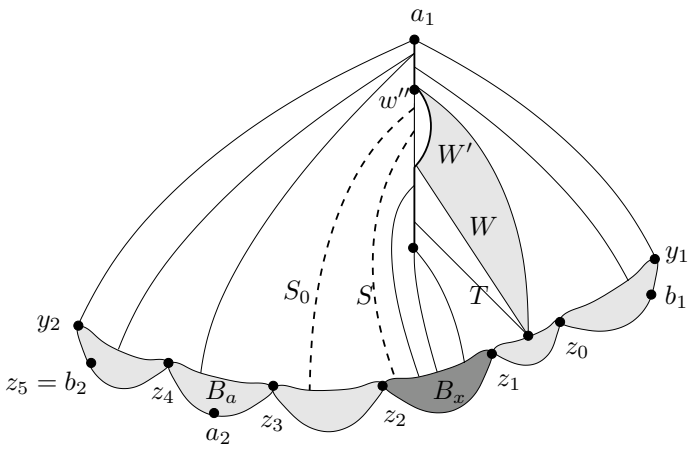


Figure 16: Lemma 3.7, Claim 1

Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, the set $\{z_1, z_2\}$ cannot be a 2-cut in G. Hence there is some path S from $s \in M_P(z_2, z_1)$ to $s' \in V(P)$ in G. We choose such a path S to minimize $P[a_1, s']$.

Note that if there is a separation (M, K) of G with $V(M \cap K) = \{s', z_1, z_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(K)$, we have $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume there is no such separation, and by our choice of S we have $s' \neq x_1$.

Claim 1. We may assume that there exist two disjoint paths from $P(s', x_1]$ to $M_P[b_2, z_2) \cup M_P(z_1, b_1]$.

Proof of Claim 1 If there is a 4-separation (H, K) of G such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X \subseteq V(K)$, and $\{z_1, z_2, s'\} \subseteq V(H \cap K)$, then (Γ, X) satisfies property (R1), since $z_1, z_2 \in C[b_1, a_2]$. We may therefore assume no such separation exists, and hence by the choice of S, there must be two disjoint paths from $P(s', x_1]$ to $M_P[b_2, z_2) \cup M_P(z_1, b_1] \cup P[a_1, s')$. Applying Lemma 3.2, we may assume that at least one of these paths ends in M_P.

Suppose Claim 1 fails. Let T be a path from $t' \in P(s', x_1]$ to $t \in M_P[b_2, z_2) \cup M_P(z_1, b_1]$ and let W' be a path from $w' \in P(s', x_1]$ to $w'' \in P[a_1, s')$ with T and W' disjoint (Figure 16). W' must lie in some $(P \cup L_P)$-bridge of G. From Lemma 3.2, this bridge has some attachment w in L_P. From the planarity of G and the choice of T, we see that $w \in M_P[b_2, z_2) \cup M_P(z_1, b_1]$, and hence there is some path W from w' to w internally disjoint

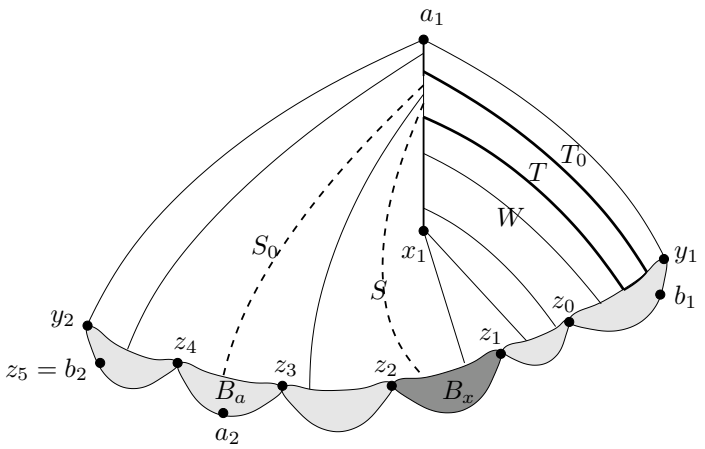


Figure 17: Lemma 3.7, Claim 2

from $P \cup L_P$.

Now for any path R from $r' \in P(s', x_1]$ to $r \in M_P[b_2, z_2) \cup M_P(z_1, b_1]$, R is not disjoint from W. Since we have $t' \neq w'$, it must be that for all such paths R, $r = w$.

If w'' is cofacial with one of z_1 or z_2, then we have a separation (H, K) of G with $V(H \cap K) = \{w'', z_1, z_2, w\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and $X \subseteq V(K)$, and we find (Γ, X) satisfies property (R1). So we may assume w'' is not cofacial with z_1 or z_2. By our choice of S, we find that there must be some path S_0 from $P(w'', s']$ to $M_P[b_2, z_2) \cup M_P(z_1, b_1]$.

Now let $Q := P[a_1, w''] \cup W' \cup P[w', x_1]$. Note that $(S \cup S_0) \subseteq L_Q$, so we have $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, $V(L_P) \subseteq V(L_Q)$, but $V(L_P) \neq V(L_Q)$, contradicting the assumption that P is a maximizing path. This proves Claim 1.

By Claim 1, let T and W be disjoint paths from $t' \in P(s', x_1]$ to $t \in M_P[b_2, z_2) \cup M_P(z_1, b_1]$ and $w' \in P(s', x_1]$ to $w \in M_P[b_2, z_2) \cup M_P(z_1, b_1]$ respectively, with s', t', w', x_1 appearing in order on P ($w' = x_1$ is possible). Assume that T is chosen to minimize $P[s', t']$, and subject to this to maximize $M_P[t, s]$. From planarity, we have either $\{t, w\} \subseteq M_P[y_2, z_2)$ or $\{t, w\} \subseteq M_P(z_1, y_1]$.

Claim 2. We may assume that $t, w \in M_P[y_2, z_2)$.

Proof of Claim 2 Suppose on the contrary that $t, w \in M_P(z_1, y_1]$, as illustrated in Figure 17. Note that this can occur only if $b_1 \notin V(B_x)$. By planarity, S is to the left of P, and

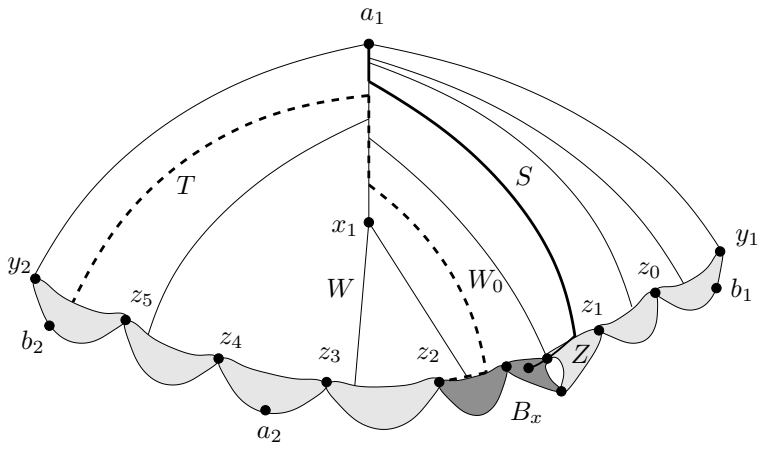


Figure 18: Lemma 3.7, choice of path W

hence no path from $P(s', x_1]$ to $M_P[y_2, z_2]$ exists. We may further assume S is chosen with $M_P[z_2, s]$ minimal. Therefore z_2 and s' are cofacial.

We may assume (Γ, X) does not satisfy property (R1), so $\{z_2, s', z_0\}$ cannot be three vertices in a 4-cut in G. Hence T can be chosen such that $t \in M_P(z_0, y_1]$. Also since (Γ, X) does not satisfy property (R1), $\{z_2, s', y_1, b_1\}$ cannot be a 4-cut in G, so there must be some path T_0 from $t'_0 \in P[a_1, s')$ to $t_0 \in M_P[t, y_1)$. Choose such a path to minimize $P[a_1, t'_0]$ and $M_P[t_0, y_1]$. Then t_0, t'_0, and y_1 are cofacial. Note from our choice of z_0 that $M_P[t, t_0] \cap C[b_1, b_2] = \emptyset$. Now $\{z_3, t'_0, y_1, b_1\}$ cannot be a 4-cut in G, since (Γ, X) does not satisfy property (R1), so there must be some path S_0 from $s'_0 \in P(t'_0, s']$ to $s_0 \in M_P[y_2, z_3)$.

Let $Q := P[a_1, t'_0] \cup T_0 \cup M_P[t, t_0] \cup T \cup P[t', x_1]$. Then paths S_0 and S demonstrate $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $\mathcal{A}(P) \neq \mathcal{A}(Q)$. This contradicts the assumption that P is a maximizing path, and hence proves the claim.

By Claim 2 and planarity, no path from $P(s', x_1]$ to $M_P[z_1, y_1]$ exists, and S lies to the right of P (as in Figure 18). We may assume S is chosen with $M_P[s, z_1]$ minimal, so z_1 and s' are cofacial. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, s must belong to the endblock Z of B_x containing z_1, and s is not a cutvertex of B_x. Since $X \cap V(C) = \emptyset$, Z contains a vertex of X which is not a cutvertex of B_x. So there is a path S' from s to X' in B_x which is disjoint from $C[b_1, b_2]$.

If $\{z_1, z_3, s'\}$ is a 3-cut in G separating X from $\{a_1, b_1, a_2, b_2\}$, we see that $(\Gamma, X) \in \mathcal{O}_2$,

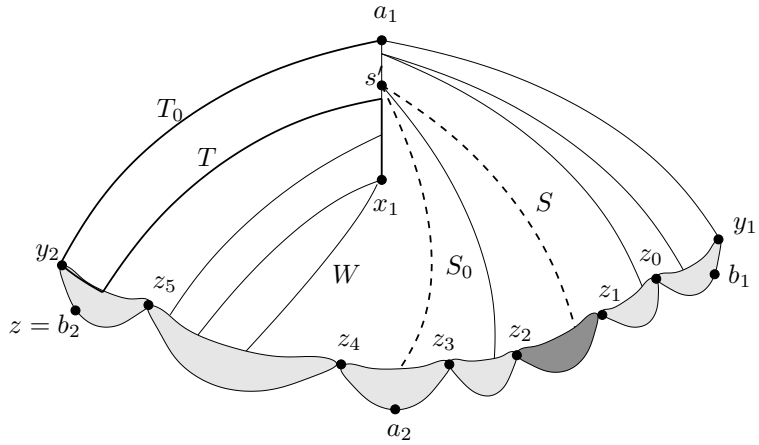


Figure 19: Lemma 3.7, Claim 4

so we may assume that it is not. Therefore, $t \in M_P[y_2, z_3)$. Assume further that W is chosen to minimize $M_P[w, z_2]$.

Claim 3. Suppose W_0 is a path from $w_0' \in P(s', x_1]$ to $w_0 \in M_P(z_4, s)$. Then $w_0 \in M_P[z_2, s]$ and for any choice of the path S' from s to X' in B_x, disjoint from $C[b_1, b_2]$, we have $V(S') \cap V(M_P[z_2, w_0]) \neq \emptyset$.

Proof of Claim 3 Let W_0 be a path from $w_0' \in P(s', x_1]$ to $w_0 \in M_P(z_4, s)$. If w_0 is left of z_2, let $M_W = M_P[w_0, z_2]$. If w_0 is right of z_2, let $M_W := M_P[z_2, w_0]$. Suppose, contrary to our claim, that there is some choice of path S' with $V(S') \cap V(M_W) = \emptyset$ (as illustrated in Figure 18). Then there is a path $Q := P[a_1, s'] \cup S \cup S'$ which is disjoint from $C[b_1, b_2] \cup T \cup P[t', w'] \cup W_0 \cup M_W$. The paths T and W_0 demonstrate that $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ and $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, but $\mathcal{A}(P) \neq \mathcal{A}(Q)$ contradicting the choice of P as a maximizing path. This proves Claim 3.

By Claim 3 and our choice of W, there is no path from $P(s', x_1]$ to $M_P(z_4, s)$, and so $w \in M_P(t, z_4]$. From planarity and $t \neq w$, we find $t \in M_P[y_2, z_4)$.

Claim 4. We may assume there is a facial cycle D of G with $M_P[z_3, z_4] \cup W \subseteq D$, and hence $\{z_3, z_4\}$ is a 2-cut in G.

Proof of Claim 4 Suppose there is no such facial cycle. By Claim 3, we see that there must be some path S_0 from s' to $s_0 \in M_P[z_4, z_3]$. (See Figure 19.)

If $s' = a_1$, we have 3-cuts $\{z_1, z_2, a_1\}$ and $\{z_4, b_2, a_1\}$ demonstrating that $\Gamma \in \mathcal{O}_3$. Hence we may assume $s' \neq a_1$. Recall that T is chosen to minimize $M_P[b_2, t]$. If b_2 lies in the same block of L_P as t, let $z = b_2$. Otherwise, let z be the cutvertex of L_P separating t from b_2 which minimizes $M_P[z, t]$. If $\{z, s', z_1\}$ is a 3-cut in G, we have a 4-separation (M, K) of G with $V(M \cap K) = \{z, s', z_1, a_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and $X \subseteq V(K)$, showing that (Γ, X) satisfies property (R1). Hence we may assume this is not a 3-cut, and there is some path T_0 from $t'_0 \in P[a_1, s')$ to $t_0 \in M_P(z, t]$.

Let $Q = P[a_1, t'_0] \cup T_0 \cup M_P[t_0, t] \cup T \cup P[t, x_1]$. Then the paths S_0 and S demonstrate that $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ and $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, but $\mathcal{A}(P) \neq \mathcal{A}(Q)$, contradicting the choice of P as a maximizing path.

So the path S_0 must not exist, and we see that all vertices of $M_P[z_3, z_4] \cup W$ are indeed incident with some common finite face of G. This proves Claim 4.

We also note by our choices of T and W that $z_4 \neq y_2$, and hence $b_2 \notin B_a$.

Claim 5. Let D be the facial cycle of G such that $z_3, z_4 \in V(D)$ and $W \subseteq D$. Then either

(i) $X' \subseteq V(D)$, or

(ii) G has a separation (H, K) such that $(X' \setminus V(D)) \subseteq V(K)$, $\{a_1, b_1, a_2, b_2, x_1\} \subseteq V(H)$, $V(H \cap K) = \{z_3, c, v\}$, $c \in V(C)$, and $v \in V(D)$.

If $s \in V(D)$, then $\{z_1, z_3, s'\}$ is a 3-cut in G, and we have the desired separation with $v = s'$ and $c = z_1$. So we assume $s' \notin V(D)$, so there must be some path R from $r' \in P(s', x_1]$ to $r \in M_P(z_3, z_1)$. (See Figure 20.)

From our choice of W (minimizing $M_P[w, z_2]$) and our earlier finding that $w \in M_P(t, z_4]$, we see that $r \in M_P[z_2, z_1)$. Together with the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, this shows that either $z_2 = z_3$, or the block of L_P containing z_2 and z_3 is induced by a single edge $z_2 z_3 \in E(G)$. Choose the path R to minimize $M_P[z_2, r]$. Note that $M_P[z_2, r] \subseteq D$. By Claim 3, there must not be a path from s to X' in $B_x - V(M_P[z_2, r] \cup C[b_1, b_2])$.

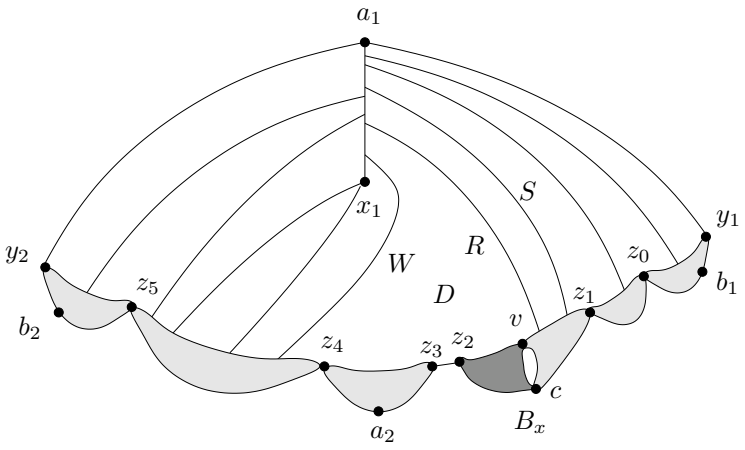


Figure 20: Lemma 3.7, Claim 5

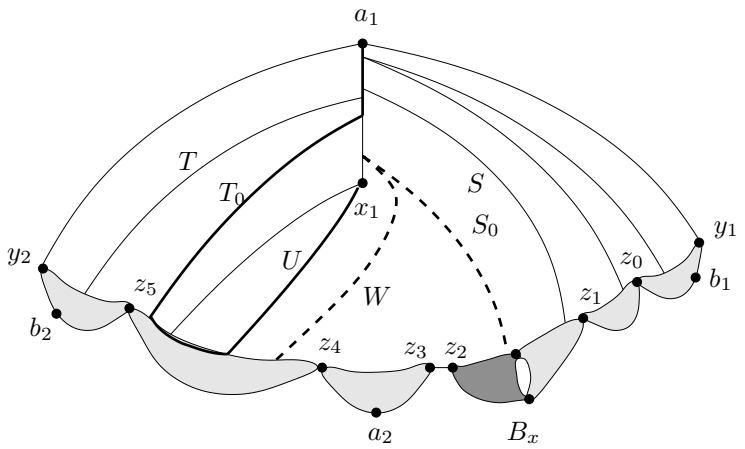
Since $X' \cap V(C) = \emptyset$, for each $x \in X'$ we have either $x \in M_P[z_2, r]$ or there is some separation (H', K') of B_x with $V(H' \cap K') \subseteq V(M_P[z_2, r] \cup C[b_1, b_2])$, $x \in V(K')$, and $\{a_1, b_1, a_2, b_2, x_1\} \subseteq V(H')$.

If $X' \subseteq V(M_P[z_2, r])$ then condition (i) of the claim is satisfied. Hence we may assume some separation (H', K') as above exists. From the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and from our choice of R we see $V(H' \cap K') \nsubseteq M_P[z_2, r]$, so we may assume $V(H' \cap K') = \{v, c\}$ with $v \in V(M_P[z_2, r])$ and $c \in V(C[z_1, z_2])$. We further assume that the separation is chosen to maximize K', giving $X' \setminus V(M_P[z_2, r]) \subseteq V(K')$. Again from the choice of R, we find that $\{v, c, z_3\}$ is a 3-cut in G, associated with a separation which satisfies condition (ii) of the claim. This proves Claim 4.

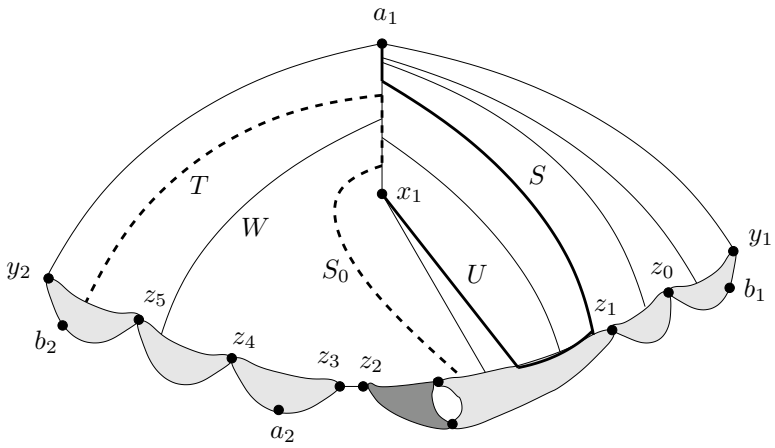
Let $S_0 \subseteq D$ be the path from $s'_0 \in P[s', w']$ to $s_0 \in M_P[z_2, s]$ maximizing $P[s'_0, x_1]$ and $M_P[z_3, s_0]$. (See Figure 21. $S_0 = S$ is possible.) So $D = W \cup P[w', s'_0] \cup S_0 \cup M_P[w, s_0]$.

Suppose $x_1 \in V(D)$. If $X' \subseteq V(D)$, then we have $(\Gamma, X) \in \mathcal{O}_1$. Otherwise, from Claim 4 we have a separation (H, K) of G such that $V(H \cap K) = \{z_3, c, v\}$, $c \in C[b_1, b_2] - a_2$, $z_3, v \in V(D)$, $z_3 \in C(c, a_2]$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, and $X \subseteq V(K \cup D)$. This gives $(\Gamma, X) \in \mathcal{O}_7$. Hence we may assume that $x_1 \notin V(D)$, so $w' \neq x_1$.

From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, neither the set $\{s'_0, s_0\}$ nor the set $\{w', w\}$ is a 2-cut in G, so there is some path U from $u' \in P(s'_0, x_1] \cap P(w', x_1]$ to $u \in M_P[b_2, w) \cup M_P(s_0, s]$, as shown in Figure 21.



(a) W right of P



(b) W left of P

Figure 21: Lemma 3.7 final cases

Suppose W lies to the right of P. Here $u \in M_P[b_2, w)$. If $\{z_4, z_5, s_0'\}$ is a 3-cut in G, we find $\Gamma \in \mathcal{O}_5$. We assume, then, that this is not a 3-cut, and hence there is some path T_0 from $t_0' \in P[a_1, s_0')$ to $t_0 \in M_P(z_5, u]$. Letting $Q := P[a_1, t_0'] \cup T_0 \cup M_P[t_0, u] \cup U \cup P[u', x_1]$, we see that $V(D[w, s_0]) \subseteq \mathcal{A}(Q)$. Hence $\mathcal{A}(P) \neq \mathcal{A}(Q)$ although $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, contradicting the choice of P as a maximizing path.

Now suppose W lies to the left of P. Here $u \in M_P(s_0, s]$. From Claim 4, we see that $X' \cap M_P[u, s] = \emptyset$. Let $Q := P[a_1, s'] \cup S \cup M_P[u, s] \cup U \cup P[u', x_1]$. Then $V(T \cup S_0) \subseteq \mathcal{A}(Q)$. Hence $\mathcal{A}(P) \neq \mathcal{A}(Q)$ although $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, and $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$, again contradicting the choice of P as a maximizing path. □

Corollary 3.8. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X = \{x_1, x_2\} \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $x_1, x_2 \notin V(C)$ and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1. Then either $x_2 \in \mathcal{A}(P)$, or (Γ, X) satisfies property (R1), or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.*

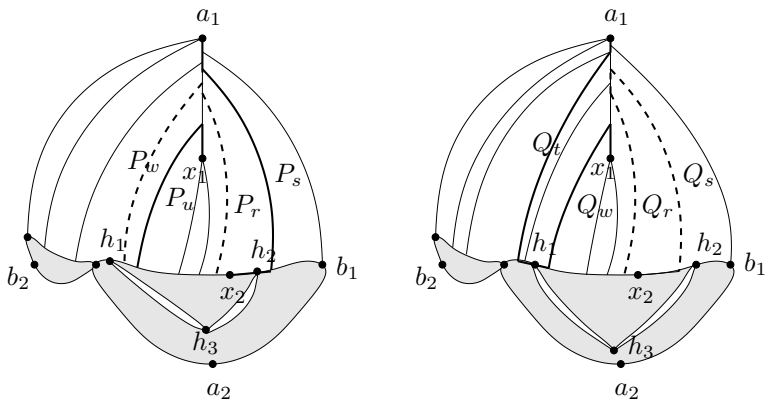
Proof. From Corollary 3.6, we see that either $(\Gamma, X) \in \mathcal{O}_1 \cap \mathcal{O}_2$ or $x_2 \in V(L_P)$, so we may assume $x_2 \in V(L_P)$. Now if $x_2 \notin \mathcal{A}(P)$, there is some cutvertex of L_P separating x_2 from a_2. By application of Lemma 3.7, either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.

□

3.4 Location of X relative to path

Assuming we have a maximizing path P, we now consider the location of X within L_P. We begin with two propositions which illustrate potential path routings. Propositions 3.9 and 3.10 are illustrated in Figure 22.

Proposition 3.9. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$ such that $\{x_1, x_2\}$ is infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, \{x_1, x_2\})}$ be a path from a_1 to x_1, with $x_2 \in V(M_P)$. Suppose we have four paths P_s, P_u, P_w, and P_r from $p_s', p_u', p_w', p_r' \in V(P)$ to $p_s, p_u, p_w, p_r \in V(M_P)$ respectively, such that*



(a) Proposition 3.9 (b) Proposition 3.10

Figure 22: Establishing separations when special paths exist

1. a_1, p'_s, p'_u, x_1 *appear in order along* P,

2. $p'_w, p'_r \in P(p'_s, p'_u)$,

3. p_w, p_u, p_r, *and* x_2 *are distinct, and* $p_s \neq b_1$,

4. $b_2,\ p_w,\ p_u,\ p_r,\ p_s,\ b_1$ *appear in order along* M_P, *and*

5. $x_2 \in M_P(p_r, b_1)$.

Then there is a separation (H, K) *of* L_P *of order at most three such that* $\{b_2, a_2, b_1, p_s\} \subseteq V(H)$ *and* $\{p_w, p_u, p_r, x_2\} \subseteq V(K)$.

Proof. Given such a paths (as in Figure 22(a)), let $Q := P[a_1, p'_s] \cup P_s \cup M_P[x_2, p_s]$. Suppose there are three disjoint paths from b_2 to p_w, from a_2 to p_u, and from b_1 to p_r in $L_P - Q$. These paths together with P_u, P_w, and P_r would show that $x_2 \in \mathcal{R}(Q)$, and $\{x_1, x_2\}$ is feasible in Γ, a contradiction. Hence no such paths may exist, and there must be a separation as desired. □

The separations guaranteed by Proposition 3.9 may be of several types. Each produces a slightly different cut in L_P. We may assume there are vertices $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, p_w], h_2 \in M_P[x_2, p_s]$. If $|V(H \cap K)| = 3$, label the third vertex as h_3.

We may have $\{h_1, h_2\}$ a 2-cut in L_P, or we may have a 3-cut $\{h_1, h_2, h_3\}$ as illustrated in Figure 22(a). (h_1 and h_3 share a finite face of L_P, as do h_2 and h_3.) We will refer to this type of separation as a *cutting separation* in L_P. If h_1 and h_2 are cofacial with a common vertex $p' \in V(P)$, we find a separation (H', K') in G with $V(H' \cap K') = V(H \cap K) \cup \{p'\}$.

However, $V(H \cap K)$ may also be a 2-cut in G, with an additional vertex in M_P sharing no finite face of G with the vertices of the 2-cut, as in Figure 23. (In this case h_1 and h_2 are both incident with the infinite face of L_P.) Then either $h_3 \in M_P[p_r, x_2) \cup M_P(s, b_1]$, and $\{h_1, h_3\}$ is a 2-cut in L_P, or $h_3 \in [p_w, p_u]$ and $\{h_2, h_3\}$ is a 2-cut in L_P. We will refer to this type of separation as a *non-cutting separation* in L_P.

Proposition 3.10. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$ such that $\{x_1, x_2\}$ is infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, \{x_1, x_2\})}$ be a path from a_1 to x_1, with $x_2 \in V(M_P)$. Suppose we have four paths Q_t, Q_w, Q_r, and Q_s from $q'_t, q'_w, q'_r, q'_s \in V(P)$ to $q_t, q_w, q_r, q_s \in V(M_P)$ respectively, such that*

1. *$q'_t \neq q'_s$, $q_w \neq q_r$*
2. *$a_1,\ q'_t,\ q'_s,\ q'_r,\ q'_w,\ x_1$ appear in order along P*
3. *$q_t \neq b_2$, q_w, q_r, x_2, and q_s are distinct, and*
4. *$b_2,\ q_t,\ q_w,\ q_r,\ x_2,\ q_s,\ b_1$ appear in order along M_P*

Then there is a separation (H, K) of L_P of order at most three such that $\{q_t, b_2, a_2, b_1\} \subseteq V(H)$ and $\{q_w, q_r, x_2, q_s\} \subseteq V(K)$.

Proof. Given such a paths (as in Figure 22(b)), let $Q := P[a_1, q'_t] \cup Q_t \cup M_P[q_t, q_w] \cup Q_w \cup P[q'_w, x_1]$. Suppose there are three disjoint paths from b_1 to q_r, from a_2 to x_2, and from b_2 to q_s in $L_P - Q$. These paths together with Q_r and Q_s would show that $x_2 \in \mathcal{R}(Q)$, and $\{x_1, x_2\}$ is feasible in Γ, a contradiction. Hence no such paths may exist, and there must be a separation as desired. □

As with the separations of Proposition 3.9, those of Proposition 3.10 may take several forms. We assume there are vertices $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[q_t, q_w], h_2 \in$

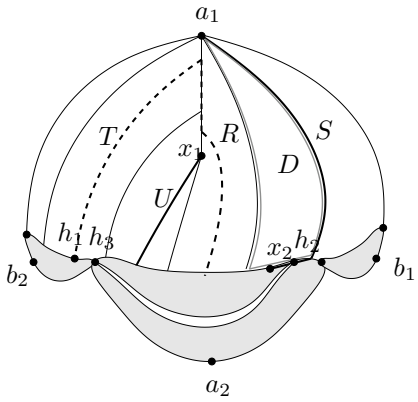


Figure 23: x_2 cofacial with a_1

$M_P[q_s, b_1]$. We then have $\{h_1, h_2\}$ a 2-cut in L_P; or each of h_1, h_2 shares a finite face with $h_3 \in V(H \cap K)$; or $h_3 \in M_P[x_2, q_s]$, and $\{h_1, h_3\}$ is a 2-cut in L_P; or $h_3 \in M_P[b_2, q_t) \cup (q_w, q_r]$ and $\{h_2, h_3\}$ is a 2-cut in L_P.

Lemma 3.11. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ such that $X \cap V(C) = \emptyset$, and X is infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma,X)}$ be a path from a_1 to x_1. Suppose there is a facial cycle D of G with $X \setminus \{x_1\} \subseteq V(D \cap M_P)$ and $a_1 \in V(D)$. Then $(\Gamma, X) \in \mathcal{O}_3$ or (Γ, X) satisfies property (R1).*

Proof. Let $X' = X \setminus \{x_1\}$, and choose $x_2 \in X'$ to minimize $M_P[x_2, b_1]$. We assume x_1 is not cofacial with a_1; otherwise $(\Gamma, X) \in \mathcal{O}_3$. So $x_1 \notin V(D)$. Assume by symmetry that $D[a_1, x_2]$ is to the right of P (Figure 23). Let $S \subseteq D[a_1, x_2]$ be a path from a_1 to $s \in M_P[x_2, b_1]$. From Proposition 2.9 there is no edge a_1b_1, and together with $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G this gives $b_1 \notin V(D)$.

Since x_1 is not cofacial with a_1, there is some path T to the left of P, from $t' \in P(a_1, x_1)$ to $t \in M_P[b_2, x_2)$. Choose T such that both of t', t are cofacial with a_1. Also since x_1 is not cofacial with a_1, there is some path R to the right of P, from $r' \in P(a_1, x_1)$ to $r \in M_P[t, x_2)$. Choose R so that both of r', r are cofacial with a_1. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, we see that $r \neq t$.

If $\{t, r, a_1\}$ is a 3-cut, this defines a 3-separation of G demonstrating that $(\Gamma, X) \in \mathcal{O}_3$. Hence we may assume this is not a cut, and there must be some path U from $u' \in P(t', x_1] \cap$

$P(r', x_1]$ to $u \in M_P(t, r)$. Choose such a path U to minimize $M_P[t, u]$. If $t' \in P[r, x_1)$ (T falls at or below R), then by this choice each vertex of $M_P[t, u]$ is cofacial with t'. If $t' \in P(a_1, r')$ (T falls above R), then each vertex of $M_P[t, u]$ is cofacial with some vertex $v \in P[t', r']$, which in turn is cofacial with a_1.

We may assume $D[a_1, x_2]$ is disjoint from $C[b_1, b_2]$. Otherwise, there is some vertex $v \in C(b_1, b_2)$ sharing a finite face with a_1. So $\{a_1, v\}$ is a 2-cut in G, and we have a 4-separation (H, K) of G with $V(H \cap K) = \{a_1, v, a_2, b_2\}$, such that $\{a_1, b_2, a_2, b_1\} \subseteq V(H)$ and $X \subseteq V(K)$. Hence (Γ, X) satisfies property (R1)

Apply Proposition 3.9 with $P_s = S$, $P_u = U$, $P_w = T$, $P_r = R$, to find that there must be a separation (H, K) of L_P of order at most three such that $\{s, b_1, a_2, b_2\} \subseteq V(H)$ and $\{x_2, r, u, t\} \subseteq V(K)$. We may assume there are $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, t], h_2 \in M_P[x_2, s]$. From our choices of T and S, we see that each of h_1, h_2 are cofacial with a_1. If $|V(H \cap K)| = 3$, label the third vertex as h_3.

Suppose $\{h_1, h_2\}$ is a 2-cut in L_P, or each of h_1, h_2 shares a finite face with h_3. Then there is a separation (H', K') of G of order at most 4 with $V(H' \cap K') = V(H \cap K) \cup \{a_1\}$, such that $\{a_1, b_2, a_2, b_1\} \subseteq V(H)$ and $X \subseteq V(K)$. If the separation is of order 4, then (Γ, X) satisfies property (R1). If it is of order 3, then $(\Gamma, X) \in \mathcal{O}_2 \cap \mathcal{O}_3$.

If $\{h_1, h_3\}$ is a 2-cut in L_P with $h_3 \in M_P[r, x_2) \cup (s, b_1]$, then we have a 3-cut $\{h_1, h_3, a_1\}$ in G, and $(\Gamma, X) \in \mathcal{O}_3$. If $\{h_2, h_3\}$ is a 2-cut in L_P with $h_3 \in M_P[t, u]$, then we have some vertex $v \in V(P[t', r']) \cup \{t\}$ which is cofacial with each of h_3 and a_1. So there is a 4-separation (H', K') of G with $V(H' \cap K') = \{v, h_3, h_2, a_1\}$, demonstrating that (Γ, X) satisfies (R1).

□

Lemma 3.12. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ such that $X \cap V(C) = \emptyset$, and X is infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1. Suppose there is a facial cycle D of G with $X \setminus \{x_1\} \subseteq V(D \cap M_P)$, and $D \cap P \neq \emptyset$. Then either $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_7$, or (Γ, X) satisfies property (R1).*

Proof. Let $X' := X \setminus \{x_1\}$, and choose $x_2 \in X'$ to minimize $M_P[x_2, b_1]$. By symmetry,

assume the face bounded by D is to the right of P. Let d_x, d_a, d_2, d_1 be vertices in $V(D)$ in that clockwise order such that $P \cap D = P[d_a, d_x]$ and $M_P \cap D = M_P[d_1, d_2]$ (see Figure 24(a)).

We claim that we may choose D such that $x_2 \neq d_1$. To see this, note by our choice of x_2 that if $x_2 = d_1$, we have $X = \{x_1, x_2\}$, and there is another facial D' with $D[d_1, d_x] = D'[d_x, d_1]$. If $x_1 \in V(D')$ then $(\Gamma, X) \in \mathcal{O}_1$, so we assume $x_1 \notin V(D')$, and the face bounded by D' is to the right of P.

We assume $a_1 \notin V(D)$, so $d_a \neq a_1$; otherwise by Lemma 3.11, $(\Gamma, X) \in \mathcal{O}_3$ or (Γ, X) satisfies property (R1). We also assume $x_1 \notin V(D)$; otherwise, $(\Gamma, X) \in \mathcal{O}_1$. From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, together with Lemma 3.2 (each bridge has an attachment on M_P), there is some path W from $w' \in P(d_x, x_1]$ to $w \in M_P[b_2, d_1)$. Choose such a path W to minimize $M_P[b_2, w]$ and, subject to this, to minimize $P[d_x, w']$. By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and Lemma 3.3 (limiting paths within $(P \cup L_P)$-bridges), we find that W is not to the right of P. Choose W_0 from $w_0' \in P[w', x_1]$ to w minimizing $P[w_0', x_1]$. ($W_0 = W$ is possible.)

We will consider five cases. We first suppose $b_1 \in V(D)$, and then suppose that there is some cutvertex of L_P in $V(D)$. These situations would prevent any useful routing from a_1 to x_2 along $D[d_a, x_2]$. We next consider the case that $d_a \in C[b_2, a_1]$, and then that there is some cutvertex of L_P left of w which is cofacial with d_a. We close with the case that d_a is not cofacial with any vertex of $C[b_2, a_1]$. In this case, we will route a path from a_1 to x_1 along some path left of P, avoiding D.

Case 1 Suppose $d_2 = b_1$.

This case is illustrated in Figure 24. In this case, $b_1 \in V(U_P)$, so by Lemma 3.4, we may assume $b_2 \notin V(U_P)$. If $\{d_a, b_1\}$ is a 2-cut in G, then we have a separation (H, K) of G with $V(H \cap K) = \{d_a, b_1, a_2, b_2\}$ which demonstrates that (Γ, X) satisfies property (R1). We may assume this is not the case, so there is some path T from $t' \in P[a_1, d_a)$ to $t \in M_P(b_2, d_1]$. Choose such a path T to minimize $P[t', d_a]$ and $M_P[t, d_1]$.

Applying Proposition 3.10 with $Q_t = T$, $Q_W = W_0$, $Q_r = D[d_1, d_x]$, and $Q_s = D[d_a, d_2]$

(Figure 24(a)), we see that there is a separation (H, K) of L_P of order at most three such that $\{t, b_2, a_2, b_1\} \subseteq V(H)$ and $\{w, d_1, d_2\} \cup X' \subseteq V(K)$. We may assume $b_1 = d_2 \in V(H \cap K)$, and there is some $h_1 \in V(H \cap K)$ with $h_1 \in M_P[t, w]$. If $|V(H \cap K)| = 3$, label the third vertex h_3.

By our choice of T and W, h_1 must be cofacial with some vertex $v \in P[d_a, d_x]$, which is also cofacial with b_1. If $h_1 \in V(C)$, and so is a cutvertex of L_P, then there is a separation (H', K') of G with $V(H' \cap K') = \{h_1, v, b_1, a_1\}$. This separation shows (Γ, X) satisfies (R1). Similarly, suppose $\{h_1, b_1\}$ is a 2-cut in L_P, or each of h_1, b_1 shares a finite face with h_3. Then there is a separation (H', K') of G of order at most 4 with $V(H' \cap K') = V(H \cap K) \cup \{v\}$, such that $\{a_1, b_2, a_2, b_1\} \subseteq V(H)$ and $X \subseteq V(K)$. If the separation is of order 4, then (Γ, X) satisfies property (R1). If it is of order 3, then $(\Gamma, X) \in \mathcal{O}_2$.

If $\{h_1, h_3\}$ is a 2-cut in L_P with $h_3 \in M_P[x_2, b_1]$, then we have a 3-cut $\{h_1, h_3, v\}$ in G, demonstrating $(\Gamma, X) \in \mathcal{O}_2$. We are left with the case that $\{b_1, h_3\}$ is a 2-cut in L_P with $h_3 \in M_P[b_2, t) \cup (w, d_1]$, as illustrated in Figure 24(a). Assume this 2-cut is chosen with $M_P[b_1, h_3]$ maximal. Suppose $h_3 \in M_P[b_2, t)$. Then there is some vertex $v' \in P[a_1, d_a]$ which is cofacial with both h_3 and b_1, and there is a 3-cut $\{h_1, h_3, v'\}$ in G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume $h_3 \in M_P(w, d_1]$.

If w is cofacial with h_3, we have a 4-separation (H', K') of G with $V(H' \cap K') = \{b_1, v, w, h_3\}$, and (Γ, X) satisfies (R1). Similarly, if x_1 shares a finite face with each of v and h_3, which can occur if $w' = x_1$, we have such a separation with $V(H' \cap K') = \{b_1, v, x_1, h_3\}$. Hence we may assume there is some path R from $r' \in P(v, x_1)$ to $M_P(w, h_3)$. Now since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $\{r', r\}$ is not a 2-cut in G, and there is a path U from x_1 to $u \in M_P[t, r)$.

Apply Proposition 3.10 with $Q_t = T$, $Q_W = U$, $Q_r = R$, and $Q_s = D[d_a, d_2]$ (Figure 24(b)). There is a separation (M, N) of L_P of order at most three such that $\{t, b_2, a_2, b_1\} \subseteq V(M)$ and $\{u, r, b_1\} \cup X' \subseteq V(N)$. We may assume $b_1 = d_2 \in V(M \cap N)$, and there is some $m_1 \in V(M \cap N)$ with $m_1 \in M_P[t, w]$. If $|V(M \cap N)| = 3$, label the third vertex m_3. Assume the separation is chosen with $M_P[b_2, m_1]$ minimal.

If $m_3 \in M_P[d_1, b_1]$ is cofacial with m_1, then m_1 is also cofacial with b_1, contradicting

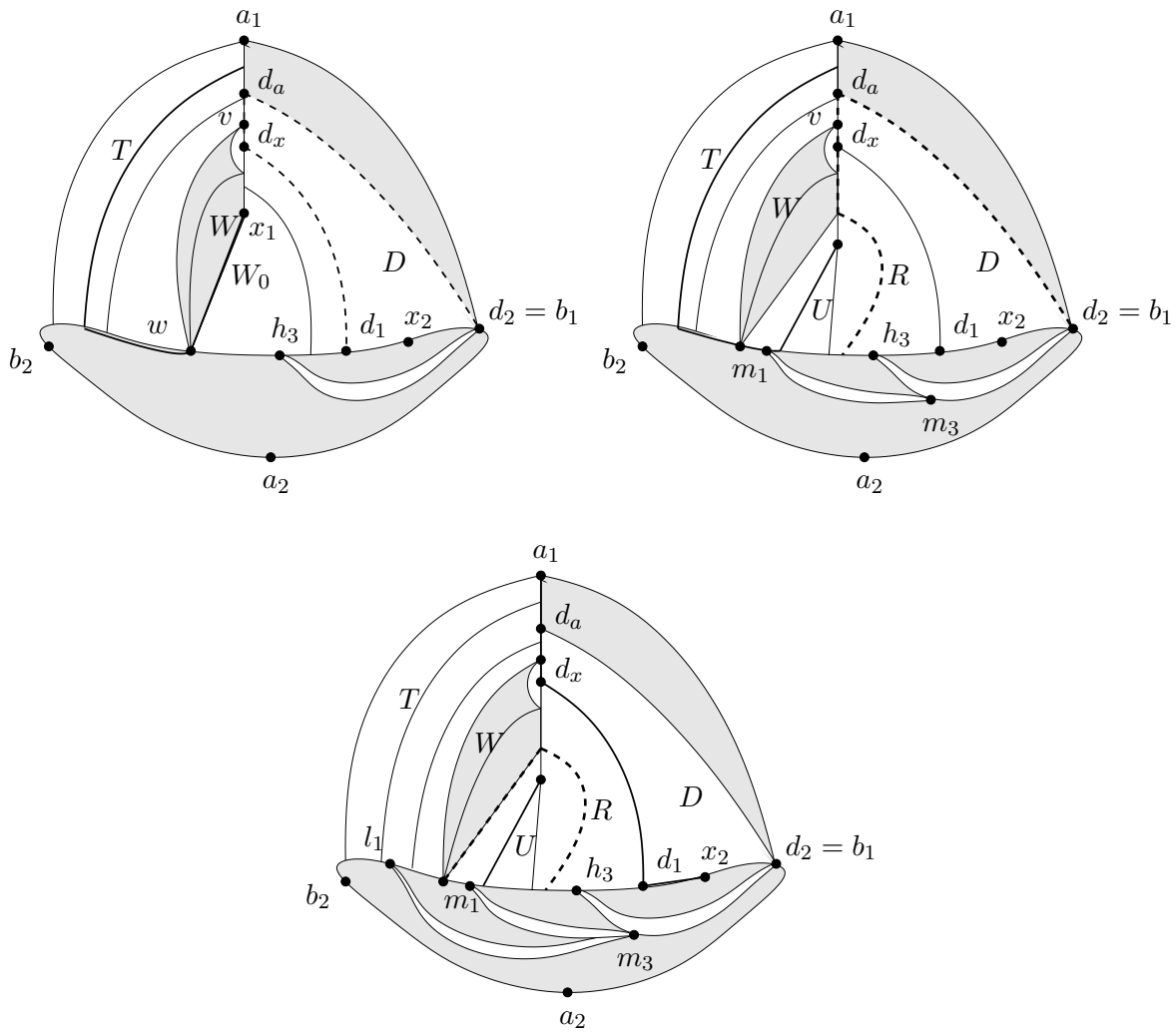


Figure 24: Separations when $b_1 \in V(D)$

our choice of the maximal 2-cut $\{b_1, h_3\}$, so this case does not occur. Also by our choice of h_3, we do not have $m_3 \in M_P[b_2, r]$ cofacial with b_1.

If m_1 is a cutvertex of L_P, we must have $m_3 \in M_P(w, u]$ by earlier arguments. Let $Q := P[a_1, t'] \cup T \cup M_P[t, w] \cup W \cup P[w', x_1]$. Now $D \subseteq \mathcal{A}(Q)$, and $\mathcal{A}(P) \subseteq \mathcal{A}(Q)$, but $D \nsubseteq \mathcal{A}(P)$. This contradicts the maximality of P.

Hence we must have either a 2-cut $\{m_1, b_1\}$, or each of m_1, b_1 sharing a finite face with m_3 in L_P. If m_1 is cofacial with any vertex of $P[d_a, d_x]$, we have a separation (M', N') of G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$ or satisfies property (R1). Hence we may assume $m_1 \in M_P(w, u]$.

Finally, choose $x_3 \in X'$ to minimize $M_P[d_1, x_3]$. Apply the arguments of Proposition 3.9 to path $P_s = P[a_1, d_x] \cup D[x_3, d_a]$, $P_u = U$, $P_w = W$, and $P_r = R$. (We need a slight, but obvious, modification to deal with x_3 being right of p_s. See Figure 24(c).) This gives some separation (L_1, L_2) of L_P of order at most three such that $\{b_2, a_2, b_1\} \cup X' \subseteq V(L_1)$ and $\{w, u, r, d_1\} \subseteq V(L_2)$. We may assume $b_1 \in V(L_1 \cap L_2)$, and some $l_1 \in V(L_1 \cap L_2)$ with $l_1 \in M_P[b_2, w]$.

By our choices of m_1 and h_3, we have either a 2-cut $\{l_1, b_1\}$, or each of l_1, b_1 sharing a finite face with l_3 in L_P. Note that l_1 is cofacial either with some vertex $v' \in V(P[d_a, d_x]) \cup \{a_1\}$. Hence we have a separation (L_1', L_2') in G with $V(L_1' \cap L_2') = V(L_1 \cap L_2) \cup \{v\}$. If this separation is of order 3, then $(\Gamma, X) \in \mathcal{O}_2$. If it is of order 4, then (Γ, X) satisfies (R1).

Case 2 Suppose there is a vertex $c \in V(M_P[x_2, d_2] \cap C)$.

If $\{d_a, c\}$ is a 2-cut in G, or $\{d_a, c, b_2\}$ is a 3-cut in G, then we have a separation (H, K) of G with $V(H \cap K) = \{d_a, c, a_2, b_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and $X \subseteq V(K)$, so (Γ, X) satisfies property (R1). We may assume this is not the case, so there is some path T from $t' \in P[a_1, d_a)$ to $t \in M_P(b_2, d_1]$. Choose such a path T to minimize $P[t', d_a]$ and $M_P[t, d_1]$.

Applying Proposition 3.10 with $Q_t = T$, $Q_w = U$, $Q_r = D[d_1, d_x]$, and $Q_s = D[d_a, d_2]$, we see that there is a separation (H, K) of L_P of order at most three such that $\{t, b_2, a_2, b_1\} \subseteq V(H)$ and $\{u, r, d_2\} \subseteq V(K)$.

Note that $d_2 \notin \mathcal{A}(P)$, and by Lemma 3.7 we may assume $X' \cap \mathcal{A}(P) \neq \emptyset$. So we

may assume there are $h_i \in V(H \cap K)$ with $h_1 \in M_P[t,u]$ and one of the following: (i) $h_1 \in V(C)$, or (ii) there is $h_2 \in M_P[x_2,c]$ sharing a finite face with h_1 in G, or (iii) there is $h_3 \in V(H \cap K)$ with $h_3 \in C[c,a_2]$ sharing a finite face with h_1. Note that h_1 is cofacial with some vertex $v' \in P[d_a,d_x]$. In case (i), we have a 4-cut $\{h_1,a_1,c,v'\}$, and in case (ii) we have 4-cut $\{h_1,h_2,c,v'\}$, each showing that (Γ,X) satisfies (R1). In case (iii), 3-cut $\{h_1,h_2,v'\}$ demonstrates that $(\Gamma,X) \in \mathcal{O}_2$ (Γ,X) satisfies (R1).

Case 3 Suppose $d_a \in C[b_2,a_1]$.

Recall the assumption that $a_1 \notin V(D)$, so $a_1 \neq d_a$. Therefore, by 2-connectivity of G, there is a path S from $s' \in P[a_1,d_a)$ to $s \in M_P[d_2,b_1]$. If $\{d_a,b_1\}$ is a 2-cut in G, we have a separation (G_1,G_2) of G with $V(G_1 \cap G_2) = \{d_a,b_1,a_2,b_2\}$, $\{a_1,b_1,a_2,b_2\} \subseteq V(G_1)$, and $X' \subseteq V(G_2)$, which demonstrates that (Γ,X) satisfies (R1). So we assume $\{d_a,b_1\}$ is not a 2-cut in G, and we may assume $s \neq b_1$. Choose S so that each vertex of $M_P[d_2,s]$ is cofacial with d_a.

Define $P_w \subseteq C[b_2,a_1]$, a path from $p'_w \in V(P)$ to $p_w \in V(M_P)$. Note that $x_1 \notin V(C)$, so we may assume $p'_w \in P[d_a,x_1)$. If $\{p_w,d_a,d_1\}$ is a 3-cut in G, we have $(\Gamma,X) \in \mathcal{O}_7$. So we may assume not, and there is a path U from $u' \in V(P)$ to $u \in M_P(p_w,d_1)$. By $(3,\{a_1,b_1,a_2,b_2\})$-connectivity of G, we may assume $u' \in P(p'_w,x_1] \cap P(d_x,x_1]$. Choose such a path U so that each vertex of $M_P[w,u]$ is cofacial with some vertex of $P \cap C$.

Apply Proposition 3.9 with $P_s = S$, $P_u = U$, P_w, and $P_r = D[d_1,d_x]$. (We could use $P_s = D[d_a,d_2]$, but for the possibility that $d_a = d_x$.) We find a separation (H,K) of L_P of order at most three such that $\{b_2,a_2,b_1,s\} \subseteq V(H)$ and $\{p_w,u,x_2,d_1\} \subseteq V(K)$. We may assume there are vertices $h_1,h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2,p_w]$ and $h_2 \in M_P[x_2,s]$. If $|V(H \cap K)| = 3$, label the third vertex as h_3.

By 2-connectivity of G, there is no cutvertex of L_P between b_2 and p_w. Suppose h_2 shares a finite face with h_1 in G, or h_3 shares a finite face with each of h_1,h_2, so (H,K) is a cutting separation. Then we have a separation (H',K') of G with $V(H' \cap K') = V(H \cap K) \cup \{d_a\}$. If this is of order 3, $(\Gamma,X) \in \mathcal{O}_2$. If it is of order 4, (Γ,X) satisfies (R1).

Now suppose $h_3 \in M_P[d_1,x_2)$ shares a finite face with h_1. Then $\{h_1,h_3,d_a\}$ is a 3-cut

in G, and $(\Gamma, X) \in \mathcal{O}_7$. If $h_3 \in M_P(x_2, b_1]$ shares a finite face with h_1, then there is some vertex $v' \in P[a_1, d_a]$ cofacial with each of h_1 and h_3. Hence $\{h_1, h_3, v'\}$ is a 3-cut in G, and $(\Gamma, X) \in \mathcal{O}_2$.

Finally, if $h_3 \in M_P(p_w, u]$ shares a finite face with h_2, then by our choice of U, h_3 is cofacial with some vertex $p \in V(P \cap C)$. Then we have a 4-separation (H', K') of G with $V(H' \cap K') = \{h_2, h_3, d_a, p\}$, and (Γ, X) satisfies (R1).

Case 4 Suppose $d_a \notin C[b_2, a_1]$ is cofacial with some $b \in M_P[b_2, w] \cap C[b_1, b_2]$.

Assume b is chosen to minimize $M_P[b, w]$. Recall the assumption that $a_1 \notin V(D)$. If $\{d_a, b\}$ is a 2-cut in G, or $\{d_a, b, b_1\}$ is a 3-cut in G, we have a separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{d_a, b, b_1, a_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, and $X \subseteq V(G_2)$. Hence (Γ, X) satisfies (R1). So we may assume no such cut exists, and $d_a \notin V(C)$. Furthermore, there are paths S_0 from $s_0' \in P[a_1, d_a)$ to $s_0 \in M_P[d_2, b_1)$, and T_0 from $t_0' \in P(s_0', d_a]$ to $t_0 \in M_P[b_2, b]$, where $t_0 = b$ only if $b = b_2$; otherwise $\{b, a_1, b_1, a_2\}$ is a 4-cut in G, producing a separation of G which shows that (Γ, X) satisfies property (R1). Choose such a pair S_0 and T_0 so that each vertex of $M_P[t_0, w]$ is cofacial with some vertex of $P[d_a, d_x]$, and each vertex of $M_P[d_2, s_0]$ is cofacial with some vertex of $P[t_0', d_a]$.

Suppose there is no path from $P(d_a, w_0')$ to $M_P[b, w)$. Then there is a facial cycle of G containing $V(M_P[t, w]) \cup V(P[t', d_a])$. Let $Q := P[a_1, s_0'] \cup S_0 \cup M_P[x_2, s_0]$. If there are disjoint paths from b_2 to t_0, from a_2 to w, and from b_1 to d_1 in L_P, each disjoint from $M_P[x_2, s_0]$, then we find $x_1 \in \mathcal{R}(Q)$, and X is feasible in Γ, a contradiction. Hence no such disjoint paths may exist. So we have either (i) a cutvertex z of L_P with $z \in M_P[x_2, s_0]$, or (ii) a 2-separation (H, K) of L_P with $\{a_2, b_1, s_0\} \subseteq H$ and $\{w, d_1, x_2\} \subseteq V(K)$.

If $z \in M_P[x_2, s_0]$ is a cutvertex of L_P, note that z is cofacial with some vertex $v \in P[t_0', d_a]$, which is in turn cofacial with b. So there is a 4-separation (M, N) of G with $V(M \cap N) = \{b, a_1, z, v\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and $X \subseteq V(N)$, and (Γ, X) satisfies property (R1).

So we assume there is a 2-separation (H, K) of L_P with $\{a_2, b_1, s_0\} \subseteq H$ and $\{w, d_1, x_2\} \subseteq V(K)$. Letting $V(H \cap K) = \{h_1, h_2\}$, we may assume $h_2 \in M_P[x_2, s_0]$. Here h_2 is cofacial

with some vertex $v \in P[t_0', d_a]$, which is in turn cofacial with b. If $h_1 \in C[a_2, b)$, then we have a 4-separation (H', K') of G with $V(H' \cap K') = V(H \cap K) \cup \{b, v\}$, demonstrating that (Γ, X) satisfies (R1). If $h_1 \in M_P[b, w]$, then h_1 is cofacial with v, and we have a 3-separation (H', K') of G with $V(H' \cap K') = V(H \cap K) \cup \{v\}$, demonstrating that $(\Gamma, X) \in \mathcal{O}_3$.

Now suppose there is some path T from $t' \in P(d_a, w_0')$ to $t \in M_P[b, w)$. Let T be chosen to minimize $P[d_a, t']$ and $M_P[b, t]$, and let S be a path from $s' \in P[a_1, d_a]$ to $s \in M_P[d_1, b_1)$, chosen to minimize $P[s', d_a]$ and $M_P[d_1, s]$. Now each vertex of $M_P[x_2, s]$ is cofacial with d_a, as is each vertex of $M_P[b, t]$.

Apply Proposition 3.9 with $P_s = S$, $P_u = W_0$, $P_w = T$, and $P_r = D[d_1, d_x]$. We find a separation (H, K) of L_P of order at most three such that $\{b_2, a_2, b_1, s\} \subseteq V(H)$ and $\{t, w, d_1, x_2\} \subseteq V(K)$. We may assume there are vertices $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, t]$, $h_2 \in M_P[x_2, s]$. If $|V(H \cap K)| = 3$, label the third vertex as h_3. We now have one of (i) h_2 is a cutvertex in L_P, (ii) (H, K) is a cutting separationin L_P, (iii) $h_3 \in M_P[b, w]$ shares finite face of L_P with h_2, or (iv) $h_3 \in M_P[d_1, x_2]$ shares a finite face of L_P with h_1.

If h_2 is a cutvertex of L_P, then there is a 4-separation (H', K') of G with $V(H' \cap K') = \{b, a_1, h_2, d_a\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$, and $X \subseteq V(K')$, and (Γ, X) satisfies property (R1).

If (H, K) is a cutting separation, then there is a separation (H', K') of G with $V(H' \cap K') = V(H \cap K) \cup \{d_a\}$. If this is of order three, then $(\Gamma, X) \in \mathcal{O}_2$. If of order 4, then (Γ, X) satisfies (R1).

Suppose $h_3 \in M_P[b, w]$ shares finite face with h_2. If h_3 shares a finite face with d_a, we have a 3-cut $\{h_3, h_2, d_a\}$ in G demonstrating $(\Gamma, X) \in \mathcal{O}_2$. Otherwise, h_3 is cofacial with some $p \in P(d_a, d_x]$, and we have a 4-cut $\{h_2, h_3, d_a, p\}$. An application of Lemma 2.11 gives $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) satisfies property (R1).

We now assume $h_3 \in M_P[d_1, x_2) \cup M_P(s, b_1]$ shares a finite face of L_P with h_1. We claim that we may assume $t_0 \in M_P[b_2, h_1)$. To see this, first note that $t_0 \notin M_P[b_2, h_1)$ if and only if $t_0 = h_1 = b$. Recalling that $t_0 = b$ only if $b = b_2$, this occurs only if $h_1 = b_2$. Suppose first that $h_3 \in M_P[d_1, x_2)$. So there is a 3-cut $\{h_1, h_3, d_a\}$ in G. If $h_1 = b_2$, then $(\Gamma, X) \in \mathcal{O}_7$.

Now suppose $h_3 \in M_P(s, b_1]$. If $h_1 = b_2$, then we have some vertex $v' \in P[a_1, d_a]$ which is cofacial with each of h_3 and b_2, so $\{h_1, h_3, v'\}$ is a 3-cut in G, and $(\Gamma, X) \in \mathcal{O}_2$.

Let $Q := P[a_1, s_0'] \cup S_0 \cup M_P[x_2, s_0]$. If there are disjoint paths from b_2 to t_0, from a_2 to w, and from b_1 to d_1 in L_P, each disjoint from $M_P[x_2, s_0]$, then we find $x_1 \in \mathcal{R}(Q)$, and X is feasible in Γ, a contradiction. Hence no such disjoint paths may exist. So we have either (i) a cutvertex z of L_P with $z \in M_P[x_2, s_0]$, or (ii) a 2-separation (M, N) of L_P with $\{a_2, b_1, s_0\} \subseteq M$ and $\{w, d_1, x_2\} \subseteq V(N)$.

If $z \in M_P[x_2, s_0]$ is a cutvertex of L_P, note that z is cofacial with some vertex $v \in P[t_0', d_a]$, which is in turn cofacial with b. So there is a 4-separation (M', N') of G with $V(M' \cap N') = \{b, a_1, z, v\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, and $X \subseteq V(N')$, and (Γ, X) satisfies property (R1).

So we assume there is a 2-separation (M, N) of L_P with $\{a_2, b_1, s_0\} \subseteq M$ and $\{w, d_1, x_2\} \subseteq V(N)$. Letting $V(M \cap N) = \{m_1, m_2\}$, we may assume $m_2 \in M_P[x_2, s_0]$. Here m_2 is cofacial with some vertex $v \in P[t_0', d_a]$, which is in turn cofacial with b. If $m_1 \in C[a_2, b)$, then we have a 4-separation (M', N') of G with $V(M' \cap N') = V(M \cap N) \cup \{b, v\}$, demonstrating that (Γ, X) satisfies (R1). Finally, if $m_1 \in M_P[b, w]$, then either m_1 is cofacial with v, or h_1, m_2 is a 2-cut in L_P, with each of h_1, m_2 cofacial with v. Hence we have a 3-separation (M', N') of G demonstrating that $(\Gamma, X) \in \mathcal{O}_3$.

Case 5 Suppose none of Cases 1-4 hold.

Here d_a is not cofacial with any vertex of $M_P[b_2, w] \cap C[b_1, b_2]$, and there is some path T from $t' \in P[a_1, d_a)$ to $t \in M_P(b_2, d_1]$. Choose such a path T to minimize $P[t', d_a]$ and $M_P[t, d_1]$. Choose a path D_x from $d_x' \in P[d_x, w_0')$ to $d_1' \in M_P(w, d_1]$ to minimize $P[d_x', w_0']$ and $M_P[w, d_1']$. ($D_x = D[d_1, d_x]$ is possible.)

Applying Proposition 3.10 with $Q_t = T$, $Q_w = W_0$, $Q_r = D_x$, and $Q_s = D[d_a, d_2]$, we find that there must be a separation (M, N) of L_P of order at most three such that $\{t, b_2, a_2, b_1\} \subseteq V(M)$ and $\{w, d_1', d_2\} \cup X' \subseteq V(N)$. We may assume there are vertices $m_1, m_2 \in V(M \cap N)$ with $m_1 \in M_P[t, w], m_2 \in M_P[d_2, b_1]$. If $|V(M \cap N)| = 3$, label the third vertex as m_3. Notice from our choice of T and W_0 that m_1 must be cofacial with

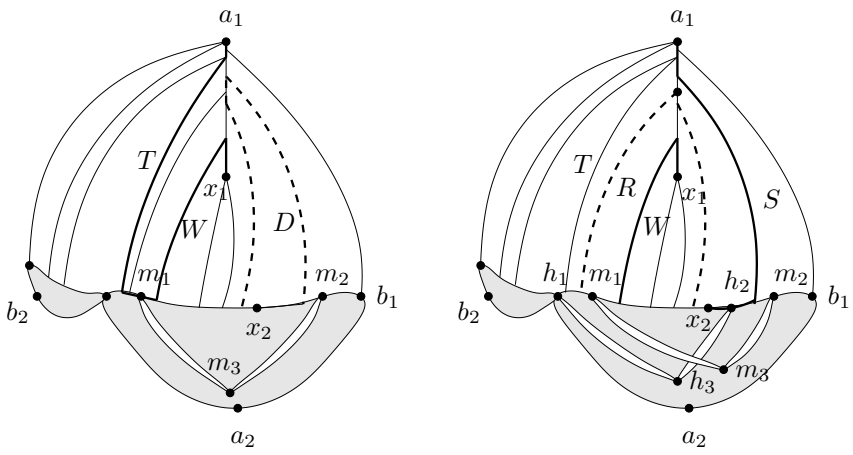


Figure 25: Separations when s' not cofacial with b_2

some $v \in P[d_a, d_x]$ (Figure 25(a)). We have one of the following situations:

(i) m_2 shares a finite face of L_P with m_1 in G, or

(ii) m_3 shares a finite face L_P with each of m_1, m_2, or

(iii) $m_3 \in M_P[x_2, d_2]$ shares a finite face L_P with m_1, or

(iv) $m_3 \in M_P[b_2, t) \cup M_P(w, d_1']$ shares a finite face L_P with m_2.

In case (iii), we have each of m_1, m_3 cofacial with v, so $\{m_1, m_3, v\}$ is a 3-cut demonstrating that $(\Gamma, X) \in \mathcal{O}_2$. We may assume, then, that case (iii) does not occur.

(5.1) Suppose (i) or (ii) holds, so (M, N) is a cutting separation.

Assume S and R are chosen to minimize $P[a_1, s']$, $M_P[s, b_1]$, $P[a_1, r']$, and $M_P[b_2, r]$.

Suppose m_1 and m_2 are both cofacial with some $v' \in P[t', s']$. Then we have a separation (M', N') of G with $V(M' \cap N') = V(M \cap N) \cup \{v'\}$. In case (i), (M', N') is a 3-separation, and $(\Gamma, X) \in \mathcal{O}_2$. In case (ii), (M', N') is a 4-separation, and by Lemma 2.11, $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies (R1).

Hence we may assume that any vertex $v' \in P[t', s']$ which is cofacial with m_1 is not also cofacial with m_2. So there are paths S from $s' \in P[a_1, d_a]$ to $s \in M_P[d_2, m_2)$ and R from $r' \in P(s', w']$ to $r \in M_P[t, m_1)$. (If there is a path "between" T and W, we get

$S = D[d_a, d_2]$. If not, we get some S above T, and $R = T$.) From our choice of W, we may assume $r' \in P(s', d_x]$, and R is thus disjoint from W.

Apply Proposition 3.9 with $P_s = S$, $P_u = W$, $P_w = R$, and $P_r = D[d_1, d_x]$, to find that there is a separation (H, K) of L_P of order at most three such that $\{b_2, a_2, b_1, s\} \subseteq V(H)$ and $\{r, w, d_1\} \cup X' \subseteq V(K)$. We may assume there are vertices $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, r]$ and $h_2 \in M_P[x_2, s]$. If $|V(H \cap K)| = 3$, label the third vertex as h_3 (Figure 25(b)). We have one of the following situations:

(a) h_2 shares a finite face of L_P with h_1 in G, or

(b) h_3 shares a finite face of L_P with each of h_1, h_2, or

(c) $h_3 \in M_P[d_1, x_2) \cup M_P(s, b_1]$ shares a finite face of L_P with h_1, or

(d) $h_3 \in M_P(r, w]$ shares a finite face of L_P with h_2.

In case (d), h_3 is cofacial with some $v' \in P[d_a, d_x]$, which is also cofacial with h_2. Hence there is a 3-cut $\{h_2, h_3, v'\}$, demonstrating that $(\Gamma, X) \in \mathcal{O}_2$. We may assume, then, that case (d) does not occur.

Suppose $h_3 \in M_P(m_2, b_1]$ shares a finite face of L_P with h_1. By our choice of S and R, h_1 and h_3 are cofacial with a common vertex $p \in P[a_1, r']$. So we have a separation (M', N') of G with $V(M' \cap N') = \{h_1, h_3, p\}$, $\{a_1, b_2, a_2, b_1\} \subseteq V(M')$ and $\{x_1, x_2\} \subseteq V(N')$, and $(\Gamma, X) \in \mathcal{O}_2$. So we may assume that if (c) occurs, $h_3 \in M_P[d_1, m_2]$.

We now have the vertices $b_2, h_1, r, m_1, w, d_1', d_1, x_2, h_2, s, m_2, b_1$ in that order along M_P. (See Figure 26.) By planarity, either $\{h_1, m_2\}$ is a 2-cut in L_P, or there is some $m \in \{h_3, m_3\}$ such that $\{h_1, m_2, m\}$ is a 3-cut in L_P. By our choice of S and R, h_1 and m_2 are cofacial with a common vertex $p \in P[a_1, r']$. So we have a separation (M', N') of G with $V(M' \cap N') = \{h_1, m_2, m, p\}$, $\{a_1, b_2, a_2, b_1\} \subseteq V(M')$ and $\{x_1, x_2\} \subseteq V(N')$. If $|V(M' \cap N')| = 3$, then $(\Gamma, X) \in \mathcal{O}_2$. If $|V(M' \cap N')| = 4$, note that there are disjoint paths from h_1 to x_1 (along W), from p to m (along D), and from m_2 to x_2 in N'. By Lemma 2.11, either $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) satisfies property (R1).

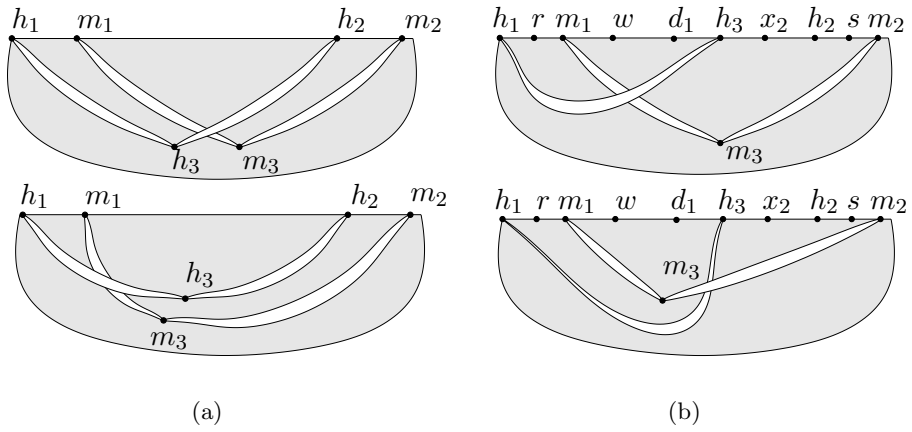


Figure 26: Combinations of separations, Case (5.1)

(5.2) Suppose (iv) holds: $m_3 \in M_P[b_2, t) \cup M_P(w, d_1']$ shares a finite face L_P with m_2.

We assume (M, N) is chosen to maximize $M_P[m_3, m_2]$.

Choose $b_1' \in M_P[m_2, b_1] \cap C[b_1, b_2]$ to minimize $M_P[m_2, b_1']$. So either b_1' is a cutvertex of L_P separating a_2 from b_1, or $b_1 \in \mathcal{A}(P)$ and $b_1' = b_1$.

Let S be a path from $s' \in P[a_1, x_1]$ to $s \in M_P[d_2, b_1')$ ($D[d_a, d_2]$ is one such path), chosen to minimize $P[a_1, s']$ and $M_P[s, b_1']$, so each vertex of $M_P[s, b_1']$ is cofacial with s'. Let T_0 be a path from $t_0' \in P(s', w']$ to $t_0 \in M_P[b_2, w]$, chosen to minimize $P[s', t_0']$ and $M_P[b_2, t_0]$ ($T_0 = W$ is possible). So each vertex of $M_P[b_2, t_0]$ is cofacial with some vertex $v \in P[a_1, s']$, and by an application of Lemma 3.3 (controlling paths within $(P \cup L_P)$-bridges), v is in turn cofacial with b_1'.

Claim 1. We may assume T_0 is left of P.

Proof of Claim 1 Since W is not right of P, we have either T_0 left of P or $t_0' = x_1$. Suppose $t_0' = x_1$. Then x_1 is cofacial with each vertex of $P[s', x_1]$, and x_1 is cofacial with m_3. Letting $v \in P[s', x_1]$ be some vertex cofacial with m_2, we have a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{m_2, m_3, x_1, v\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $X' \subseteq V(G_2)$. Note that $D[d_a, d_x]$ provides a path from v to m_3 in G_2, disjoint from $M_P[x_n, m_2]$. By Lemma 2.11, $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). Hence we may assume $t_0' \neq x_1$, and this proves Claim 1.

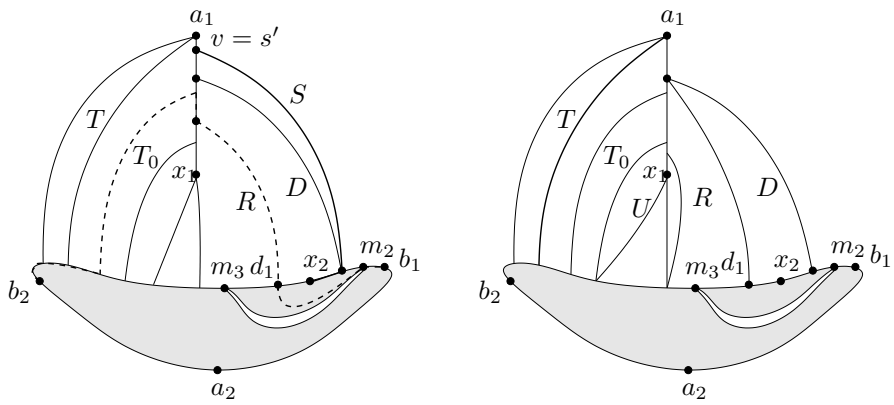


Figure 27: Case (5.2)

Choose a path R from $r' \in P(s', x_1]$ to $r \in M_P[w, d_1]$ to minimize $M_P[r, d_1]$, and subject to this to minimize $P[s', r']$.

Claim 2. We may assume R is right of P.

Proof of Claim 2 If R is left of P, we find that one of $\{r', r\}$ or $\{s', r\}$ is a 2-cut in G, contradicting $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. So R is not left of P. Suppose $r' = x_1$. Then x_1 shares a facial cycle F to the right of P with s', and hence $s' = d_a = d_x$. Moreover, there is vertex $p \in V(P[s', x_1]) \cap V(F)$ which is cofacial with m_3. So there is a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{m_3, m_2, s', p\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, and $X \subseteq V(G_2)$. By Lemma 2.11, $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). Hence we may assume $r' \neq x_1$. This proves Claim 2.

Now by Claim 2 and $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, there is some path U from $u' \in P(r', x_1] \cap P(t_0', x_1]$ to $u \in M_P[t_0, r]$.

Claim 3. We may assume U is chosen with $u \in M_P(t_0, m_3)$, $u \neq r$.

Proof of Claim 3 Suppose r' and m_3 share a finite face left of P. By our choice of R and S, we have each of r' and m_2 cofacial with s', and there is a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{m_3, m_2, r', s'\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, and $X \subseteq V(G_2)$. By Lemma

2.11, $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). Hence we may assume r' and m_3 do not share a finite face left of P, so $u \in M_P[t_0, m_3)$.

By our choice of T_0, together with Lemma 3.3, we have some vertex $v \in P[a_1, t_0']$ which is cofacial with both of t_0 and m_2. If m_3 is cofacial with t_0, then there is a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{m_3, m_2, v, t_0\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, and $X \subseteq V(G_2)$. By Lemma 2.11, $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). Hence we may assume t_0 is not cofacial with m_3, and thus $u \in M_P(t_0, m_3)$.

Now suppose we must choose $u = r$, so $r \in M_P(t_0, m_3)$. For any vertex $v \in P[s', r']$, $\{v, r\}$ is not a 2-cut in G. Hence there must be some path U_0 from x_1 to $u_0 \in V(P[a_1, s')) \cup \{t_0\}$. By Lemma 3.3, we find $u_0 = t_0$. Together with our choice of T_0, this gives $T_0 = W$, $U_0 = W_0$. But then $w_0' = x_1$, $r' \in P[d_x, w_0')$, and $r \in M_P(w, m_3)$. This contradicts the choice of D_x and $\{m_2, m_3\}$. Hence we may assume $u \neq r$, which completes the proof of Claim 3.

Apply Proposition 3.9 with $P_s = S$, $P_u = U$, $P_w = T_0$, and $P_r = R$, to find that there is a separation (H, K) of L_P of order at most three such that $\{b_2, a_2, b_1, s\} \subseteq V(H)$ and $\{t_0, u, r, x_2\} \cup X' \subseteq V(K)$. We may assume there are vertices $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, t_0]$ and $h_2 \in M_P[x_2, s]$. If $|V(H \cap K)| = 3$, label the third vertex as h_3. (See Figure 28.) We have one of the following situations:

(a) h_2 shares a finite face of L_P with h_1 in G, or

(b) h_3 shares a finite face of L_P with each of h_1, h_2, or

(c) $h_3 \in M_P[r, x_2) \cup M_P(s, b_1']$ shares a finite face of L_P with h_1, or

(d) $h_3 \in M_P(t_0, u]$ shares a finite face of L_P with h_2.

In case (a), we see by planarity that $\{h_1, m_2\}$ is a 2-cut in G, contradicting our choice of $\{m_2, m_3\}$. Similarly in case (d), $\{h_3, m_2\}$ is a 2-cut in G, contradicting our choice of $\{m_2, m_3\}$. Hence we may assume cases (a) and (d) do not occur.

Suppose case (c) occurs, with $h_3 \in M_P(s, b_1']$ sharing a finite face of L_P with h_1. If h_3 is right of m_2, we have a 2-cut $\{h_1, h_3\}$ in G. If h_3 is left of m_2, then $\{h_1, m_2\}$ is a 2-cut in

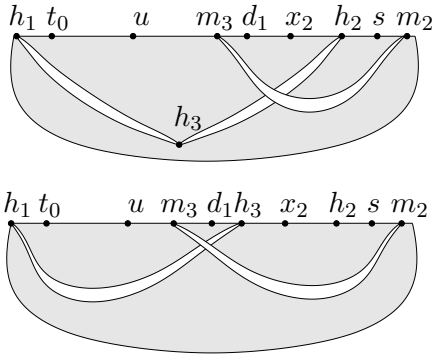


Figure 28: Combinations of separations, Case (5.2)

G. In each case, the 2-cut contradicts our choice of $\{m_2, m_3\}$. So we may assume that of (c) occurs, we have $h_3 \in M_P[r, x_2)$.

We now have the vertices $b_2, h_1, t_0, u, d_1, x_2, h_2, s, m_2, b_1'$ in that order along M_P. (See Figure 28.) By planarity, either $\{h_1, m_2\}$ is a 2-cut in L_P, or $\{h_1, m_2, h_3\}$ is a 3-cut in L_P. By our choice of $\{m_2, m_3\}$, we must have $\{h_1, m_2, h_3\}$ a 3-cut in L_P.

By our choice of paths S and T_0, we have some vertex $v' \in P[a_1, t_0']$ which is cofacial with both h_1 and m_2. Hence there is a 4-separation (H', K') of G with $V(H' \cap K') = \{h_1, h_3, m_2, v'\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H' \cap K')$, and $X \subseteq V(K)$. By an application of Lemma 2.11, either $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) satisfies property (R1).

□

Corollary 3.13. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $\{x_1, x_2\} \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ such that $\{x_1, x_2\} \cap V(C) = \emptyset$, and $\{x_1, x_2\}$ is infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1. Then $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$, or satisfies property (R1).*

3.5 Separating X from a

Here we consider the special case where X is separated from a_2 by a 2-cut in L_P.

Lemma 3.14. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 2$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1. Assume further that there is some $x_2 \in X$*

such that $x_2 \notin V(M_P)$*. If there is some 2-separation* (H, K) *of* L_P *with* $X \setminus \{x_1\} \subseteq V(K)$ *and* $\{b_1, a_2, b_2\} \subseteq H$*, then either* (Γ, X) *satisfies property (R1) or* $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.

The proof of Lemma 3.14 is given in the next several subsections. In Section 3.5.1, we define several paths from P to L_P which will exist in any counterexample to Lemma 3.14. (Recall that when we refer to a path from P to L_P, we mean a path which is internally disjoint from $P \cup L_P$.) One of these is the path T from $t' \in V(P)$ to $t \in V(M_P)$. We then show that the path T is not to the right of P in any counterexample to the lemma. In Section 3.5.2, we address the case that $t' = x_1$, and in Section 3.5.3 we address the case that T is to the left of P.

Paths and vertex labels defined within the proof of any single claim will apply only within that proof. To limit the notation required, they may be redefined in subsequent claims. Labels defined outside any claim or case will apply to all subsequent arguments in the section, unless clearly noted otherwise.

Note that since $X \cap V(C) = \emptyset$, if (Γ, X) is reducible then (Γ, X) must satisfy property (R1).

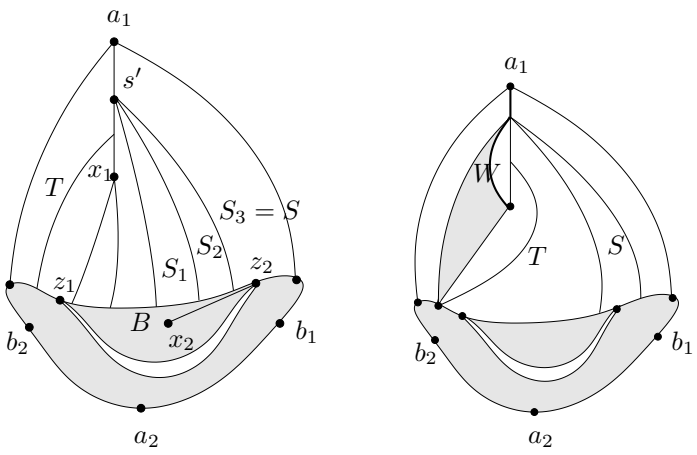
3.5.1 Defining common paths

Let $X' = X \setminus \{x_1\}$. Assume (H, K) is chosen with K maximal, and let $Z = V(H \cap K)$. Then $Z \subseteq V(M_P)$. Label $Z = \{z_1, z_2\}$ such that z_1 lies to the left of z_2. Since $x_2 \in V(K) \setminus V(M_P) \neq \emptyset$, Z is a 2-cut in L_P. Let B be the Z-bridge of L_P containing X'.

We may assume $X' \subseteq \mathcal{A}(P)$. Otherwise, we apply Lemma 3.7 to find (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.

Define paths S and T: Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, Z is not a 2-cut in G, so there must be some path S in G from $s' \in V(P)$ to $s \in M_P(z_1, z_2)$. Choose S to minimize $P[a_1, s']$. We may assume by symmetry that S is to the right of P, and we further choose S to minimize $M_P[s, z_2]$. If there are multiple paths from s' to $M_P(z_1, z_2)$, label these as $S_1, S_2, \ldots, S_k = S$ with attachments $s_1, s_2, \ldots, s_k$ in order from z_1 to z_2. (See Figure 29(a).)

Note that if there is a separation (M, N) of G with $V(M \cap N) = \{s', z_1, z_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq$



(a) $S_1, S_2, \ldots, S_k$ (b) Path T right of P

Figure 29: Initial path definitions

$V(M)$ and $X \subseteq V(N)$, we have $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume there is no such separation, so by the choice of S there is some path T from $P(s', x_1]$ to $P[a_1, s') \cup M_P[b_2, z_1)$. By Lemma 3.2 (every $(P \cup L_P)$-bridge has an attachment in L_P), there is a path T from $t' \in P(s', x_1]$ to $t \in M_P[b_2, z_1)$. Choose such a T to minimize $M_P[b_2, t]$, and subject to this minimizing $P[s', t']$.

Claim 1. We may assume either $t' = x_1$ or T is to the left of P.

Proof. Suppose not, so $t' \neq x_1$ and T is to the right of P (Figure 29(b)). From our choice of T to minimize $M_P[b_2, t]$, we find that there is no path from $P(s', x_1]$ to $M_P[b_2, t)$. From planarity, there is no path from $P(t', x_1]$ to $M_P(t, b_1]$.

Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, the set $\{t', t\}$ is not a 2-cut in G, so there must be some path W from $w' \in P[a_1, t')$ to $w \in P(t', x_1]$ internally disjoint from $P \cup L_P$. Now from Lemma 3.2 (every $P \cup L_P$-bridge has an attachment in L_P), we may assume W lies entirely in some $(P \cup L_P)$-bridge of G which has attachment t on L_P. From our choice of T to minimize $P[s', t']$, we must have $w' \in P[a_1, s']$.

Now by Lemma 3.3 (limiting paths within $(P \cup L_P)$-bridges), there can be no path from $P(w', t']$ to $M_P(t, b_1]$. Hence $w' = s'$, and $\{s', t\}$ is a 2-cut in G, violating the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. □

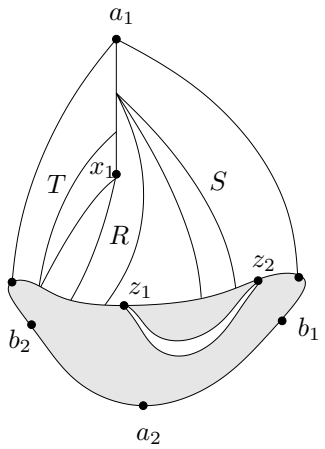


Figure 30: Defining path R

Define paths U_s and Q_x: Let U_s be a path from $u'_s \in P[s', x_1]$ to $u_s \in M_P(z_1, s]$, chosen to minimize $M_P[z_1, u_s]$, and subject to this, to minimize $P[u'_s, x_1]$. Note that if $u'_s = s'$, we have $U_s = S_1$.

Now from $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, there can be no 2-cut of L_P contained in $M_P[z_1, u_s]$ and separating x_2 from z_2 in B. Since we assume $x_2 \notin V(M_P)$, there are disjoint paths $Q_b = M_P[z_1, u_s]$ and Q_x from z_2 to x_2 in B.

Define path R: Note from our choices of S and T that t is cofacial with s', and s' is cofacial with z_2. We also have z_2 cofacial with z_1.

Suppose t is cofacial with z_1. Then there is 4-separation (M, N) of G with $V(M \cap N) = \{z_1, z_2, s', t\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. If $P[t', x_1] \subseteq V(N)$, we have the following disjoint paths in N: $S_1 \cup M_P[z_1, s_1]$ from s' to z_1, $T \cup P[t', x_1]$ from t to x_1, and Q_x from z_2 to x_2. Applying Lemma 2.11 (4-separations in G), we find that $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. So we may assume that either t is not cofacial with z_1, or for any such separation $x_1 \notin V(N)$. Therefore there is some path R from $r' \in P[s', x_1]$ to $r \in M_P(t, z_1)$.

We claim that we may choose R such that $r' \neq s'$. (See Figure 30.) To see this, suppose not. Then any path from $P(s', x_1]$ to $M_P[t, s_1)$ has endpoint t, and $\{s', t\}$ is a 2-cut in G violating $(3, \{a_1, b_1, a_2, b_2\})$-connectivity.

3.5.2 The case $t' = x_1$.

Claim 2. We may assume R falls to the right of P

Proof. From our choice of T, we see that $P[s', x_1]$ is contained in the boundary of a face to the left of P. Suppose x_1 is cofacial with z_1. Then $\{z_1, z_2, s', x_1\}$ is a 4-cut in G. Recalling the existence of the paths Q_b and Q_x, from u_s to z_1 and x_2 to z_2 respectively, we apply Lemma 2.11 (4-separations in G) to see that either (Γ, X) is reducible, or $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume x_1 is not cofacial with z_1.

Similarly, if s' is cofacial with z_1, then $\{z_1, z_2, s'\}$ is a 3-cut in G, and $\{z_1, z_2, s', x_1\}$ is a 4-cut in G. An application of Lemma 2.11 shows (Γ, X) is reducible, or $(\Gamma, X) \in \mathcal{O}_2$. So we may assume s' is not cofacial with z_1.

We have shown that neither x_1 nor s' is cofacial with z_1. Together with our choice of R and the planarity of G, this proves the claim. □

Define path R_0: Furthermore, since s' is not cofacial with z_1, there is some path R_0 from $r'_0 \in P(s', r']$ to $r_0 \in M_P(z_1, s]$. Choose such a path R_0 to minimize $P[s', r'_0]$ and $M_P[r_0, s]$.

Define path T_0: Noting that $x_1 \notin V(C)$, we see that there is a path T_0 from $t'_0 \in P[a_1, s']$ to $t_0 \in M_P[b_2, t]$. Choose such a path with $t'_0 \neq s'$ if possible, and then to minimize $P[t'_0, s']$ and $M_P[t_0, t]$. Observe that each vertex of $T_0 \cup M_P[t_0, t] \cup T \cup P[t'_0, x_1]$ is cofacial with s'. (This will in fact be a face boundary unless there is some path from s' to $M_P(t_0, t)$.) Moreover, if $t'_0 = s'$, we have $P[a_1, s'] \subseteq V(C)$.

Claim 3. We may assume $C[a_2, b_2] \cap M_P[t_0, t] = \emptyset$.

Proof. To see this, suppose there is some vertex $z_t \in C[a_2, b_2] \cap M_P[t_0, t]$. (The example $z_t = t_0 = b_2$ is illustrated in Figure 31.) If $t_0 = b_2$, we recall Proposition 2.9, $a_1 b_2$ is not an edge of G, so $t'_0 \neq a_1$. If $\{z_t, s'\}$ is a 2-cut separating $\{a_1, b_2\}$ from $\{b_1, a_2\} \cup X$ in G, then we have a 4-separation (M, N) of G with $V(M \cap N) = \{z_t, s', b_1, a_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and $X \subseteq V(N)$, and (Γ, X) satisfies property (R1). So, by our choice of S, there must be some path S_0 from $s'_0 \in P[a_1, t'_0)$ to $s_0 \in M_P[z_2, b_1]$. Choose such an S_0 to minimize $P[s'_0, t'_0]$ and $M_P[z_2, s_0]$, so each of s'_0, s_0 is cofacial with t'_0.

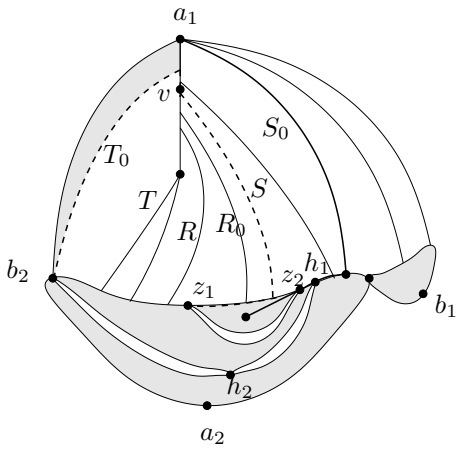


Figure 31: Where $t_0 = b_2$

For any vertex $v \in M_P[z_2, s_0]$, the set $\{v, z_t, s'\}$ is not a 3-cut separating $\{a_1, b_1\}$ from $\{a_2\} \cup X$ in G, or G would have a 4-separation (H, K) with $V(H \cap K) = \{v, z_t, s', a_2\}$ demonstrating that (Γ, X) satisfies (R1). So we may assume $s_0 \notin C[b_1, a_2]$, and $s_0 \in \mathcal{A}(P)$.

Define $Q := P[a_1, s'_0] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, a path from a_1 to x_2.

We first establish that $Q \in \mathcal{P}_{(\Gamma,X)}$. Suppose not. Then a_2 is a cutvertex of L_Q, and a_2 is cofacial with some vertex $q \in V(Q)$. We may assume $q \in M_P[z_2, s_0]$ (because of the 2-cut Z in L_P). But then we have a 4-separation (M, N) of G with $V(M \cap N) = \{a_2, q, t'_0, z_t\}$, and we find (Γ, X) satisfies property (R1). Hence we may assume a_2 is not a cutvertex of L_Q, and $Q \in \mathcal{P}_{(\Gamma,X)}$.

Note that $(T_0 \cup U_s \cup Q_b) \subseteq L_Q$. If there are disjoint paths in L_P from a_2 to r and from b_1 to z_1 respectively, each disjoint from $M_P[z_2, s_0]$ and b_2, then we find that $x_1 \in \mathcal{R}(Q)$. Hence no such paths may exist. By our choice of Z as a maximal 2-cut, we see that there must be some 3-cut $\{z_t, h_1, h_2\}$ in L_P separating $\{a_2, b_1, s_0\}$ from $\{r, z_1, z_2\}$, with $h_1 \in M_P[z_2, s_0]$.

Now h_1 is cofacial with some vertex $v \in P[s', x_1]$, and v is also cofacial with z_t. So G has a separation (H', K') of order 4 with $V(H' \cap K') = \{z_t, h_1, h_2, v\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$, and $X \subseteq V(K')$. Note that we have disjoint paths from z_t to x_1, from v to h_2, and from h_1 to x_2 in K'. Applying Lemma 2.11, we find that $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). □

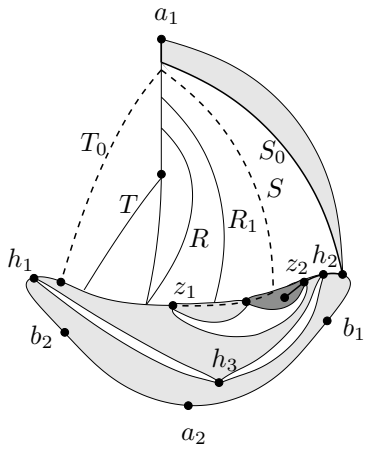


Figure 32: Where path O_x does not exist

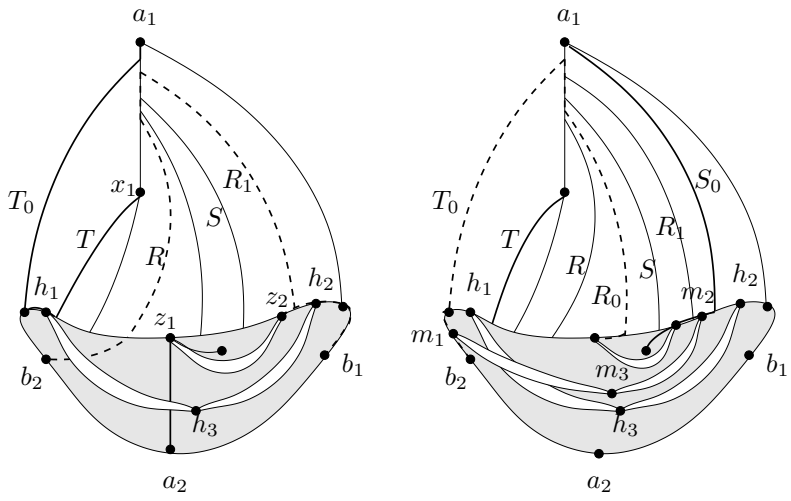
Define path R_1: Let R_1 be a path from $r_1' \in P(t_0', s']$ to $r_1 \in M_P[s, b_1]$, if such a path exists, chosen to minimize $P[t_0', r_1']$ and $M_P[r_1, b_1]$. (Note that $R_1 = S$ is possible.) If such a path does not exist, we must have $t_0' = s'$, and we let $R_1 = R_0$. By our choice of S, we see that either $R_1 = R_0$, or $R_1 = S$, or $r_1 \in M_P[z_2, b_1]$.

Claim 4. We may assume there is some path O_x from z_1 to X' in $B - M_P[r_1, z_2]$.

Proof. Since $x_2 \notin V(M_P)$, we see from $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and our choice of S that such a path must exist if $r_1 \in M_P[s, b_1]$. Suppose, then, that $r_1 \notin M_P[s, b_1]$. We must then have $R_1 = R_0$, and $t_0' = s'$. (See Figure 32.) Then for each $x \in X'$, we must have either $x \in M_P[r_1, b_1]$, or some 2-cut $W \subseteq M_P[r_1, b_1]$ separates x from z_1 in B. (Recall that B is the Z-bridge of L_P containing X'.) Hence for each $x \in X'$, either x is cofacial with s', or $W \cup \{s'\}$ is a 3-cut in G separating x from $\{b_1, a_2, b_2\}$. We recall that x_1 is also cofacial with s'. We may assume, then, that $s' \neq a_1$; otherwise we have $(\Gamma, X) \in \mathcal{O}_3$.

Since $t_0' = s'$, we know $P[a_1, s'] \subseteq V(C)$. Hence there must be some path S_0 from $s_0' \in P[a_1, t_0')$ to $s_0 \in M_P[z_2, b_1]$ (by our choice of S). Choose such an S_0 to minimize $P[s_0', t_0']$ and $M_P[z_2, s_0]$, so each of s_0', s_0 is cofacial with t_0'. For any vertex $v \in M_P[z_2, s_0]$, the set $\{v, b_2, s'\}$ is not a 3-cut separating $\{a_1, b_1\}$ from $\{a_2\} \cup X$ in G, or G would have a 4-separation (H, K) with $V(H \cap K) = \{v, b_2, s', a_2\}$ demonstrating that (Γ, X) satisfies (R1). So we may assume $s_0 \notin C[b_1, a_2]$ and $s_0 \in \mathcal{A}(P)$.

Define $Q := P[a_1, s_0'] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, a path from a_1 to x_2.



(a) Path O_1 (b) Path O_2

Figure 33: Forcing crossed 3-separations when $t' = x_1$

We claim that $Q \in \mathcal{P}_{(\Gamma,X)}$. To see this, suppose not. Then a_2 is a cutvertex of L_Q, and a_2 is cofacial with some vertex $q \in V(Q)$. We may assume $q \in M_P[z_2, s_0]$ (because of the 2-cut Z in L_P). But then we have a 4-separation (M, N) of G with $V(M \cap N) = \{a_2, q, t'_0, b_2\}$, and we find (Γ, X) satisfies property (R1). Hence we may assume a_2 is not a cutvertex of L_Q, and $Q \in \mathcal{P}_{(\Gamma,X)}$.

Now $(T_0 \cup U_s \cup Q_b) \subseteq L_Q$. If there are disjoint paths in L_P from a_2 to r and from b_1 to z_1 respectively, each disjoint from $M_P[z_2, s_0]$ and $M_P[b_2, t_0]$, then we find that $x_1 \in \mathcal{R}(Q)$. Hence no such paths may exist. By our choice of Z as a maximal 2-cut, we see that there must be some 3-cut $\{h_1, h_2, h_3\}$ in L_P separating $\{b_2, a_2, b_1, s_0\}$ from $\{t_0, r, z_1, z_2\}$. We may assume $h_1 \in M_P[b_2, t_0]$ and $h_2 \in M_P[z_2, s_0]$.

Now h_2 is cofacial with s', so G has a separation (H', K') of order 4 with $V(H' \cap K') = \{h_1, h_2, h_3, s'\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$, and $X \subseteq V(K')$. Since $h_1, s' \in C[b_2, a_1]$, (Γ, X) satisfies property (R1). □

Define $O_1 := P[a_1, t'_0] \cup T_0 \cup M_P[t_0, t] \cup T$, a path from a_1 to x_1 in G. (See Figure 33(a).)

Suppose there are four disjoint paths in L_P: from t_0 to t, from b_2 to r, from a_2 to z_1, and from r_1 to b_1 in L_P. These paths together with R, R_1, and O_x would show that X is

feasible in Γ, a contradiction. Hence no such paths may exist. There must be a separation (M, N) of L_P of order at most 3 such that $\{t_0, b_2, a_2, b_1\} \subseteq V(M)$ and $\{t, r, z_1, r_1\} \subseteq V(N)$. By Claim 3, there is no cutvertex of L_P in $M_P[t_0, t]$. From this, and our choice of Z as a maximal cut, we see that $|V(M \cap N)| = 3$. We may assume $V(M \cap N) = \{h_1, h_2, h_3\}$ with $h_1 \in M_P[t_0, t], h_2 \in M_P[r_1, b_1] \cap M_P[z_2, b_1]$. We further assume that (M, N) is chosen to minimize $M_P[z_2, h_2]$.

If h_2 is cofacial with t_0', there is a separation (M', N') of G of order 4 such that $\{a_1, b_2, a_2, b_1\} \subseteq V(M')$ and $X \subseteq V(N')$, and $V(M' \cap N') = V(M \cap N) \cup \{t_0'\}$. By maximality of Z, N' has disjoint paths from h_1 to x_1, from t_0' to h_3, and from h_2 to x_2. An application of Lemma 2.11 shows that either (Γ, X) satisfies property (R1), or $(\Gamma, X) \in \mathcal{O}_2$. So we may assume h_2 is not cofacial with t_0'.

From our choice of R_1, this means there is some path S_0 from $s_0' \in P[a_1, t_0')$ to $s_0 \in M_P[r_1, h_2)$. Choose such an S_0 to minimize $P[s_0', t_0']$ and $M_P[r_1, s_0]$. Note by our choice of S that $s_0 \in M_P[z_2, b_1]$.

Define $O_2 := P[a_1, s_0'] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, a path from a_1 to x_2 in G.

If there are three disjoint paths, from b_1 to z_1, from a_2 to t, and from b_2 to t_0 in $L_P - O_2$, then we see that $T \cup R_0 \subseteq L_{O_2}$, $x_1 \in \mathcal{R}(O_2)$, and X is feasible in Γ, a contradiction. Hence no such paths exist, and there must be a 3-separation $(M'N')$ of L_P such that $\{s_0, b_1, a_2, b_2\} \subseteq V(M')$ and $\{z_2, z_1, t, t_0\} \subseteq V(N')$. We may assume $V(M' \cap N') = \{m_1, m_2, m_3\}$ with $m_1 \in M_P[b_2, t_0]$ and $m_2 \in M_P[z_2, s_0]$. (See Figure 33(b).)

Suppose m_2 is a cutvertex of L_P. By planarity (and since $z_2 \in \mathcal{A}(P)$), m_2 is right of z_2. But then there is a separation (L_1, L_2) of L_P with $V(L_1 \cap L_2) = \{h_1, m_2, h_3\}$, contradicting the choice of separation (M, N) is to minimize $M_P[z_2, h_2]$. So we may assume m_2 is not a cutvertex of L_P.

If m_1 shares a finite face with m_2, then $\{m_1, m_2\}$ is a 2-cut in L_P contradicting the maximality of Z. If $\{m_1, m_3\}$ separates $\{b_2, a_2, b_1\}$ from $\{t_0, t, r_0\}$ in L_P, then $m_3 \in M_P[z_1, b_1]$. By maximality of Z, we find $m_3 \in M_P[z_1, z_2)$, and we may assume $m_2 = z_2$, sharing a finite face with m_3. So we may in any case assume m_3 shares a finite face of L_P with each of m_1 and m_2.

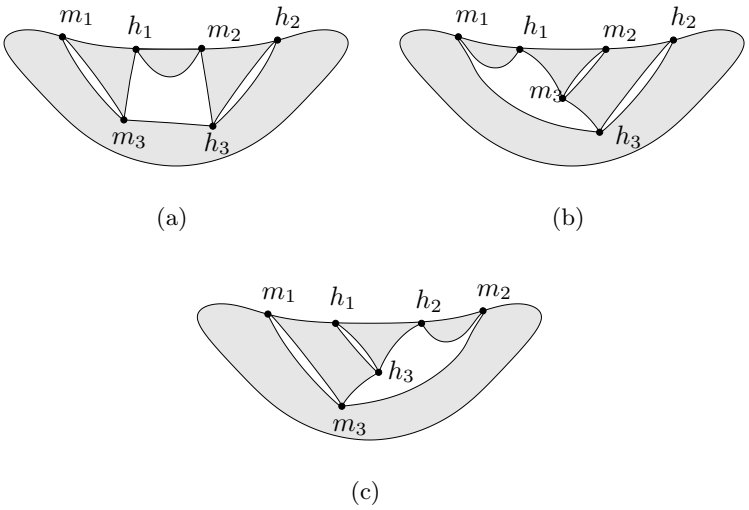


Figure 34: Cases of crossed 3-separations

Note that the vertices $(b_2, m_1, t_0, h_1, t, z_2, m_2, s_0, h_2, b_1)$ occur in order along M_P. Observe from the planarity of L_P that one of the following must hold:

(a) $\{h_1, m_2\}$ is a 2-cut in L_P (Figure 34(a)), or

(b) $\{m_1, h_1\}$ is a 2-cut in L_P, $\{h_1, m_2, m_3\}$ is a 3-cut in L_P, and h_3 is cofacial with each of m_1, h_2 (Figure 34(b)), or

(c) $\{m_2, h_2\}$ is a 2-cut in L_P, $\{h_1, m_2, h_3\}$ is a 3-cut in L_P, and m_3 is cofacial with each of m_1, h_2 (Figure 34(c)).

But (a) contradicts the choice of Z. Hence (b) or (c) must hold, and we have some $v \in \{h_3, m_3\}$ which is cofacial with each of m_1 and h_2.

Suppose $\{m_1, v, h_2\} = \{b_2, a_2, b_1\}$. By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, $\{b_2, a_2, b_1\}$ is not a 3-cut in L_P, so $h_3, m_3 \in C[b_1, b_2]$. In case (b), h_1 is a cutvertex of L_P, contradicting Claim 3. In case (c), m_2 is a cutvertex of L_P, contradicting our choice of the separation (M, N) (minimizing $M_P[z_2, h_2]$).

We may now assume that $\{m_1, v, h_2\}$ is a 3-cut in L_P. Note that h_2 is cofacial with some vertex of $P[a_1, s'_o] = C[s'_0, a_1]$. So we have a separation (E, F) of G with $V(E \cap F) = \{m_1, h_2, v_1, s'_0\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(E)$, and $X' \subseteq V(F)$. Since $m_1, s'_0 \in C[b_2, a_1]$, we find

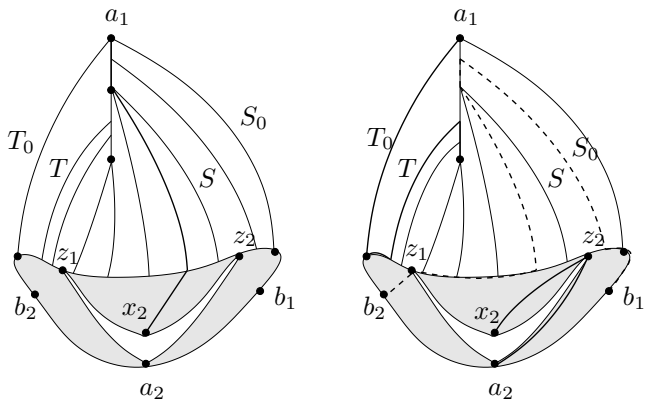


Figure 35: Where a_2 is a cutvertex of L_Q

that (Γ, X) satisfies property (R1).

3.5.3 The case that T falls to the left of P

Define paths Q_1 and Q: For any path O from $s_i \in \{s_1, \dots s_k\}$ to X' in $B - \{z_1, z_2\}$, let $m(O) \in V(M_P)$ be the vertex of $O \cap M_P$ which lies leftmost along M_P. Choose a path Q_1 from $s_i \in \{s_1, \dots s_k\}$ to X' in $B - \{z_1, z_2\}$ minimizing $M_P[m(Q_1), z_2]$. Define $Q := P[a_1, s'] \cup S_i \cup Q_1$, a path from a_1 to X'. Recalling that path R is from $r' \in P(s', x_1]$ to $r \in M_P(t, z_1)$, we note that $(T \cup R) \subseteq L_Q$.

Claim 5. We may assume $Q \in \mathcal{P}_{(\Gamma,X)}$.

Proof. We first show that Q_1 is disjoint from $C[b_1, b_2]$. If not, we have $M_P(z_1, z_2) \cap C[b_1, b_2] \neq \emptyset$. This forces $\{z_1, z_2\} \subseteq V(C)$. So one of z_1 or z_2 is a cutvertex of L_P separating X' from a_2, violating our assumption that $X' \subseteq \mathcal{A}(P)$.

Now suppose a_2 is a cutvertex of L_Q, and hence a_2 is cofacial with some vertex of Q. Noting that $Q \cap L_P \subseteq B$, we find that a_2, z_1, and z_2 are cofacial. (See Figure 35.) Now $z_2 \neq b_1$. To see this, suppose $z_2 = b_1$. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $\{b_1, a_2\}$ is not a 2-cut in G, and we must have $z_1 \in C(b_1, a_2)$. So z_1 is a cutvertex of L_P separating X' from a_2. This contradicts our observation that $x_2 \in \mathcal{A}(P)$.

Suppose some vertex $p \in P[a_1, x_1)$ is cofacial with both z_2 and b_2. Then $\{b_2, p, z_2, a_2\}$

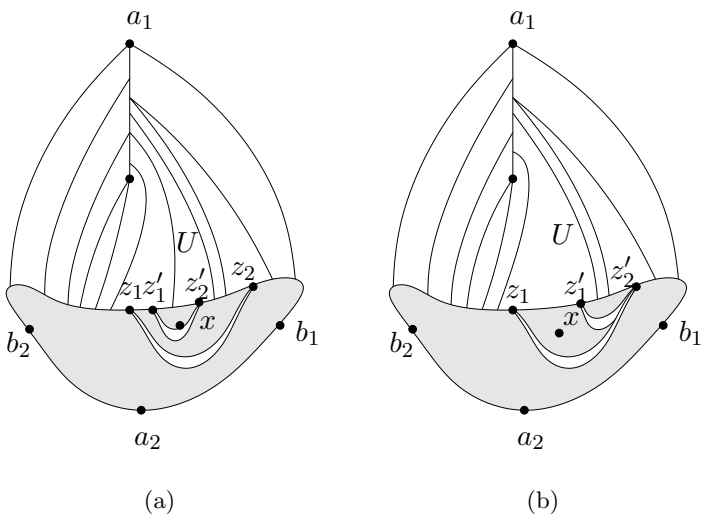


Figure 36: Defining $Z'(U)$

is a 4-cut in G, and we find (Γ, X) satisfies property (R1). Hence we may assume there is no such vertex p, and we must have paths T_0 from $t_0' \in P[a_1, s')$ to $t_0 \in M_P(b_2, t]$ and S_0 from $s_0' \in P(t_0', s]$ to $s_0 \in M_P(z_2, b_1]$. Choose such paths to minimize $P[t_0', s']$, $P[t_0', s_0']$, $M_P[t_0, t]$, and $M_P[s_0, b_1]$.

Let $O := P[a_1, t_0'] \cup T_0 \cup M_P[t_0, t] \cup T \cup P[t, x_1]$. (Here we are using a routing similar to that in Proposition 3.10.) If there are disjoint paths from b_2 to z_1, from a_2 to z_2, and from b_1 to s_0 in $L_P - V(M_P[t_0, t])$, then X is feasible in Γ, a contradiction. Hence such paths do not exist, and there must be some separation (M, N) of L_P of order at most 3, with $\{t, b_2, a_2, b_1\} \subseteq V(M)$ and $\{t_0, z_1, z_2, s_0\} \subseteq V(N)$. By maximality of Z, we have $|V(M \cap N)| = 3$. Moreover, since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, we have $V(M \cap N) = \{m_1, m_2, a_2\}$ with $m_1 \in M_P[t_0, t]$ and $m_2 \in M_P[s_0, b_1]$. Hence $\{m_1, a_2, z_2\}$ is a 3-cut in L_P. By our choice of T and T_0, we have m_1 cofacial with s', so there is a separation (G_1, G_2) of G with $V(G_1 \cap G_2) = \{m_1, s', z_2, a_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $X' \subseteq V(G_2)$. Hence (Γ, X) satisfies (R1). □

Define general $Z'(U)$: Given any path U from $u' \in P(s', x_1]$ to $u \in M_P[z_1, s)$ such that $U \not\subseteq L_Q$, we define a 2-cut $Z'(U)$ of L_P as follows. (See Figure 36.)

Since $T \cup R \subseteq L_Q$ and $z_1 \in L_Q$, but $U \not\subseteq L_Q$, we must have $M_P(z_1, u] \cap Q \neq \emptyset$.

Recalling our choice of $Q_1 \subseteq Q$ to minimize $M_P[m(Q_1), z_2]$, we have $m(Q_1) \in M_P(z_1, u]$. Moreover, there is no path from $\{s_1, \dots s_k\}$ to x_2 in $B - \{z_1, z_2\}$ disjoint from $M_P(z_1, u]$. Since $x_2 \notin V(M_P)$, we find that there must be some 2-cut $Z'(U) \subseteq (V(M_P[z_1, u]) \cup \{z_2\})$ separating $\{s_1, \dots s_k\}$ from x_2 in B. Again from our choice of Q_1, we may assume that $Z'(U)$ separates X' from s_1 in B. Let $Z'(U) = \{z'_1, z'_2\}$ with z'_1 left of z'_2. We may have $Z'(U) \subseteq V(M_P[z_1, u])$, as illustrated in Figure 36(a), or $z'_2 = z_2$, as in Figure 36(b).

Claim 6. We may assume $x_1 \in L_Q$.

Proof. Suppose $x_1 \notin L_Q$. Since $(T \cup R) \subseteq L_Q$, it must be that R falls to the left of P. (See Figure 37.) From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, together with Lemma 3.3 (limiting paths within bridges) we know there are at least two internally disjoint paths from x_1 to M_P. Let U_1 and U_2 be paths from x_1 to $u_1, u_2 \in M_P$ respectively, chosen to minimize $M_P[z_1, u_1]$ and $M_P[u_1, u_2]$. Since $x_1 \notin L_Q$, we have $U_1 \not\subseteq L_Q$. Let $Z' := Z'(U_1)$, and label $Z' = \{z'_1, z'_2\}$, with z'_1 left of z'_2.

We now proceed to show that there is some path R_0 from $r'_0 \in P[r', x_1)$ to $r_0 \in M_P(z_1, u_1]$. Suppose first that $Z' \subseteq V(M_P[z_1, u_1])$ (Figure 37(a)). Then from the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, Z' is not a 2-cut in G. Hence there must be a path R_0 as desired. Now suppose $z'_2 = z_2$ (Figure 37(b)). If no path R_0 exists, then $Z' \cup \{z_1\}$ is a 3-cut in G, separating X' from s_1. Furthermore, z_1 is cofacial with x_1. Hence we have a 4-separation (M, N) of G with $V(M \cap N) = \{z_1, x_1, z'_1, z'_2\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. An application of Lemma 2.11 gives $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. Hence we may assume the path R_0 exists. Choose such a path R_0 to minimize $P[r', r'_0]$ and $M_P[z_1, r'_0]$, so z_1 and r'_0 are cofacial.

Now choose a path S_0 from $s'_0 \in P[a_1, r')$ to $s_0 \in M_P[u_2, s_1]$ to minimize $M_P[u_2, s_0]$ and, subject to this, to minimize $P[s'_0, r']$ (where $S_0 = S_1$ is possible). Choose a path U from $u' \in P(r'_0, x_1]$ to $u \in M_P[u, s_1]$ to minimize $P[r'_0, u']$ and $M_P[u, s_0]$ (where $U = U_2$ is possible). Note that each vertex of $M_P[u, s_0]$ is cofacial with r'.

Define $O := P[a_1, s'_0] \cup S_0 \cup M_P[u, s_0] \cup U \cup P[u', x_1]$ (Figure 37(c)). If there is a path Q_O from r_0 to z_2 in $B - M_P[u, s_0]$, then we find $R \cup R_0 \cup Q_0 \subseteq L_O$. Hence $z_1 \in \mathcal{R}(O)$.

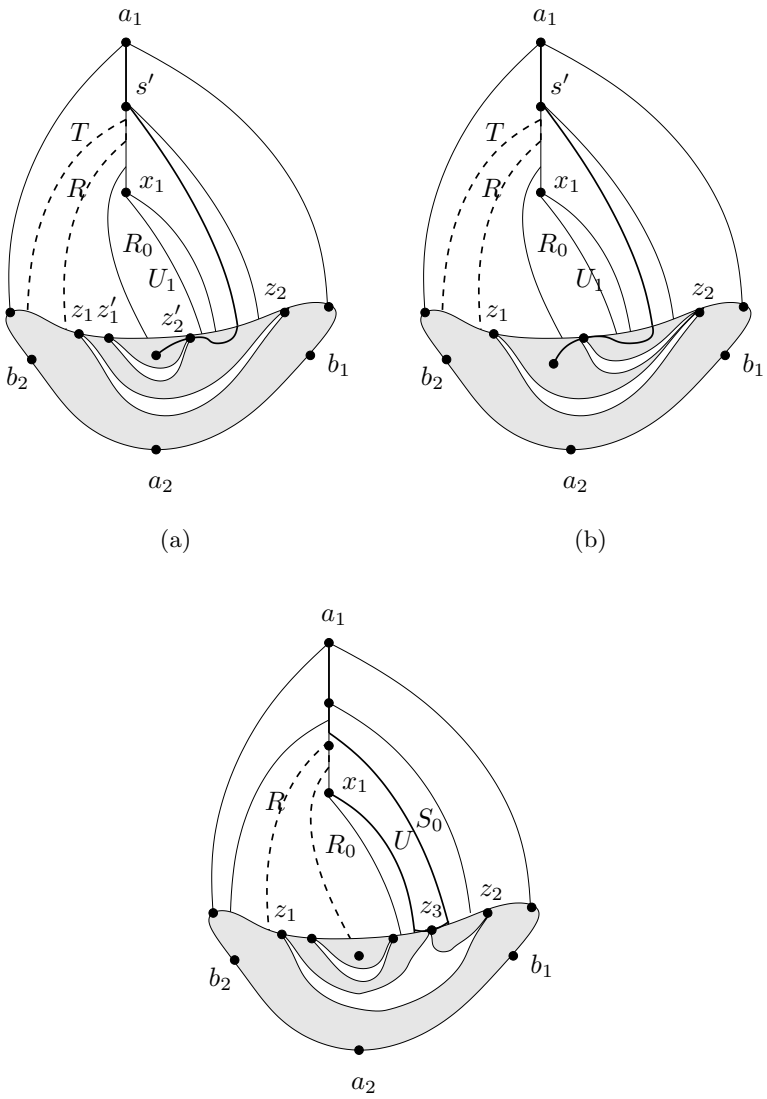


Figure 37: Where x_1 not in L_Q

But $\mathcal{R}(P) \subseteq \mathcal{R}(O)$, so this violates the choice of P as a maximizing path. So no such path Q_O may exist, and there must be some cutvertex v of B with $v \in M_P[u, s_0]$. Now there is a 3-separation (M, N) in G with $V(M \cap N) = \{z_1, v, r'\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and we have $(\Gamma, X) \in \mathcal{O}_2$. □

Claim 7. We may assume $x_1 \notin M_Q$.

Proof. Suppose $x_1 \in M_Q$. Here we nearly repeat the arguments of Claim 6, with a slightly different choice of paths.

Consider first the case that x_1 is cofacial with z_1. If x_1 is cofacial with one of s' or s, we have a 4-separaton (M, N) in G with $V(M \cap N) = \{z_1, z_2, x_1, s\}$ or $V(M \cap N) = \{z_1, z_2, x_1, s'\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. Applying Lemma 2.11 (4-separations in G), we find that $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. Hence we may assume x_1 is not cofacial with either s' or s, and there is some path U_1 from $u_1' \in P(s', x_1)$ to $u_1 \in M_P[z_1, s)$, falling to the right of P. Choose such a path to minimize $P[u_1', x_1]$ and $M_P[z_1, u_1]$.

Since $x_1 \in M_Q$, we have $U_1 \nsubseteq L_Q$. Let $Z' := Z'(U_1)$, and label $Z' = \{z_1', z_2'\}$, with z_1' left of z_2'. If $Z' \subseteq V(M_P[z_1, u_1])$ (Figure 38(a)), we see from our choice of U_1 that $Z' \cup \{x_1\}$ is a 3-cut in G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$. Therefore we may assume $z_2' = z_2$ (Figure 38(b)). But then $Z' \cup \{z_1\}$ is a 3-cut in G, separating X' from s_1, and we have a 4-separation (M, N) of G with $V(M \cap N) = \{z_1, x_1, z_1', z_2'\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. An application of Lemma 2.11 gives $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible.

Now consider the case where x_1 is not cofacial with z_1 (Figure 38(c)). Because $x_1 \in V(M_Q)$, let R_0 be a path from $r_0' \in P[r', x_1)$ to $r_0 \in M_P(z_1, s]$, chosen to minimize $P[r', r_0']$ and $M_P[z_1, r_0']$, so z_1 and r_0' are cofacial. From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, together with Lemma 3.3 (limiting paths within bridges) we know there are at least two internally disjoint paths from x_1 to M_P. From this together with planarity, we may assume $r_0 \notin \{s_1, \ldots, s_k\}$.

Now choose a path S_0 from $s_0' \in P[a_1, r')$ to $s_0 \in M_P(r_0, s_1]$ to minimize $M_P[r_0, s_0]$ and, subject to this, to minimize $P[s_0', r']$ (where $S_0 = S_1$ is possible). Choose a path U from

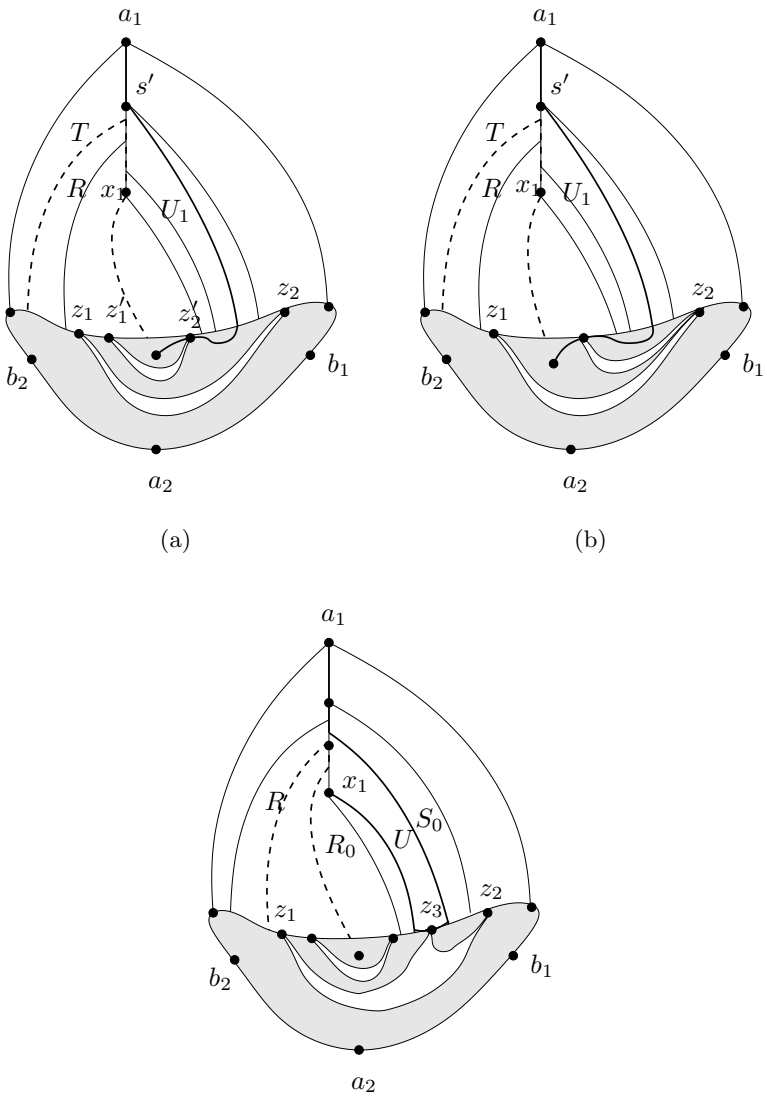
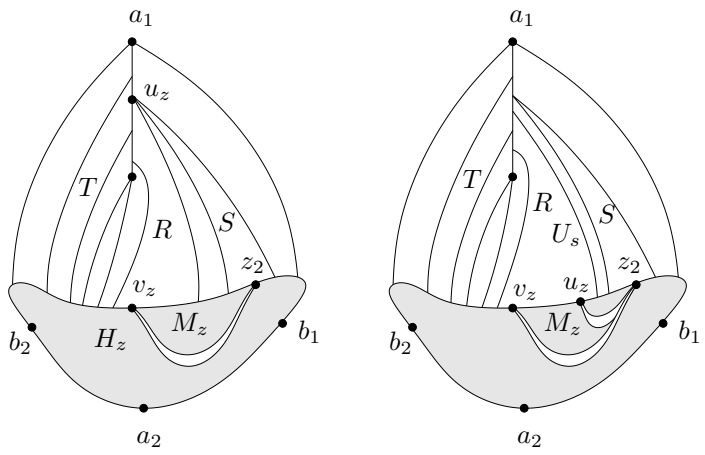


Figure 38: Where $x_1 \in M_Q$



(a) External (b) Internal

Figure 39: Defining separation (H_z, M_z)

$u' \in P(r'_0, x_1]$ to $u \in M_P(r_0, s_1]$ to minimize $M_P[u, s_0]$ and then $P[r'_0, u']$. Note that each vertex of $M_P[u, s_0]$ is cofacial with r'_0.

Define $O := P[a_1, s'_0] \cup S_0 \cup M_P[u, s_0] \cup U \cup P[u', x_1]$. If there is a path Q_O from r_0 to z_2 in $B - M_P[u, s_0]$, then we find $R \cup R_0 \cup Q_0 \subseteq L_O$. Hence $z_1 \in \mathcal{R}(O)$. But $\mathcal{R}(P) \subseteq \mathcal{R}(O)$, so this violates the choice of P as a maximizing path. So no such path Q_O may exist, and there must be some cutvertex z_3 of B with $v \in M_P[u, s_0]$. Now there is a 3-separation (M, N) in G with $V(M \cap N) = \{z_1, z_3, r'_0\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and we have $(\Gamma, X) \in \mathcal{O}_2$. □

Refine choice of path R: Assume R is chosen such that $R \subseteq M_Q$. By Claims 6 and 7, we may assume R is from $r' \in P(s', x_1)$ to $r \in M_P(t, s')$, and falls to the right of P.

Define Paths U_t and U_r: From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, $\{r, r'\}$ is not a 2-cut in G, so there must be some path U'_t from $u'_t \in P(r', x_1]$ to $u_t \in M_P[t, r)$. Choose such a path U'_t to minimize $M_P[t, u_t]$ and then $P[r', u'_t]$, and let $U_t = U'_t \cup P[u'_t, x_1]$. Similarly, $\{t, t'\}$ is not a 2-cut in G, so there must be some path U'_r from $u'_r \in P(t', x_1]$ to $u_r \in M_P(t, r]$. Choose such a path U'_r to minimize $M_P[u_r, r]$ and then $P[t', u'_r]$, and let $U_r = U'_r \cup P[u'_r, x_1]$.

Define separation (H_z, M_z)**:** We proceed to define a 3-separation (H_z, M_z) of G with $X' \subseteq V(M_z)$, $\{a_1, b_1, a_2, b_2, x_1\} \subseteq V(H_z)$, and $V(H_z \cap M_z) = \{u_z, v_z, z_2\}$.

If z_1 is cofacial with some vertex $u_z \in \{s, s'\}$, then we have a 3-cut $Z \cup \{u_z\}$ in G, separating X' from $\{a_1, b_1, a_2, b_2, x_1\}$. Let (H_z, M_z) be the associated 3-separation, with $v_z = z_1$. We will call this as an *external* separation (Figure 39(a)).

Now suppose z_1 is not cofacial with either of s', s. Then path U_s (defined in Section 3.5.1) has $u'_s \in P(s', x_1]$ and $u_s \in M_P(z_1, s)$, as in Figure 39(b). From our choice of $R \subseteq M_Q$, we know $U_s \not\subseteq L_Q$. Let $Z' := Z'(U_s)$, and label $Z' = \{z'_1, z'_2\}$, with z'_1 left of z'_2. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, and by our choice of U_s to minimize $M_P[z_1, u_s]$, we see that $Z' \not\subseteq M_P[z_1, u_s]$, so we must have $z'_2 = z_2$. Again by our choice of U_s, we have $M_P[z_1, u_s]$ contained in a facial cycle of G, so there is a 3-cut $\{z_1, z'_1, z_2\}$ in G separating X' from $\{a_1, b_1, a_2, b_2, x_1\}$. Let (H_z, M_z) be the associated 3-separation, with $v_z = z_1$, $u_z = z'_1$. We call this as an *internal* separation.

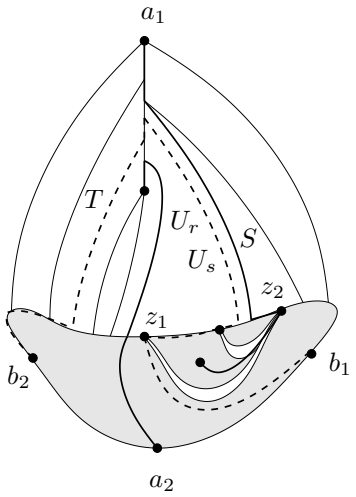
Claim 8. If $z_2 \neq b_1$, we may assume z_1 is cofacial with one of s' or s.

Proof. Suppose $z_2 \neq b_1$, but z_1 is cofacial with neither s nor s'. Then $u'_s \in P(s', x_1]$ and $u_s \in M_P(z_1, s)$, and (H_z, M_z) is an internal separation. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, and by planarity of G, we note that there is no cutvertex of B in $M_P[z_1, u_s]$. So we have two disjoint paths in B: $Q_b = M_P[z_1, u_s]$ and Q'_x from z_2 to X'.

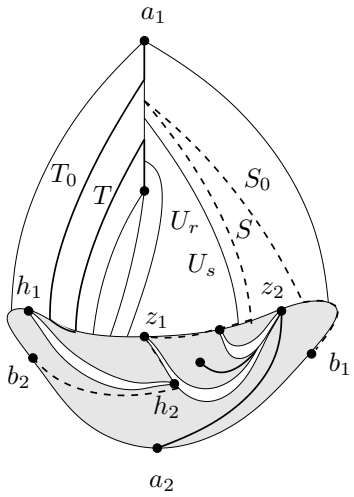
Let $O_1 := P[a_1, s'] \cup S \cup M_P[s, z_2] \cup Q_x$, a path from a_1 to x_2. Suppose there are disjoint paths in L_P from b_1 to z_1 and from a_2 to u_r, each disjoint from $M_P[b_2, t]$ and z_2. These paths, together with $M_P[b_1, t] \cup T \cup U_s \cup M_P[z_1, u_s]$ and U_r, show that X is feasible in Γ, as shown in Figure 40(a).

Hence no such paths exist. There must be a separation (M, N) of L_P of order at most three such that $\{b_1, a_2, b_2\} \subseteq V(M)$, $\{z_1, u_r, t\} \subseteq V(N)$, and $z_2 \in V(M \cap N)$. From our choice of Z as a maximal cut, we see that $|V(M \cap N)| = 3$. We may assume $V(M \cap N) = \{h_1, h_2, z_2\}$ with $h_1 \in M_P[b_2, t]$ (Figure 40(b)).

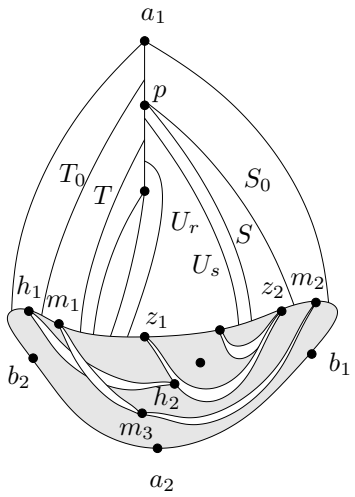
If this extends to a 4-separation (M', N') of G with $V(M' \cap N') = \{h_1, h_2, z_2, v\}$ for some $v \in P[a_1, s']$, then by Lemma 2.11, (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_2$. So



(a) X feasible in Γ



(b) Separation (M, N)



(c) Separation (M', N')

Figure 40: Example of $z_2 \neq b_1$, not cofacial s or s'

we may assume there is no such 4-separation. By our choice of S, there are paths T_0 from $t_0' \in P[a_1, s')$ to $t_0 \in M_P(h_1, t]$ and S_0 from $s_0' \in P(t_0', s']$ to $s_0 \in M_P(z_2, b_1]$. Choose such paths T_0, S_0 to minimize $P[t_0', s']$, $M_P[t_0, t]$, $P[s_0', s']$, and $M_P[z_2, s_0]$. Note from this choice that each vertex of $M_P[t_0, t]$ is cofacial with some vertex of $P[s_0', s']$. Furthermore, each vertex of $P[s_0', s']$ is cofacial with z_2.

Let $O_2 := P[a_1, t_0'] \cup T_0 \cup M_P[t_0, t] \cup T \cup P[t', x_1]$, from a_1 to x_1 in G. Suppose there are disjoint paths from b_1 to z_1 and from a_2 to z_2, each disjoint from $M_P[t_0, t]$ and $M_P[s_0, b_1]$. These paths, together with $S \cup S_0$, Q_b, and Q_x' show that X is feasible in Γ, a contradiction. Hence no such paths exist, and there must be a separation $(M'N')$ of L_P of order at most three such that $\{t_0, b_1, a_2, b_2\} \subseteq V(M')$ and $\{t, z_1, z_2, s_0\} \subseteq V(N')$. From our choice of Z as a maximal cut, $|V(M' \cap N')| = 3$. We may assume $V(M' \cap N') = \{m_1, m_2, m_3\}$ with $m_1 \in M_P[t_0, t]$ and $m_2 \in M_P[s_0, b_1]$ (Figure 40(c)).

We now have 3-cuts $\{h_1, h_2, z_2\}$ and $\{m_1, m_2, m_3\}$ in L_P, with the vertices $b_2, h_1, t_0, m_1, t, z_2, s_0, m_2, b_1$ in order along M_P. Observe from the planarity of L_P that one of the following must hold:

(i) $\{m_1, z_2\}$ is a 2-cut in L_P

(ii) There is some $v \in \{h_2, m_3\}$ such that $\{m_1, z_2, v\}$ is 3-cuts in L_P.

But (i) contradicts the choice of Z, so (ii) must hold. Note that m_1 is cofacial with some vertex $p \in P[s_0', s']$. So we have a separation (E, F) of G with $V(E \cap F) = \{m_1, z_2, v, p\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(E)$, and $X' \subseteq V(F)$. Applying Lemma 2.11 we find that $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. □

Define 2-cut W: Since we assume $x_1 \notin \mathcal{R}(Q)$, there is some 2-separation (L_1', L_2') of L_Q with $x_1 \in V(L_1')$ and $\{b_1, a_2, b_2\} \subseteq V(L_2')$. Let $W = V(L_1' \cap L_2') = \{w_1, w_2\}$. Since $T \cup R \subseteq L_Q$, we may assume $w_1 \in M_Q[b_2, t] = M_P[b_2, t]$ and $w_2 \in M_Q[r, b_1]$. Assume also that W is chosen to minimize $M_Q[w_1, t]$ and $M_Q[w_2, b_1]$.

Observe that if $w_2 \in M_Q[z_2, b_1] = M_P[z_2, b_1]$ then W is a 2-cut in L_P separating a_2 from x_2, contradicting our choice of Z as a maximal 2-cut. Hence $w_2 \in M_Q[r, z_2)$. Note also that $w_1 = b_2$ is possible. In this case, w_2 is a cutvertex of L_Q.

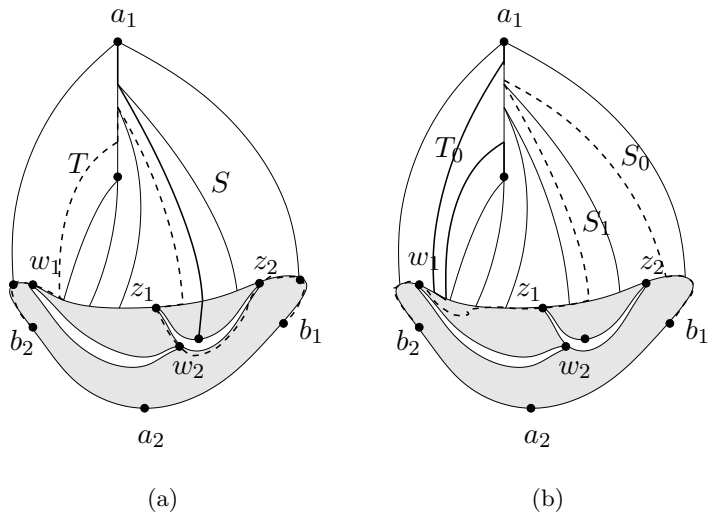


Figure 41: 2-cut in L_Q corresponding to 3-cut in L_P

Define vertices z_0 and z_3: Let $z_0 \in V(M_P[b_2, w_1] \cap C[b_1, b_2])$ be chosen to minimize $M_P[z_0, w_1]$. Note that either z_0 is a cutvertex of L_P, or $z_0 = b_2$. Let $z_3 \in V(M_P[z_2, b_1] \cap C[b_1, b_2])$ be chosen to minimize $M_P[z_2, z_3]$. Note that either z_3 is a cutvertex of L_P, or $z_3 = b_1$.

Claim 9. We may assume $w_2 \in M_Q[r, z_1)$.

Proof. If Claim 9 is false, there is a 3-separation (L_1, L_2) of L_P with $L_1 \cap L_2 = \{w_1, w_2, z_2\}$, $\{b_1, a_2, b_2\} \subseteq V(L_1)$, $\{t, r\} \cup X' \subseteq V(L_2)$. (See Figure 41(a).) Suppose this corresponds to a 4-separation (M, N) of G with $V(M \cap N) = \{w_1, w_2, z_2, v\}$ for some $v \in P[a_1, s']$, with $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$. An application of Lemma 2.11 (4-separations in G) gives $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. Hence we may assume there is no such 4-separation of G.

By our choice of S, there are paths T_0 from $t'_0 \in P[a_1, s')$ to $t_0 \in M_P(w_1, t]$ and S_0 from $s'_0 \in P(t'_0, s']$ to $s_0 \in M_P(z_2, b_1]$ (Figure 41(b)). Choose T_0 to minimize $P[a_1, t'_0]$ and $M_P[b_2, t_0]$. Subject to this, choose S_0 to minimize $P[t'_0, s'_0]$ and $M_P[s_0, b_1]$.

Let $O := P[a_1, t'_0] \cup T_0 \cup M_P[t_0, t] \cup T \cup P[t', x_1]$, a path from a_1 to x_1. From our choice of w_1, and assumption that Claim 9 is false, we see that there is a path from w_1 to z_1 in L_2 disjoint from $M_P[t_0, t]$, and hence disjoint from O. Moreover, this path is disjoint from a_2;

otherwise we have $a_2 = w_2 = z_1$, and a_2 is a cutvertex of L_P. So there is a path from b_2 to z_1, disjoint from O and $C[b_1, a_2]$, and we find that $S_1 \cup S_0 \subseteq L_O$, and $x_2 \in (L_O \setminus M_O)$.

Recall that we have disjoint paths the paths $Q_b = M_P[z_1, u_s]$ and Q_x from z_2 to x_2 in B. If L_1 contains a path from a_2 to z_2 which is disjoint from $M_P[b_2, w_1] \cup M_P[s_0, b_1]$, then we find that X is feasible in Γ. Hence no such path exists, and there must be a 2-cut $Z' = \{z'_1, z'_2\}$ in L_P separating x_2 from a_2, with $z'_1 \in M_P[b_2, w_1]$ and $z'_2 \in M_P[s_0, b_1]$. But such a cut also separates X' from a_2, contradicting our choice of Z. □

Define $\mathcal{C}, \mathcal{C}_w$: We call a pair of paths (T_i, R_i) *consistent* if the following conditions hold:

1. T_i is from $t'_i \in P[a_1, r'_i)$ to $t_i \in M_P(b_2, t]$, and
2. $M_P[t_i, u_t] \cap V(C) = \emptyset$, and
3. R_i is from $r'_i \in P(t'_i, r']$ to $r_i \in M_P[z_1, b_1]$.

Let $\mathcal{C}$ be the collection of all consistent pairs of paths. Let $\mathcal{C}_w \subseteq \mathcal{C}$ be the collection of all consistent pairs (T_i, R_i) such that $t_i \in M_P(w_1, t]$.

Claim 10. If $w_2 \in V(C)$, then for all $(T_i, R_i) \in \mathcal{C}$, we have $r_i = z_1$.

Proof. Suppose there is some pair of paths $(T_0, R_0) \in \mathcal{C}$ with $r_0 \in M_P(z_1, b_1]$. Letting $O := P[a_1, t'_0] \cup M_P[t_0, u_t] \cup U_t$, we have $R \cup R_0 \subseteq L_O$. From our choice of Z, we find that no 2-cut of L_P contained in $M_P[s, b_1] \cup \{w_2\}$ separates X' from a_2. Hence there is a path from a_2 to z_1 in L_P which is disjoint from O. So $z_1 \in \mathcal{R}(O)$. Since $\mathcal{R}(P) \subseteq \mathcal{R}(O)$ and $z_1 \notin \mathcal{R}(P)$, this contradicts the assumption that P is a maximal path. □

Define paths S_0 and T_0: If there is a path S_0 from $s'_0 \in P[a_1, s')$ to $s_0 \in M_P[z_2, b_1]$, choose such an S_0 to minimize $P[s'_0, s']$ and $M_P[s, s_0]$, so each of s'_0, s_0 is cofacial with s'. If there is a path T_0 from $t'_0 \in P[a_1, s')$ to $t_0 \in M_P[b_2, t]$, choose such a T_0 to minimize $P[t'_0, s']$ and $M_P[t_0, t]$, so each of t'_0, t_0 is cofacial with s'. Note by our choice of S and T that if $s' \notin V(C)$, both paths S_0 and T_0 exist. If $s' = a_1$, neither path exists, and if $s' \in V(C) \setminus \{a_1\}$, exactly one of S_0, T_0 exists.

Claim 11. We may assume $\{w_2, s'\} \not\subseteq V(C)$.

Proof. Let $\{w_2, s'\} \subseteq V(C)$. Then $w_2 \in C[b_1, a_2]$, is shown in Figure 42.

Since $s \neq z_1$, Claim 10 tells us that there is no path T_0 for which $(T_0, S) \in \mathcal{C}$. From our choice of T and z_0, we find that either z_0 shares a finite face with s', or $z_0 = b_2$ with $a_1 \notin C(b_2, s')$. Since $z_0, s' \in V(C)$, there is a 2-separation (G_l, G_r) of G with $V(G_l \cap G_r) = \{z_0, s'\}$, $b_2 \in V(G_l)$, and $\{b_1, a_2\} \cup X \subseteq V(G_r)$.

Suppose $a_1 \in V(G_l)$. Then we have a 4-separation (M, N) of G with $V(M \cap N) = \{z_0, s', b_1, a_2\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$. If either $z_0 \neq b_2$ or $s' \neq a_1$, then the separation is non-trivial and (Γ, X) satisfies property (R1). So we may assume that one of the following holds:

(a1) $a_1 \notin V(G_l)$, so $a_1 \in C(s', b_1)$, or

(a2) $z_0 = b_2$ and $s' = a_1$.

If s' shares a finite face with z_3, we have a 4-separation (M', N') of G with $V(M' \cap N') = \{s', z_3, a_2, b_2\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$ and $X \subseteq V(N')$. Again (Γ, X) satisfies property (R1). Hence we may assume s' does not share a finite face with z_3. So one of the following holds:

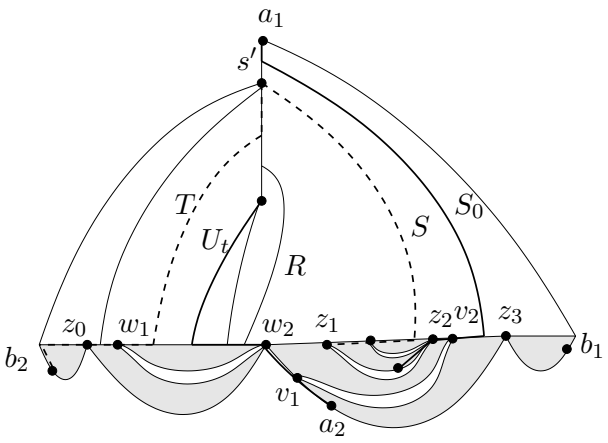
(b1) the path S_0 exists, with $s'_0 \in P[a_1, s')$ to $s_0 \in M_P[z_2, z_3)$, or

(b2) $s' = a_1$ and $z_3 = b_1$.

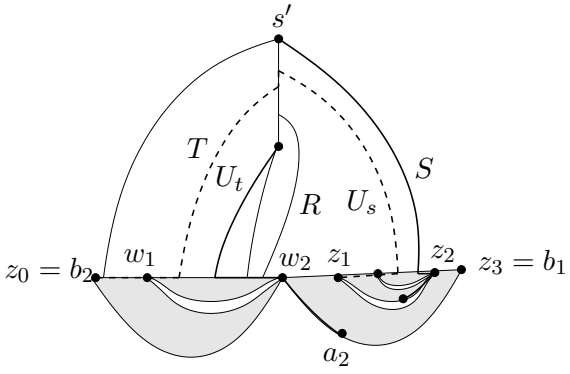
Case 1 (b1) is satisfied (Figure 42(a)).

Let $O := P[a_1, s'_0] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, a path from a_1 to x_2. By our choice of z_3, we find $O \cap C[b_1, b_2] = \emptyset$. Note that $M_P[b_2, t] \cup T \cup S \subseteq L_O$.

Suppose there is a path O_b from b_1 to z_1 in L_P which is disjoint from $M_P[z_2, s_0] \cup C[a_2, w_2]$. Then we have $O_b \subseteq L_O$. Furthermore, the path $U_r \cup M_P[u_r, w_2] \cup C[a_2, w_2]$ shows that $x_1 \in \mathcal{R}(O)$. This contradicts the assumption that X is infeasible in Γ, and hence cannot occur.



(a)



(b)

Figure 42: Where $\{w_2, s'\} \subseteq V(C)$.

So there must be some 2-cut $V = \{v_1, v_2\}$ in L_P with $v_1 \in C[a_2, w_2]$ and $v_2 \in M_P[z_2, s_0]$, separating b_1 from z_1. From our choice of S_0, we have v_2 cofacial with s', and there is a 4-separation (M, N) of G with $V(M \cap N) = \{v_1, v_2, s', z_0\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$. Hence (Γ, X) satisfies property (R1).

Case 2 (b2) is satisfied: $s' = a_1$ and $z_3 = b_1$.

Since (a1) cannot hold when $s' = a_1$, we must also have $z_0 = b_2$ (Figure 42(b)).

Suppose the path U_s has $u'_s = s'$. Then we see that $\{z_1, z_2, a_1\}$ is a 3-cut in G separating $\{b_1, a_2, b_2\}$ from X', and $\{w_1, w_2, a_1\}$ is a 3-cut in G separating $\{b_1, a_2, b_2\}$ from x_1. That is, $(\Gamma, X) \in \mathcal{O}_3$. Hence we may assume $u'_s \neq s'$.

Now suppose there is a path $S_c \subseteq C$ from s' to $s_c \in M_P(u_s, b_1)$. Define $O_1 := S_c \cup M_P[s, z_2] \cup Q_x$, a path from a_1 to x_2. Note that $M_P[b_2, t] \cup T \cup U_s \subseteq L_O$. Applying the same argument as we did for path O above, we find that there must be some 2-cut $V = \{v_1, v_2\}$ in L_P with $v_1 \in C[a_2, w_2]$ and $v_2 \in M_P[z_2, s_c]$, separating b_1 from z_1. Again v_2 is cofacial with s', leading to a 4-separation (M, N) of G which shows that (Γ, X) satisfies property (R1).

Now we may assume there is no path S_c. Applying Proposition 2.9, we assume there is no edge a_1b_1, so we must have $s \in V(C)$ and $u_s = s$. From Claim 10, we find that $(T, U_s) \notin \mathcal{C}$, so we must have $t' \in P[u_s, x_1)$.

We now have the following in G:

- A finite facial cycle D with $w_2, s' \in V(D)$
- A separation (H_1, M_1) of G with $V(H_1 \cap M_1) = \{s, z_1, z_2\}$, $s \in V(C \cap D)$, $z_1 \in V(D)$, $z_2 \in V(C)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,
- A separation (H_2, M_2) of G with $V(H_2 \cap M_2) = \{w_2, t', b_2\}$, $w_2 \in V(C \cap D)$, $t' \in V(D)$, $b_2 \in V(C)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$

where $X \subseteq V(M_1 \cup M_2 \cup D \cup C[z_2, b_1] \cup C[b_2, b_2])$, $X \cap M_1 \neq \emptyset$, and $X \cap M_2 \neq \emptyset$. So we have $(\Gamma, X) \in \mathcal{O}_6$. □

Claim 12. We may assume $w_2 \notin V(C)$.

Proof. Suppose $w_2 \in V(C)$. From Claim 11, we may assume $s' \notin V(C)$. So in particular, paths S_0 and T_0 both exist.

Consider first the case that $s_0 \in M_P[z_2, z_3)$. Let $O := P[a_1, s'_0] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, a path from a_1 to x_2. Note that $M_P[b_2, t] \cup T \cup S \subseteq L_O$.

Suppose there is a path O_b from b_1 to z_1 in L_P which is disjoint from $M_P[z_2, s_0] \cup C[a_2, w_2]$. Then we have $O_b \subseteq L_O$. Furthermore, the path $U_r \cup M_P[u_r, w_2] \cup C[a_2, w_2]$ shows that $x_1 \in \mathcal{R}(O)$. This contradicts the assumption that X is infeasible in Γ, and hence cannot occur.

So there must be some 2-cut $V = \{v_1, v_2\}$ in L_P with $v_1 \in C[a_2, w_2]$ and $v_2 \in M_P[z_2, s_0]$, separating z_3 from z_1. From our choice of S_0, we have v_2 cofacial with s', and there is a 4-separation (M, N) of G with $V(M \cap N) = \{v_1, v_2, s', z_0\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$. Hence (Γ, X) satisfies property (R1).

Now consider the case that $s_0 \in M_P[z_3, b_1]$. By our choice of S_0, and assumption that there is no edge a_1b_1, we note that either $z_3 \neq b_1$ or $s'_0 \neq a_1$. We may assume $t_0 \neq b_2$; otherwise, we have a 3-cut $\{b_1, b_2, s'\}$ in G, and a 4-separation (M, N) of G with $V(M \cap N) = \{b_1, a_2, b_2, s'\}$ demonstrating that (Γ, X) satisfies property (R1).

From Claim 10, and since $s \neq z_1$, we see that $(T_0, S) \notin \mathcal{C}$. Hence there must be some vertex $v_3 \in M_P[t_0, u_t] \cap C[b_1, b_2]$, separating t_0 from u_t in L_P. By planarity, this forces $w_1 \in C[b_1, b_2]$. By our choice of T_0 and T, w_1 is cofacial with s'. Hence $\{w_1, s', z_3\}$ is a 3-cut in G, and there is a 4-separation (M, N) of G with $V(M \cap N) = \{w_1, s', z_3, a_2\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$. Hence (Γ, X) satisfies property (R1). □

Claim 13. If S_0 exists with $s_0 \in M_P[z_2, z_3)$, then there is a 3-cut $\{h_1, h_2, h_3\}$ in L_P separating $\{b_2, a_2, b_1, s_0\}$ from $\{w_1, w_2, z_1, z_2\}$, with $h_1 \in M_P[b_2, w_1]$ and $h_2 \in M_P[z_2, s_0]$. Moreover, T_0 exists with $t_0 \in M_P(h_1, w_1]$.

Proof. Let $O_s := P[a_1, s'_0] \cup S_0 \cup M_P[z_2, s_0] \cup Q_x$, from a_1 to x_2. Suppose there are disjoint path from a_2 to w_2 and from b_1 to z_1 in L_P, each disjoint from $M_P[b_2, w_1]$ and $M_P[z_2, s_0]$. Together with $T \cup S$ and U_r, these would show $x_1 \in \mathcal{R}(O_s)$, contradicting the assumption that X is infeasible in Γ. Hence no such paths may exist.

By our choice of Z as a maximal 2-cut, we see that there must be some 3-cut $\{h_1, h_2, h_3\}$ in L_P separating $\{b_2, a_2, b_1, s_0\}$ from $\{w_1, w_2, z_1, z_2\}$. We may assume $h_1 \in M_P[b_2, w_1]$ and $h_2 \in M_P[z_2, s_0]$. From our choice of S_0, we see that h_2 is cofacial with s'. If h_1 is also cofacial with s', we have a 4-cut $\{h_1, h_2, h_3, s'\}$. Applying Lemma 2.11 to the associated 4-separation, we find that $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1).

Hence we may assume h_1 is not cofacial with s'. From our choice of T, we see that there must be a path T_0 with $t_0 \in M_P(h_1, w_1]$. □

Claim 14. If T_0 exists with $t_0 \in M_P(z_0, t]$, then there is a 3-cut $\{m_1, m_2, m_3\}$ in L_P separating $\{t_0, b_2, a_2, b_1\}$ from $\{u_t, w_2, z_1, z_2\}$, with $m_1 \in M_P[t_0, u_t]$ and $m_2 \in M_P[z_2, b_1]$. Moreover, S_0 exists with $s'_0 \in P[a_1, t'_0)$, $s_0 \in M_P[z_2, m_2)$.

Proof. Let $O_t := P[a_1, t'_0] \cup T_0 \cup M_P[t_0, u_t] \cup U_t$, from a_1 to x_1. Suppose there are disjoint path from b_2 to w_2 and from a_2 to z_1 in L_P, each disjoint from $M_P[t_0, u_t] \cup M_P[z_2, b_1]$. Together with $R \cup S$, these would show $x_2 \in \mathcal{R}(O_t)$, contradicting the assumption that X is infeasible in Γ. Hence no such paths may exist.

By our choice of Z as a maximal 2-cut, we see that there must be some 3-cut $\{m_1, m_2, m_3\}$ in L_P separating $\{t_0, b_2, a_2, b_1\}$ from $\{u_t, w_2, z_1, z_2\}$. We may assume $m_1 \in M_P[t_0, u_t]$ and $m_2 \in M_P[z_2, b_1]$, and choose such a cut to minimize $M_P[b_2, m_1]$, and subject to this to minimize $M_P[z_2, m_2]$.

Consider the case that m_1 is cofacial with s'. If m_2 is also cofacial with s', we have a 4-cut $\{m_1, m_2, m_3, s'\}$. Applying Lemma 2.11 to the associated 4-separation, we find that $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). So we may assume m_2 is not cofacial with s', and by our choice of S, S_0 exists with $s_0 \in M_P[z_2, m_2)$.

Now consider the case that m_1 is not cofacial with s'. By our choice of T and T_0, we must have $m_1 \in M_P(t, u_t]$. Letting B_w be the W-bridge of G containing $M_P(w_1, w_2)$, we see that m_1 is a cutvertex of B_w. Since m_1 is not cofacial with s', we may assume $t \in M_P[w_1, m_1)$.

Now w_1 is cofacial with some vertex of $P[a_1, t'_0]$. Let t'_v be such a vertex, chosen to minimize $P[t'_v, x_1]$. Then there is a path T_v from t'_v to $t_v \in M_P(w_1, m_1)$. Let T_v be chosen to minimize $P[w_1, t_v]$.

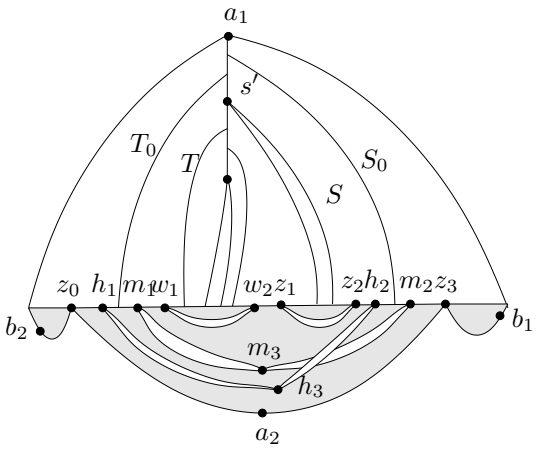


Figure 43: Where S_0 and T_0 both exist

If m_2 is cofacial with t'_v, we have a 4-cut $\{w_1, m_2, m_3, t'_v\}$ in G. (By Claim 12, $w_2 \notin V(C)$, so $\{w_1, m_2, m_3, t'_v\} \neq \{a_1, b_1, a_2, b_2\}$.) We again apply Lemma 2.11 to the associated 4-separation, to find that $(\Gamma, X) \in \mathcal{O}_2$ or (Γ, X) satisfies property (R1). Hence we may assume m_2 is not cofacial with t'_v, and either S_0 exists with $s_0 \in M_P[z_2, m_2)$, as desired, or there is a path from $P(t'_v, s']$ to $M_P(m_2, b_1]$.

We now assume there is a path V from $v' \in P(v_t, s']$ to $v \in M_P(m_2, b_1]$. Let $O_v := P[a_1, t'_v] \cup T_v \cup T \cup P[t', x_1]$, from a_1 to x_1. From our choice of m_1, we have a path O_w from w_1 to w_2 in B_w which is disjoint from O_v. Let $O_b := M_P[b_2, w_1] \cup O_w \cup M_P[w_2, s_1] \cup S_1 \cup P[v', s'] \cup V \cup M_P[v, b_1]$. Now if we have a path from a_2 to z_2 in L_P disjoint from O_b, then together with Q_x this shows $x_2 \in \mathcal{R}(O_t)$, contradicting the infeasibility of X in Γ. Hence there can be no such path, and by planarity there must be a 2-cut in L_P separating z_2 from a_2, with one vertex in $M_P[b_2, w_1] \cup M_P[w_2, z_1]$ and the other in $M_P[z_2, b_1]$. But such a 2-cut also separates X' from a_2, and thus contradicts our choice of Z. □

Claim 15. We may assume $s' = a_1$.

Proof. Suppose $s' \neq a_1$. If s' is cofacial with both z_0 and z_3, we have a 4-separation (M, N) of G with $V(M \cap N) = \{s', z_3, a_2, z_0\}$ demonstrating that (Γ, X) satisfies (R1). So we may assume that either S_0 exists with $s_0 \neq b_1$, or T_0 exists with $t_0 \neq b_2$.

Applying Claims 13 and 14, we have the following (Figure 43)

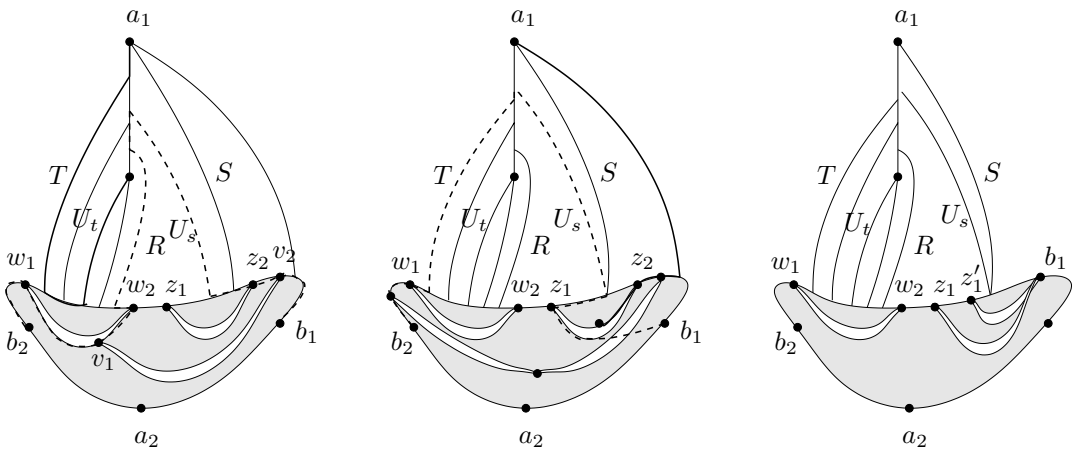


Figure 44: Where $s' = a_1$

- 3-cut $\{h_1, h_2, h_3\}$ in L_P separating $\{b_2, a_2, b_1, s_0\}$ from $\{w_1, w_2, z_1, z_2\}$
- 3-cut $\{m_1, m_2, m_3\}$ in L_P separating $\{t_0, b_2, a_2, b_1\}$ from $\{u_t, w_2, z_1, z_2\}$
- $s_0' \in P[a_1, t_0')$, $s_0 \in M_P[z_2, m_2)$, and $t_0 \in M_P(h_1, w_1]$.

So we have $z_0, h_1, t_0, m_1, w_1, z_2, h_2, s_0, m_2, z_3$ appearing in order along M_P.

We choose the cut $\{m_1, m_2, m_3\}$ to minimize $M_P[b_2, m_1]$, and then to minimize $M_P[z_2, m_2]$.

Observe from the planarity of L_P that one of the following must hold:

(i) $\{m_1, h_2\}$ is a 2-cut in L_P

(ii) there is some $v \in \{h_3, m_3\}$ such that $\{m_1, h_2, v\}$ is a 3-cut in L_P.

But (i) contradicts the choice of Z, and (ii) contradicts the choice of m_2. □

Final Case. We now assume $s' = a_1$.

From our choice of T, we find that either z_0 and a_1 share a finite face, or $z_0 = b_2$. If $z_0 \neq b_2$, we have a 4-separation (M, N) of G with $V(M \cap N) = \{z_0, a_1, b_1, a_2\}$, such that $\{a_1, b_1, a_2, b_1\} \subseteq V(M)$ and $X \subseteq V(M)$, and (Γ, X) satisfies (R1). Hence we may assume that $z_0 = b_2$. Since we assume there is no edge a_1b_2, we find $T \subseteq C$. Similarly, we may

assume $z_3 = b_1$; otherwise we find a 4-separation (M', N') with $V(M' \cap N') = \{a_1, z_3, a_2, b_2\}$ demonstrating that (Γ, X) satisfies (R1).

Recall the path U_s from $u'_s \in P[s', r']$ to $u_s \in M_P(z_1, s]$, chosen to minimize $M_P[z_1, u_s]$, and subject to this, minimize $P[u'_s, x_1]$. If $u'_s = s'$, we find that each of $\{w_1, w_2, s'\}$ and $\{z_1, z_2, s'\}$ is a 3-cut in G, and $(\Gamma, X) \in \mathcal{O}_3$. So we may assume that $u'_s \neq s'$.

Suppose $u'_s \in P(t', r']$. Let $O := P[a_1, t'] \cup T \cup M_P[t, u_t] \cup U_t$, from a_1 to x_1 (Figure 44(a)). Then $R \cup U_s \subseteq L_O$. Since $x_2 \notin \mathcal{R}(O)$, there must be some 2-cut $V = \{v_1, v_2\}$ in L_O separating $\{b_1, a_2, b_2\}$ from $\{u_s, z_1, w_2\}$. We may assume $v_2 \in M_P[z_2, b_1]$. From our choice of the 2-cut Z, we may assume $v_1 \in M_O(w_1, w_2)$. Thus we have a 4-separation (M, N) of G with $V(M \cap N) = \{v_1, v_2, s', w_1\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $X \subseteq V(N)$, and (Γ, X) satisfies property (R1).

We may now assume $u'_s \in P(s', t']$. By our choice of U_s, we have $\{w_1, w_2, t'\}$ is a 3-cut in G (Figure 44(b)).

Suppose $z_2 \neq b_1$ and there is some path Σ from a_1 to $\sigma \in M_P(u_s, b_1)$. Define $O_\sigma := \Sigma \cup M_P[z_2, \sigma] \cup Q_x$. Note that O_σ is disjoint from $C[b_1, b_2]$.

If there are disjoint paths from a_2 to w_2 and from b_1 to z_1, each disjoint from $M_P[b_2, w_1] \cup M_P[z_2, \sigma]$, then X is feasible in Γ. Hence no such paths may exist, so we must have some separation (M, N) of L_P of order at most three with $\{b_1, a_2, a_1\} \subseteq V(M)$ and $\{z_2, z_1, w_2, w_1\} \subseteq V(N)$. If $\sigma \in M_P(z_2, b_1)$, then $\sigma \in V(M)$.

From our choice of Z, we may assume this separation is of order three, and label $V(M \cap N) = \{h_1, h_2, h_3\}$, with $h_1 \in M_P[b_2, w_1]$ and $h_3 \in M_P[z_2, \sigma]$. But now we have a separation (M', N') of G with $V(M' \cap N') = \{h_1, h_2, h_3, s'\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$ and $X' \subseteq V(N')$, and we find that (Γ, X) satisfies property (R1).

Now suppose $z_2 \neq b_1$, and there is no path from a_1 to $M_P(u_s, b_1)$. Since we assume there is no edge $a_1 b_1$, we have $u_s \in V(C)$ and $u_s = s$. So $\{z_1, z_2, s\}$ is a 3-cut in G. We now have a facial cycle $D = P[u'_s, r'] \cup R \cup M_P[r, u_s] \cup S$, and separations $(H_1, M_1), (H_2, M_2)$, and (H_3, M_3) with

1. $V(H_1 \cap M_1) = \{u_s, z_1, z_2\}$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,

2. $V(H_2 \cap M_2) = \{t', w_2, w_1\}$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$,

3. $V(H_3 \cap M_3) = \{t', u'_s\}$, $a_1 \in V(M_3)$, $\{a_2, b_1, b_2\} \in V(H_3)$,

4. $X \subseteq V(M_1 \cup M_2 \cup D \cup C[b_2, w_1] \cup C[u_s, b_1])$, $X \cap M_1 \neq \emptyset$, and $X \cap M_2 \neq \emptyset$.

So we have $(\Gamma, X) \in \mathcal{O}_5$.

Finally, consider the case that $z_2 = b_1$. If $u_s = s$, we have separations as above, and $(\Gamma, X) \in \mathcal{O}_5$. Let us assume, then, that $u_s \neq s$. Let $Z' = Z'(U_s)$ be labeled as $\{z'_1, z'_2\}$. Then we have an internal separation (H_z, M_z) with $H_z \cap M_z = \{z_1, z'_1, b_1\}$ (Figure 44(c)). Let $H_1 = H_z$ and $M_1 = M_z$, and let (H_2, M_2) and (H_3, M_3) be the separations associated with the cuts $\{t', w_2, w_1\}$ and $\{t', z'_1, b_1\}$. These separations again demonstrate that $(\Gamma, X) \in \mathcal{O}_5$.

This completes the proof of Lemma 3.14.

CHAPTER IV

SPECIAL CASE $|X| = 2$

In this chapter, we address a limited class of disk graphs with $|X| = 2$. We prove a modified version of our main theorem, which will later be applied in the proof of the general case.

4.1 Obstructions

For convenience, we describe the obstructions $\Theta_1 \dots \Theta_6$ when $|X| = 2$. Note that for each $i = 1, \dots, 6$, we have $\Theta_i \subseteq \mathcal{O}_i$.

Definition 4.1. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We say $(\Gamma, \{x_1, x_2\}) \in \Theta_1$ if x_1 and x_2 are incident with a common finite face, or $\{x_1, x_2\} \subseteq V(C[b_1, b_2])$, or $\{x_1, x_2\} \subseteq V(C[b_2, b_1])$.

Definition 4.2. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We say $(\Gamma, \{x_1, x_2\}) \in \Theta_2$ if G has a separation (H, K) of order 3 such that $\{x_1, x_2\} \subseteq V(K)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$.

Definition 4.3. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We say $(\Gamma, \{x_1, x_2\}) \in \Theta_3$ if there is some $i \in \{1, 2\}$ such that for each $j \in \{1, 2\}$, one of the following holds:

(i) $x_j \in C[b_{3-i}, b_i]$, or

(ii) there is a finite face of G incident with both a_i and x_j, or

(iii) G has a separation (H, K) of order 3 such that $a_i \in V(H \cap K)$, $x_j \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$

Definition 4.4. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We say $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ if $\exists i, j \in \{1, 2\}$, $x_j \in V(C)$,

and $C_a = C[a_i, x_j]$ if $x_j \in C[a_i, b_i]$, $C_a = C[x_j, a_i]$ if $x_j \in C[b_{3-i}, a_i]$ such that one of the following holds:

1. $\exists$ a finite face of G incident with both x_{3-j} and some vertex in $V(C_a)$; or
2. G has a separation (H, K) of order at most 4 with $\{x_j, a_1, b_1, a_2, b_2\} \subseteq V(H)$, $x_{3-j} \in V(K)$ and $|V(H \cap K) - V(C_a)| \leq 2$.

Definition 4.5. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We say $(\Gamma, \{x_1, x_2\}) \in \Theta_5$ if G has separations $(H_1, M_1), (H_2, M_2)$, and (H_3, M_3), and a cycle D bounding a finite face of G such that the following hold for some $i, j \in \{1, 2\}$:

1. $V(H_1 \cap M_1) = \{u_1, v_1, w_1\}$, $w_1 \in V(C)$, $\{u_1, v_1\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,
2. $V(H_2 \cap M_2) = \{u_2, v_2, w_2\}$, $w_2 \in V(C)$, $\{u_2, v_2\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$,
3. $V(H_3 \cap M_3) = \{u_1, u_2\} \cup (\{w_1, w_2\} \cap \{b_1, b_2\})$, $a_i \in V(M_3)$, $\{a_{3-i}, b_1, b_2\} \in V(H_3)$,
4. $X \subseteq V(M_1 \cup M_2 \cup D \cup C[b_{3-i}, w_1] \cup C[w_2, b_i])$, $x_j \in V(M_1)$, and $x_{3-j} \in V(M_2)$.

Definition 4.6. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We say $(\Gamma, X) \in \Theta_6$ if G has separations (H_1, M_1) and (H_2, M_2) and a cycle D bounding a finite face of G such that the following hold for some $i \in \{1, 2\}$:

1. $V(D \cap C) = \{u_1, u_2\}$
2. For each $j \in \{1, 2\}$ there is a path $C_j \subseteq C$ from c_j to $\{b_1, b_2\}$ with $V(C_j) \cap \{a_1, a_2\} = \emptyset$
3. $V(H_j \cap M_j) = \{u_j, v_j, w_j\}$, $v_j \in V(D)$, $w_1 \in V(C_j)$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H_j)$, and
4. $X \subseteq V(M_1 \cup M_2 \cup D \cup C_1 \cup C_2)$

4.2 Theorem 2

The results we state here are nearly identical to those of Theorem 1. Our only adjustments come in the consideration of obstruction class $\mathcal{O}_7$, and (Γ, X) satisfying reduction property (R2).

If $(\Gamma, X) \in \mathcal{O}_7$ and $|X| < 3$, we have $(\Gamma, X) \in \Theta_1 \cup \Theta_2$ or (Γ, X) satisfies (R1).

Recall for disk graph $\Gamma = (G, a_1, b_1, a_2, b_2)$ and vertex set X with (Γ, X) satisfying (R2), X can be partitioned into $X_1 = X \cap V(C(b_{3-i}, a_i))$, $X_2 = X \cap V(C(a_i, b_i))$, and $X_3 = X \setminus (X_1 \cup X_2) \neq \emptyset$, and G has a separation (M, K) of order at most 4 such that one of the following holds:

(i) X_1, X_2 are nonempty, $X_1 \cup X_2 \cup \{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K - C[b_{3-i}, b_i])| \leq 1$

(ii) $X_2 = \emptyset$, $X_1 \cup \{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K - C[b_{3-i}, a_i])| \leq 2$

(iv) $X_1 = \emptyset$, $X_2 \cup \{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$, and $|V(H \cap K - C[a_i, b_i])| \leq 2$

Suppose $X = \{x_1, x_2\}$. Then in case (i), we have $(\Gamma, \{x_1, x_2\}) \in \Theta_1$; in cases (ii) and (iii), we have $(\Gamma, \{x_1, x_2\}) \in \Theta_4$.

Theorem 2. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. Then one of the following holds.*

(C1) *$\{x_1, x_2\}$ is feasible in Γ, or*

(C2) *$(\Gamma, \{x_1, x_2\})$ satisfies property (R1), or*

(C3) *$(\Gamma, \{x_1, x_2\}) \in \bigcup_{i=1}^{6} \Theta_i$.*

4.3 A special case

We will approach the proof of Theorem 2 using maximizing paths, and the lemmas of the previous chapter. First, however, we deal with the special case when some vertex x_i lies on the outer cycle of G.

Lemma 4.1. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph. Let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $x_1 \neq x_2$, and $\{x_1, x_2\}$ infeasible in Γ. Suppose $\{x_1, x_2\} \cap V(C) \neq \emptyset$. Then $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cup \Theta_4$.*

Proof. We assume without loss of generality that $x_1 \in C(b_2, a_1)$. If $x_2 \in C(b_2, b_1)$, then $(\Gamma, \{x_1, x_2\}) \in \Theta_1$, so we may further assume that $x_2 \notin C(b_2, b_1)$.

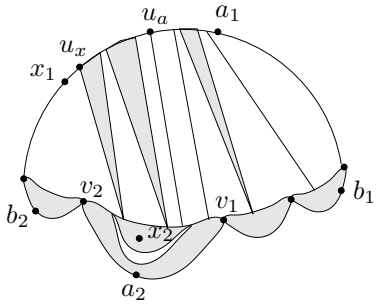


Figure 45: $x_1 \in V(C), \quad (\Gamma, \{x_1, x_2\}) \in \Theta_4$

Let $P := C[x_1, a_1]$, and consider the graph $H = G - P$. Note that H is connected; otherwise, there is some 2-cut of G contained in P, which contradicts $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G.

If x_2 is a cutvertex of H, then x_2 is cofacial with some vertex of P, and $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ with $i = j = 1$, $C_a = P$. Hence we may assume that there is a unique block B_x of H for which $x_2 \in V(B_x)$.

Suppose first that $x_2 \notin V(L_P)$. Let B be the $(L_P \cup P)$-bridge of G which includes x_2, and let $\{v\} = V(B \cap L_P)$. Let u_x, u_a be the elements of $V(B \cap P)$ which are closest to (and possibly equal to) x_1 and a_1 respectively. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $u_x \neq u_a$. Then G has a separation (M, K) with $V(M \cap K) = \{u_x, u_a, v\}$ with $\{x_1, a_1, b_1, a_2, b_2\} \subseteq V(M)$ and $x_2 \in V(K)$, and we see that $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ with $i = j = 1$, $C_a = P$.

Now suppose $x_2 \in V(L_P)$. (See Figure 45.) When S is a subgraph of L_P, we call a vertex $v \in P$ a *neighbor* of S if there is a path from some $s \in V(S)$ to v in G which is internally disjoint from $P \cup L_P$.

First consider the case that either $a_2 \notin B_x$ or $a_2 \in B_x$ is a cutvertex of H. Then there is some block $B_a \neq B_x$ of H with $a_2 \in V(B_a)$ and $B_a \subseteq L_P$. Let $v \in V(B_a)$ be a cutvertex in H separating B_x, B_a (where $v = a_2$ is possible). If B_x and B_a occur in clockwise order along $C \cap L_P$, then let u be the neighbor of B_x closest to x_1 on P, and let $T = \{u, v, a_1, b_1\}$. Otherwise, let u be the neighbor of B_x closest to a_1 on P, and let $T = \{u, v, x_1, b_2\}$. In either case, we see that G has a separation (M, K) with $M \cap K = T$ and $\{x_1, a_1, a_2, b_1, b_2\} \subseteq V(M)$, $x_2 \in V(K)$, and hence $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ with $i = j = 1$ and $C_a = P$.

We now assume $a_2 \in V(B_x)$ and a_2 is not a cutvertex of H. Let D be the outer cycle of the block B_x. Let $v_1, v_2 \in V(C) \cap V(D)$ such that $C[v_1, v_2] = D[v_1, v_2]$ is maximal. We see that if $x_2 \in V(D[v_2, v_1])$, then x_2 is cofacial with a vertex of P and $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ with $i = j = 1$ and $C_a = P$. Let us assume, then, that $x_2 \notin V(D[v_2, v_1])$.

Let $Q = (C[b_1, b_2] - C(v_1, v_2)) \cup D[v_2, v_1]$, so $D[v_2, v_1] \subseteq Q \subseteq (C \cup D)$. Then $B_x - Q$ has no x_2 to a_2 path, since such a path along with P and Q would show that $\{x_1, x_2\}$ is feasible in Γ. So H has a separation (H_1, H_2) of order 2 with $V(H_1 \cap H_2) \subseteq V(Q)$, $a_2 \in V(H_1), x_2 \in V(H_2)$.

Let u_x be the neighbor of H_2 closest to x_1 on P, and u_a the neighbor of H_2 closest to a_1 on P. Then G has a separation (M, K) with $V(M \cap K) = V(H_1 \cap H_2) \cup \{u_x, u_a\}$, $\{x_1, a_1, a_2, b_1, b_2\} \subseteq V(M)$, and $x_2 \in V(H_2) \subseteq V(K)$. Hence $(\Gamma, \{x_1, x_2\}) \in \Theta_4$ with $i = j = 1$ and $C_a = P$. □

4.4 Proof of Theorem 2

Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We wish to prove that one of the following holds:

(C1) $\{x_1, x_2\}$ is feasible in Γ, or

(C2) $(\Gamma, \{x_1, x_2\})$ satisfies property (R1), or

(C3) $(\Gamma, \{x_1, x_2\}) \in \bigcup_{i=1}^{5} \Theta_i$.

Proof. We assume that Γ does not satisfy (C1) or (C2), and show that $(\Gamma, \{x_1, x_2\}) \in \bigcup_{i=1}^{5} \Theta_i$.

From Lemma 4.1, if $\{x_1, x_2\} \cap V(C) \neq \emptyset$, then $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cup \Theta_4$. We may assume, then, that $\{x_1, x_2\} \cap V(C) = \emptyset$.

Recall that $\mathcal{P}_{(\Gamma, \{x_1, x_2\})}$ is defined as the collection of all paths P from $a_i \in \{a_1, a_2\}$ to $x_j \in \{x_1, x_2\}$ in $G - (C[b_i, b_{3-i}] \cup \{x_{3-j}\})$ such that a_{3-i} is not a cutvertex of L_P. From Lemma 3.1, either $\mathcal{P}_{(\Gamma, \{x_1, x_2\})} \neq \emptyset$ or $(\Gamma, \{x_1, x_2\}) \in \Theta_3$, so we may assume that there is a maximizing path $P \in \mathcal{P}_{(\Gamma, \{x_1, x_2\})}$. We further assume without loss of generality that the path P is from a_1 to x_1.

Now suppose $x_2 \notin V(L_P)$. From Lemma 3.5, we find that $(\Gamma, \{x_1, x_2\}) \in \mathcal{O}_1 \cap \mathcal{O}_2$, so $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cap \Theta_2$. Hence we may assume that $x_2 \in V(L_P)$. Applying Lemma 3.7, we find that either $(\Gamma, \{x_1, x_2\}) \in \Theta_2 \cup \Theta_5$ or $x_2 \in \mathcal{A}(P)$, so we assume that $x_2 \in \mathcal{A}(P)$. We now apply Lemma 3.12 to see that if $x_2 \in M_P$ then $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$ or satisfies (R1), so we may assume $x_2 \notin M_P$.

If there is some 2-separation (H, K) of L_P with $x_2 \in V(H) \setminus V(K)$ and $\{b_1, b_2, a_2\} \in K$, then by Lemma 3.14, $(\Gamma, \{x_1, x_2\}) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$. We may assume that no such 2-separation exists. Together with $x_2 \notin M_P$, this shows that there must be some path P_2 from a_2 to x_2 in $L_P - M_P$, and $x_2 \in \mathcal{R}(P)$. Then M_P provides a path from b_1 to b_2 disjoint from P and P_2, and we find that $\{x_1, x_2\}$ is feasible in Γ, a contradiction. □

4.5 1-connected case

While we have considered graphs which are 2-connected, a slight modification gives similar results for graphs which are connected, but not 2-connected.

Definition 4.7. Let G be a graph and let a_1, b_1, a_2, b_2 be distinct vertices of G. We call $\Gamma = (G, a_1, b_1, a_2, b_2)$ a *weak disk graph* if G is a connected graph drawn in a disk in the plane with no pair of edges crossing such that $a_1, b_1, a_2, b_2 \in V(G)$ appear in clockwise order along the boundary of the disk, and G is $(3, \{a_1, b_1, a_2, b_2\})$-connected.

Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a weak disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$. We again say X is *feasible* in Γ if there exist three disjoint paths in G: from a_1 to X, from a_2 to X, and from b_1 to b_2 respectively. Otherwise, X is infeasible in Γ.

Note that the definitions of property (R1) and of Θ_2 do not depend upon the 2-connectivity of disk graphs, and hence can also be applied to weak disk graphs.

Lemma 4.2. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a weak disk graph such that G is not 2-connected. Let $x_1, x_2 \in V(G) \setminus \{a_1, b_1, a_2, b_2\}$. Then one of the following holds:*

(1) $\{x_1, x_2\}$ is feasible in Γ, or

(2) $(\Gamma, \{x_1, x_2\})$ satisfies property (R1), or

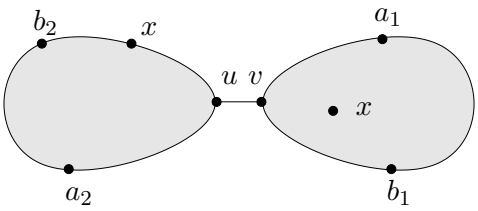


Figure 46: Condition (4)

(3) $(\Gamma, \{x_1, x_2\}) \in \Theta_2$, *or*

(4) G *is a chain of at most three blocks such that*

- *(a) each endblock of* G *contains exactly two vertices of* $\{a_1, b_1, a_2, b_2\}$, *and one of* $\{x_1, x_2\}$,
- *(b) if* G *has three blocks, one of these is a single edge,*
- *(c) there is a unique path* P *from* b_1 *to* b_2 *in* $\partial G - \{a_1, a_2\}$, *and*
- *(d)* $\{x_1, x_2\} \cap V(P) \neq \emptyset$.

Proof. Let v be a cutvertex of G. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, each component of $G - v$ must contain at least one of $\{a_1, b_1, a_2, b_2\}$. Label the components of $G - v$ as $H_1, \ldots, H_k$ where $k \in \{2, 3, 4\}$, and label the associated v-bridges of G as $B_1, \ldots, B_k$.

We first claim that we may assume each component of $G - v$ contains at least two vertices of $\{a_1, b_1, a_2, b_2\}$. To see this, suppose there is some component H_j with $H_j \cap \{a_1, b_1, a_2, b_2\} = \{u\}$. Either $\{v, u\}$ is a 2-cut in G, contradicting $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of weak disk graphs, or $V(H_j) = \{u\}$ and B_j is a single edge uv. But then we have a 4-separation (H, K) of G with $V(H \cap K) = \{v, a_1, b_1, a_2, b_2\} \setminus \{u\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and $\{x_1, x_2\} \subseteq V(K)$, and $(\Gamma, \{x_1, x_2\})$ satisfies property (R1). Hence we may assume each component of $G - v$ contains exactly two vertices of $\{a_1, b_1, a_2, b_2\}$, and $G - v$ has exactly two components, H_1 and H_2.

From planarity and the order of $\{a_1, b_1, a_2, b_2\}$ along the boundary of our embedding disk, we note that for $i \in \{1, 2\}$ we cannot have $\{a_1, a_2\} \subseteq V(H_i)$ and $\{b_1, b_2\} \subseteq V(H_{3-i})$. We assume without loss of generality that $\{a_i, b_i\} \subseteq V(H_i)$.

If there is a cutvertex u separating a_i from b_i in B_i, then u is also a cutvertex of G, and we apply the arguments above (for cutvertex v) to u. So we may assume that a_i, b_i appear in a single block of B_i. If u is a cutvertex of B_i separating $\{a_i, b_i\}$ from v, then we must have the edge $uv \in E(G)$; otherwise, $\{u, v\}$ is a 2-cut in G violating $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. So we see that B_i is either 2-connected, or the union of a 2-connected block and an edge uv. If each of B_1, B_2 has a cutvertex, say u_1 and u_2, then $\{u_1, u_2\}$ is a 2-cut in G separating v from $\{a_1, b_1, a_2, b_2\}$, violating the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. So at least one of B_1, B_2 is 2-connected. (See Figure 46.)

We claim that each component of $G - v$ must contain at least one vertex of $\{x_1, x_2\}$. To see this, suppose there is some component H_i for which $V(H_i) \cap \{x_1, x_2\} = \emptyset$. Letting $H = B_i \cup \{a_{3-i}, b_{3-i}\}$ and $K = B_{3-i}$, we note that $V(H \cap K) = \{v, a_{3-i}, b_{3-i}\}$. So we have a separation (H, K) of G of order 3 such that $\{x_1, x_2\} \subseteq V(K)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, and we see $(\Gamma, \{x_1, x_2\}) \in \Theta_2$. So we assume without loss of generality that $x_1 \in H_1$ and $x_2 \in H_2$.

Now for each $i \in \{1, 2\}$, let Q_i be the path in B_i from b_i to v along ∂B_i, chosen so that $a_i \notin V(Q_i)$. Suppose there are paths P_i from a_i to x_i disjoint from Q_i in each bridge B_i. Then that paths P_1, P_2 and $Q_1 \cup Q_2$ would show that $\{x_1, x_2\}$ is feasible in Γ. Hence no such paths may exist. So some x_i must either lie on the path P_i, or be separated from a_i by a 2-cut $T_i \subseteq V(Q_i)$. But such a 2-cut T_i would violate $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, so we find $x_i \in Q_i$, and condition (4) of this lemma is satisfied. □

CHAPTER V

PROOF OF GENERAL CASE

We now extend Theorem 2 to the more general case $|X| \geq 2$. We work inductively on $|X|$, making use of maximizing paths and the lemmas of previous sections.

We begin with two cases which can be handled without the use of either maximizing paths or induction. These use direct definitions of path routings. We then handle the situation that some vertex of X is on the outer cycle of G. This makes use of the results where $|X| = 2$, considering the relative positions of vertex pairs in X. It also employs Theorem 1 inductively.

We then establish several more technical lemmas, and handle several cases by considering maximizing paths. The remaining possibilities are dispatched by considering the obstructive relationship within subsets of X, and their possible combinations, again employing Theorem 1 inductively.

We combine these individual results to prove Theorem 1.

5.1 Independent inductive cases

We first consider the cases where removal of one vertex from the set X leaves us with a disk graph in $\mathcal{O}_5$ or $\mathcal{O}_6$. We approach this by considering path routings which make use of the identified vertex.

Lemma 5.1. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X = (X' \cup \{x_1\}) \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X'| \geq 2$, and X infeasible in Γ. If $(\Gamma, X') \in \mathcal{O}_5$, then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$.*

Proof. Assume (Γ, X') satisfies the definition of $\mathcal{O}_5$ with $i = 1$, so G has separations $(H_1, M_1), (H_2, M_2)$, and (H_3, M_3), and a cycle D bounding a finite face of G such that the following hold:

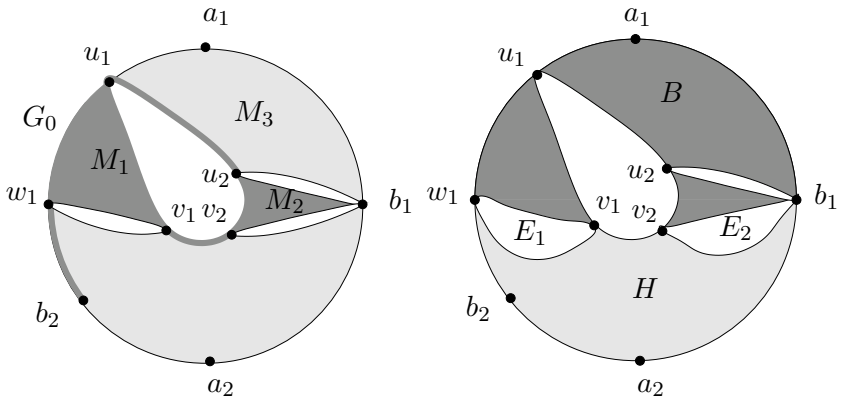


Figure 47: Subgraph definitions for $(\Gamma, X') \in \mathcal{O}_5$

1. $V(H_1 \cap M_1) = \{u_1, v_1, w_1\}$, $w_1 \in V(C)$, $\{u_1, v_1\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,
2. $V(H_2 \cap M_2) = \{u_2, v_2, w_2\}$, $w_2 \in V(C)$, $\{u_2, v_2\} \subseteq V(D)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$,
3. $V(H_3 \cap M_3) = \{u_1, u_2\} \cup (\{w_1, w_2\} \cap \{b_1, b_2\})$, $a_1 \in V(M_3)$, $\{a_2, b_1, b_2\} \in V(H_3)$,
4. if $w_1 \notin \{b_1, b_2\}$, then $u_1 \in V(C)$, and if $w_2 \notin \{b_1, b_2\}$, then $u_2 \in V(C)$, and
5. $X' \subseteq V(M_1 \cup M_2 \cup D \cup C[b_2, w_1] \cup C[w_2, b_1])$.

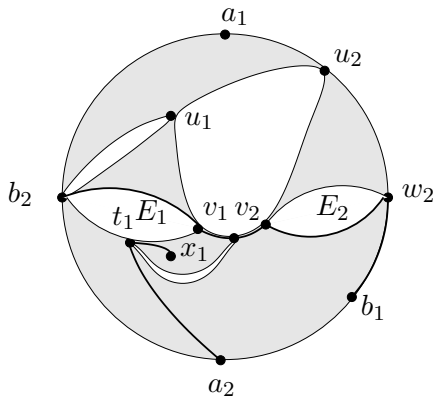
Assume further that these separations are chosen with M_1 and M_2 each maximal. See Figure 47 for illustration of the following definitions. Let $G_o = M_1 \cup M_2 \cup D \cup C[b_2, w_1] \cup C[w_2, b_1]$, so $X' \subseteq V(G_0)$. Let $U_i = H_i \cap M_i$ for $i \in \{1, 2, 3\}$. Let $H = H_1 \cap H_2 \cap H_3$, and note that G has exactly one H-bridge, with attachments $\{w_1, v_1, v_2, w_2\}$. Call this bridge B. We may assume $x_1 \notin V(G_o)$, or we have $(\Gamma, X) \in \mathcal{O}_5$. Hence $x_1 \in M_3 \cup H$.

For $i = 1, 2$, let E_i be the facial cycle of G including w_i and v_i, such that $E_1[v_1, w_1] \subseteq H_1$, $E_1[w_1, v_1] \subseteq M_1$, $E_2[v_2, w_2] \subseteq M_2$, and $E_2[w_2, v_2] \subseteq H_2$.

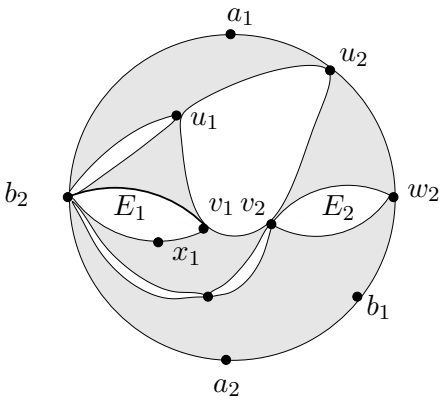
Case 1 Suppose $x_1 \in V(H)$.

We claim that we may assume there is some path Q_2 from a_2 to x_1 in H such that Q_2 is disjoint from $C[b_2, w_1] \cup D[v_1, v_2] \cup E_2[w_2, v_2]$. (See Figure 48.)

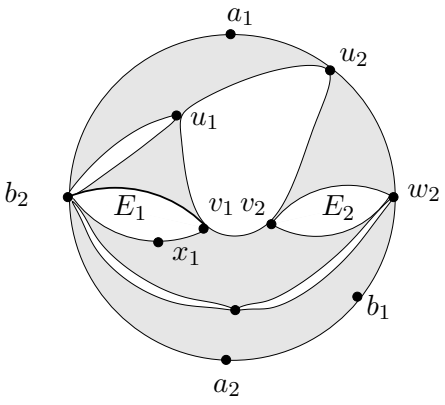
Let $Q := C[b_2, w_1] \cup E_1[v_1, w_1] \cup D[v_2, v_1] \cup E_2[w_2, v_2] \cup C[w_1, b_1]$. Since $V(E_1 \cap E_2) \subseteq V(D)$, and $x_1 \notin V(D)$, we have $x_1 \notin V(E_1)$ or $x_1 \notin V(E_2)$.



(a) Paths Q and Q_2



(b) $x_2 \in D[v_2, v_1)$



(c) $x_2 \in C[w_1, b_1)$

Figure 48: Case 1, $x_1 \in V(H)$

Suppose some 2-cut $T = \{t_1, t_2\}$ of H with $T \subseteq V(Q)$ separates x_1 from a_2. Note that since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $Q[t_1, t_2]$ must include at least one of $\{w_1, v_1, v_2, w_2\}$. If we have $t_j \in V(C)$ and $t_{3-j} \in V(D)$ for some $j \in \{1, 2\}$, then T violates the maximality of M_1 and M_2. If $T \subseteq V(Q \cap C)$, we have a 3-cut $T \cup \{a_1\}$ demonstrating that $(\Gamma, X) \in \mathcal{O}_2$. So if such a 2-cut T exists, we may assume $T \cap V(E_1 \cup E_2) \neq \emptyset$. So there is some $i \in \{1, 2\}$ such that $T \cap V(E_i) \neq \emptyset$ and $x_1 \notin V(E_{3-i})$.

Choose T to be a maximal such 2-cut, and assume by symmetry that $T \cap V(E_1) \neq \emptyset$ and $x_1 \notin V(E_2)$. Let $t_1 \in T \cap V(E_1)$. Now there is a path Q_2 from a_2 to x_1 in H which is disjoint from $Q[b_2, t_1) \cup Q[v_1, b_1]$.

Let $Q_1 := C[b_2, w_1] \cup E_1[w_1, v_1] \cup Q[v_1, b_1]$. If there is some $x_2 \in X'$ such that $x_2 \notin V(Q_1)$,

then there is a path from a_1 to x_2 in G disjoint from each of Q_1 and Q_2, and X is feasible in Γ, a contradiction. So we must have $X' \subseteq V(Q_1)$. If $X' \subseteq V(C[b_2, w_1] \cup E_1[w_1, v_1])$, we have $(\Gamma, X) \in \mathcal{O}_4$. We assume this is not the case, so there is some $x_2 \in X' \cap V(Q[v_1, b_1])$. Choose such a vertex x_2 to minimize $Q_1[x_2, b_1]$. Since $X' \subseteq V(M_1 \cup M_2 \cup D \cup C[b_2, w_1] \cup C[w_2, b_1])$, we have either $x_2 \in D[v_2, v_1)$ or $x_2 \in C[w_1, b_1)$.

Now $\{x_1, x_2\}$ is feasible in Γ if there are disjoint paths from b_2 to w_1, from a_2 to x_1, and from b_1 to v_1 in $H - x_2$. Hence no such paths may exist. So we must have some 3-separation (H', M') of H with $\{b_2, a_2, b_1\} \subseteq V(H')$, $\{w_1, x_1, v_1\} \subseteq V(M')$, and $x_2 \in V(H' \cap M')$.

If $x_2 \in D[v_2, v_1)$ (Figure 48(b)), then there is a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = V(H' \cap M') \cup \{u_1\}$ and $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$. By our choice of x_2, we have $X \subseteq V(M')$. Hence (Γ, X) satisfies property (R1).

If $x_2 \in C[w_1, b_1)$, then there is a 4-separation (G_1, G_2) of G with $V(G_1 \cap G_2) = V(H' \cap M') \cup \{a_1\}$ and $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$. Again by our choice of x_2, we have $X \subseteq V(M')$, so (Γ, X) satisfies property (R1).

Case 2 Suppose $x_1 \in V(M_3)$.

If $|U_3| = 4$, then $V(H_3 \cap M_3) = \{u_1, u_2, b_1, b_2\}$. Note that M_3 and H are now symmetric, so we simply apply the argument of Case 1 above.

(2.1) Suppose $|U_3| = 2$.

Then we have $u_1, u_2 \in V(C \cap D)$. We have assumed $x_1 \notin V(G_o)$, so $x_1 \notin V(D)$. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, there can be no 2-cut $T \subseteq V(D)$ separating x_1 from a_1. Hence there is a path P_1 from a_1 to x_1 in $H_3 - V(D)$. Let $P_2 = C[b_2, u_1] \cup D[u_1, u_2] \cup C[u_2, b_1]$, and note P_1 and P_2 are disjoint.

If $X' \subseteq V(P_2)$, then $(\Gamma, X) \in \mathcal{O}_4$, so we assume this is not the case. Let $X'' = X' \setminus V(P_2)$.

If there is some path P_3 from a_2 to X'' in $G - V(P_2)$, then X is feasible in Γ, a contradiction. Hence we may assume no such path exists. From planarity, we see that there must be a 2-cut $T \subseteq V(P_2)$ in G separating X'' from a_2. In fact, $T \subseteq V(C \cap P_2)$, and we find that $T \cup \{a_1\}$ is a 3-cut in G separating X'' from $\{a_1, b_1, a_2, b_2\}$. If $X'' = X'$, we have $(\Gamma, X) \in \mathcal{O}_2$. If $X'' \neq X'$, we have $(\Gamma, X) \in \mathcal{O}_4$.

(2.2) Suppose $|U_3| = 3$.

We assume without loss of generality that $w_2 = b_1$, so $V(H_3 \cap M_3) = \{u_1, u_2, b_1\}$, as in Figure 47. We claim that there is a path P_1 from a_1 to x_1 in $M_3 - D$. If not, then either $x_1 \in V(D)$, or some cut $T \subseteq V(D[u_1, u_2])$, $|T| \leq 2$, separates a_1 from x_1 in M_3. But we have assumed $x_1 \notin V(D)$, and any such cutset T violates the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. Let Q be any path from u_2 to b_1 in M_2, and let $P_2 = C[b_2, u_1] \cup D[u_1, u_2] \cup Q$, from b_2 to b_1 in G. Note that P_1 and P_2 are disjoint.

Notice that $\{u_1, v_2, b_1, a_1\}$ is a 4-cut in G, leading to a separation (H', M') with $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$ and $x_1 \in V(M')$. If $X' \subseteq V(M')$, we find (Γ, X) satisifes (R1). If $X' \subseteq V(M') \cup C[b_2, u_1]$, then $(\Gamma, X) \in \mathcal{O}_4$. Hence we may assume $X' \subseteq V(M') \cup C[b_2, u_1]$, so $X' \nsubseteq V(P_2)$. Let $X'' = X' \setminus V(P_2)$.

Now if there is a path P_3 from a_2 to X'' disjoint from P_2, then X is feasible in Γ, a contradiction. Hence no such path may exist. So by planarity there must be some 2-cut $T \subseteq V(P_2)$ in $H \cup M_1$ separating X'' from a_2. From planarity and $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, we see that $T = \{t_1, t_2\}$, with $t_1 \in C[b_2, w_1]$ and $t_2 \in \{v_2, b_1\}$. If $t_2 = v_2$, we have a 3-cut $T \cup \{u_1\}$ in G, violating the maximality of M_1. Hence we assume $t_2 = b_1$, and we have a separation (H_0, M_0) of G with $V(H_0 \cap M_0) = T \cup \{a_1\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H_0)$ and $X'' \cup \{x_1\} \subseteq V(M_0)$. If $X \subseteq V(M_0)$, then $(\Gamma, X) \in \mathcal{O}_2$. Otherwise, $X \subseteq V(M_0 \cup C[b_2, t_1]$, and $(\Gamma, X) \in \mathcal{O}_4$. □

Lemma 5.2. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X = (X' \cup \{x_1\}) \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X'| \geq 2$, and X infeasible in Γ. If $(\Gamma, X') \in \mathcal{O}_6$ and $X' \nsubseteq V(C)$, then either (Γ, X) is reducible or $(\Gamma, X) \in \mathcal{O}_6$.*

Proof. We may assume (Γ, X') satisfies the definition for $\mathcal{O}_6$ with $i = 1$, as shown in Figure 49. That is, G has separations (H_1, M_1) and (H_2, M_2) and a cycle D bounding a finite face of G such that:

1. $V(H_1 \cap M_1) = \{u_1, v_1, w_1\}$, $u_1 \in V(C \cap D)$, $v_1 \in V(D)$, $w_1 \in C[u_1, b_2]$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$,

2. $V(H_2 \cap M_2) = \{u_2, v_2, w_2\}$, $u_2 \in V(C \cap D)$, $v_2 \in V(D)$, $w_2 \in C[u_2, b_1]$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_2)$,

3. $X' \subseteq V(M_1 \cup M_2 \cup D \cup C[w_1, b_2] \cup C[w_2, b_1])$

Assume these separations are chosen to maximize M_1 and M_2. We may assume $x_1 \notin V(M_1 \cup M_2 \cup D \cup C[w_1, b_2] \cup C[w_2, b_1])$; otherwise we have $(\Gamma, X) \in \mathcal{O}_6$. Notice that $\{w_1, v_1, u_2\}$ is a 3-cut in G. Let H be the $\{w_1, v_1, u_2\}$-bridge of G which contains $\{a_1, b_2\}$. Let M be the $\{w_1, v_1, u_2\}$-bridge containing M_1 and M_2. By symmetry, we may assume $x_1 \in V(H) \setminus V(C[w_1, b_2] \cup D[v_1, v_2])$.

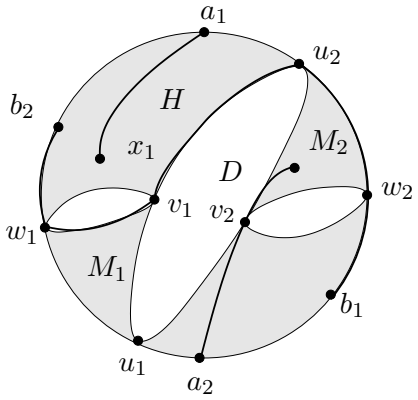


Figure 49: $(\Gamma, X') \in \mathcal{O}_6$

We claim first that we may assume there is a path P_1 from a_1 to x_1 in $H - V(C[w_1, b_2] \cup D)$. To see this, suppose not. Since we assume $x_1 \notin V(C[w_1, b_2] \cup D)$, there must be some cut T in H with $|T| \leq 2$ separating a_1 from x_1. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, we must have $T = \{t_1, t_2\}$ with $t_1 \in C[w_1, b_2]$ and $t_2 \in V(D)$. But this violates the maximality of M_1. So no such cutset exists, and we have a path P_1.

Let $Q := C[w_1, b_2] \cup \partial M_1[w_1, v_1] \cup D[v_1, u_2] \cup C[u_2, b_1]$, and note that Q is a path from b_1 to b_2 which is disjoint from P_1.

Now if there is a path P_2 from a_2 to X in M, disjoint from Q, then we would have X feasible in Γ. So there can be no such path, and for each $x \in X \cap V(M)$, either $x \in V(Q)$ or there is a 2-cut T in M separating a_2 from x, with $T \subseteq V(Q)$. From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity and planarity of G, we find that so such 2-cut exists, so we

must have $X \cap V(M) \subseteq V(Q)$. We find that there is a 4-separation (H', M') of G with $V(H' \cap M') = \{a_1, u_1, u_2, b_2\}$, such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H')$ and $X \subset V(M') \cup C[u_2, b_1]$. Since $X' \nsubseteq V(C)$, we find $|X \setminus V(C[u_2, b_1])| \leq 2$. If $X \subseteq V(M')$, then (Γ, X) satisfies property (R1). Otherwise, (Γ, X) satisfies property (R2). □

5.2 Vertex of X on outer face

Several obstruction classes require some vertex of X on the outer face of G. Where X has only two vertices, one of which is on the outer cycle, we have seen that their relative positions are extremely limited. We will make use of this by considering vertex pairs.

Remark 5.1. Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 3$, and X infeasible in Γ. Suppose $x_1 \in V(C)$, and let $X' = X \setminus \{x_1\}$. Each pair $\{x_1, x\}$ with $x \in X'$ must be infeasible in Γ. By an application of Lemma 4.1, we find that $(\Gamma, \{x_1, x\}) \in \Theta_1 \cup \Theta_4$. Letting $C_a = C[a_i, x_1]$ if $x_1 \in C[a_i, b_i]$, $C_a = C[x_1, a_i]$ if $x_1 \in C[b_{3-i}, a_i]$, one of the following holds:

(A1) $\exists$ a face of G incident with both x and some vertex in C_a, or

(A2) G has a separation (H, M) of order at most 4 with $\{x_1, a_1, a_2, b_1, b_2\} \subseteq V(H)$, $x \in V(M)$, and $|V(H \cap M) - C_a| \leq 2$

We will use this fact in several of the following arguments.

Definition 5.1. Where D is a cycle in a plane graph G, we denote by $Int(D)$ the open disk bounded by D, and by $\overline{Int}(D)$ the closed disk bounded by D.

Lemma 5.3. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 3$, X infeasible in Γ, and some $x_1 \in X \cap V(C)$. Let $X' = X \setminus \{x_1\}$. If there is a cycle D in G such that $X' \subseteq \overline{Int}(D)$ and $\{a_1, b_1, a_2, b_2\} \cap Int(D) = \emptyset$, then (Γ, X) satisfies (R2) or $(\Gamma, X) \in \mathcal{O}_4$.*

Proof. Assume by symmetry that $x_1 \in C[b_2, a_1]$, and hence $C_a = C[x_1, a_1]$. Label $X' = \{x_2, \ldots, x_n\}$. For any $x_k \in X'$ which satisfies (A2), choose a separation (H, M) with M maximal, and call it (H_k, M_k).

Suppose (H, M) is a separation of G of order at most 4 with $\{x_1, a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $|V(H \cap M - C_a)| \leq 2$. If $X' \subseteq V(M)$, then we find $(\Gamma, X) \in \mathcal{O}_4$. Hence we may assume $X' \nsubseteq V(M)$, and therefore $V(D) \nsubseteq V(M)$.

Now suppose x_k is not cofacial with any vertex of C_a. Then we have the separation (H_k, M_k), and $x_k \notin V(H_k \cap M_k)$. Hence $|V(D) \cap V(M_k)| \geq 2$. But we assume $V(D) \nsubseteq V(M_k)$, so $V(M_k \cap K_k)$ is a cutset in G separating D. Furthermore, $V(H_k \cap M_k - C_a) \subseteq V(D)$, so $|V(H_k \cap M_k - C_a)| = 2$. Label $V(H_k \cap M_k - C_a) = \{v_k, w_k\}$ such that $D[v_k, w_k] \subseteq M_k$, and note that each of v_k, w_k is cofacial with a vertex of C_a. (See Figure 50.)

Recall our choice of each separation (H_k, M_k) to maximize M_k. By planarity, we find that there is a single separation (H, M) with $(H, M) = (H_k, M_k)$ for any x_k not cofacial with a vertex of C_a. Label $V(H \cap K) = \{u_x, u_a, v, w\}$ with $v = v_k$, $w = s_k$, $u_x, u_a \in V(C)$ chosen to minimize each of $C[x_1, u_x]$ and $C[u_a, a_1]$.

Now suppose there is some $x \in X'$ with $x \notin V(M)$. Then x is cofacial with some vertex $u \in V(C_a)$. If $u \in C[x_1, u_x]$, we have a separation (H', M') of G with $V(H', M') = \{u_a, w, x, u\}$, contradicting the maximality of M. Similarly, if $u \in C[u_a, a_1]$, there is a separation (H', M') of G with $V(H', M') = \{u, x, v, u_x\}$, again contradicting the maximality of M. So we find $X' \subseteq V(M_u)$, and $(\Gamma, X) \in \mathcal{O}_4$ or (Γ, X) satisfies (R2).

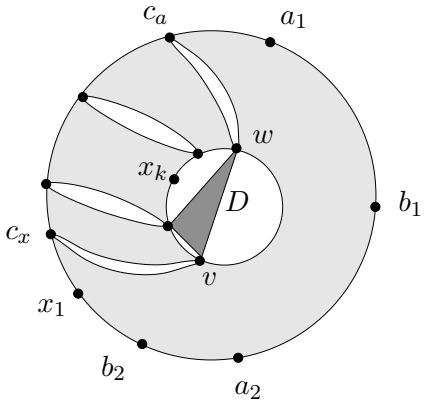


Figure 50: $x_1 \in V(C)$, $X' \subset \overline{Int}(D)$.

□

Lemma 5.4. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| = n \geq 3$, X infeasible in Γ, and $X \cap V(C) \neq \emptyset$. If Theorem*

1 holds for vertex sets of size $n-1$, then either (Γ, X) is reducible, or $(\Gamma, X') \in \bigcup_{i=1}^{7} \mathcal{O}_i$.

Proof. By symmetry, assume that $X \cap C(b_2, a_1) \neq \emptyset$. Define $X_1 = X \cap C(b_2, a_1)$, $X_2 = X \cap C(a_1, b_1)$, and $X_3 = X \setminus V(C(b_2, b_1))$. If $X \subseteq C(b_2, b_1)$, then $(\Gamma, X) \in \mathcal{O}_1$, so we assume that $X_3 \neq \emptyset$. Let $x_1 \in X_1$ be chosen to maximize $C[x_1, a_1]$, and let $X' = X \setminus \{x_1\}$.

We may assume X' is not feasible in Γ; otherwise X is feasible in Γ, a contradiction. Now $|X'| = n - 1$, so we may apply Theorem 1 to find that either (Γ, X') is reducible or $(\Gamma, X') \in \bigcup_{i=1}^{7} \mathcal{O}_i$.

If $(\Gamma, X') \in \mathcal{O}_5$, we apply Lemma 5.1. If $(\Gamma, X') \in \mathcal{O}_6$, apply Lemma 5.2. If $(\Gamma, X') \in \mathcal{O}_1$, we have some facial cycle D in G with $X' \subseteq V(D)$, and we apply Lemma 5.3. If $(\Gamma, X') \in \mathcal{O}_2$ or satisfies (R1), G has a separation (H, K) of order at most 4 such that $X' \subseteq V(K)$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$. Choose such a separation with K maximal. Let D be the facial cycle of H such that $V(H \cap K) \subseteq V(D)$. (Note from the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and maximality of K that this cycle is unique.) Apply Lemma 5.3.

We assume, then, that $(\Gamma, X') \in \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_7$ or satisfies property (R2).

Case 1 $|X \cap V(C_1)| \geq 2$ for some $C_1 \in \{C[a_1, b_1], C[b_1, a_2], C[a_2, b_2], C[b_2, a_1]\}$.

The argument is symmetric, so we may assume $|X_1| \geq 2$. By our choice of x_1, we have $X' \cap C[x_1, a_1] \neq \emptyset$.

(1.1) Suppose $(\Gamma, X') \in \mathcal{O}_3$.

If $(\Gamma, X') \in \mathcal{O}_3$ with respect to a_1, then we find $(\Gamma, X) \in \mathcal{O}_3$. Assume, then, that $(\Gamma, X') \in \mathcal{O}_3$ with respect to a_2. Let $x_2 \in X' \cap C[x_1, a_1]$ be chosen to minimize $C[x_1, x_2]$. Either x_2 is cofacial with a_2, or G has a separation (H, K) of order 3 such that $a_2 \in V(H \cap K)$, $x_2 \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$. In either case, there is some vertex $v \in C[b_2, a_1]$ which is cofacial with a_2. Hence there is a separation (H_1, K_1) of G with $V(H_1 \cap K_1) = \{v, a_2, b_2\}$, $x_1 \in V(K_1)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H_1)$. This shows $(\Gamma, X) \in \mathcal{O}_3$, with respect to a_2.

(1.2) Suppose $(\Gamma, X') \in \mathcal{O}_4$ or satisfies (R2).

We have a partition X_1', X_2', X_3' of X', some $i, j \in \{1,2\}$, $x' \in X_j'$, and $C_a' = C[a_i, x']$ if $x' \in C[a_i, b_i]$, $C_a' = C[x', a_i]$ if $x' \in C[b_{3-i}, a_1]$, and a face or separation of G satisfying the requirements of $\mathcal{O}_4$ or property (R2). Since for each $x \in (X_1 \setminus \{x_1\})$, $y \in X_3$, the pair $\{x, y\}$ must be infeasible, we may assume $i = j = 1$, and $C_a' = C[x', a_1]$. So $X_1' = X_1 \setminus \{x_1\}$, $X_2' = X_2$, $X_3' = X_3$. From our choice of x_1 to maximize $C[x_1, a_1]$, we have $C_a' \subseteq C_a$, and the face or separation which demonstrates that $(\Gamma, X') \in \mathcal{O}_4$ or satisfying property (R2) also demonstrates $(\Gamma, X) \in \mathcal{O}_4$ or satisfying (R2), respectively.

(1.3) Suppose $(\Gamma, X') \in \mathcal{O}_7$.

Let (H, K) be the separation and D the cycle of G satisfying the requirements of $\mathcal{O}_7$. That is, for some $i \in \{1,2\}$, the following hold

1. $V(H \cap K) = \{u, v, w\}$, $w \in C[b_{3-i}, b_i] - a_i$, and $u, v \in V(D)$
2. $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, $X \cap V(K) \nsubseteq V(H)$,
3. If $w \in C(b_{3-i}, a_i)$, then $u \in C(w, a_i]$ and $X \subseteq V(K \cup D \cup C(b_{3-i}, w])$
4. If $w \in C(a_i, b_i)$, then $u \in C[a_i, w)$ and $X \subseteq V(K \cup D \cup C[w, b_i))$
5. If $w = b_{3-i}$, then $X \subseteq V(K \cup D)$.

Since $X' \cap C[b_2, a_1] \neq \emptyset$, we must have $i = 1$. From 3-5 above, $u \in V(C)$, so $(X_1 \setminus \{x_1\}) \subseteq V(C(b_2, u])$. From our choice of x_1 to maximize $C[x_1, a_1]$, we find that $x_1 \in V(C(b_2, u])$, and hence $(\Gamma, X) \in \mathcal{O}_7$.

Case 2 $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$.

Label $X_3 = \{x_3, \ldots, x_n\}$, and recall that for each $x \in \{x_1, x_2\}$ and $y = x_k \in X_3$ we have either a face satisfying (A1) above, or a separation of G satisfying (A2) above.

Claim 1. For each $x_k \in X_3$, there is some separation (O_k, N_k) of G of order at most 3 such that $\{b_1, a_2, b_2, x_1, x_2\} \subseteq V(O_k)$, $\{x_k, a_1\} \subseteq V(N_k)$, and $|V(O_k \cap N_k - C[b_2, b_1])| \leq 1$.

Proof of claim 1 First suppose x_k is cofacial with some $v_1 \in C[x_1, a_1]$ and $v_2 \in C[a_1, x_2]$. Then we have the desired separation with $V(O_k \cap N_k) = \{x, v_1, v_2\}$.

Now suppose x_k is not cofacial with any vertex of $C[x_1, a_1]$. Then there is some separation (H, M) of order at most 4 with $\{x_1, a_1, a_2, b_1, b_2\} \subseteq V(H)$, $x_k \in V(M)$, and $|V(H \cap M) - C[x_1, a_1]| \leq 2$. Label $V(H \cap M) = \{c_x, c_a, m_1, m_2\}$ in clockwise order along the boundary of M, with $c_x, c_a \in V(C)$, letting $c_a = c_x$ or $m_1 = m_2$ as appropriate if the separation is of order 3 (Figure 51).

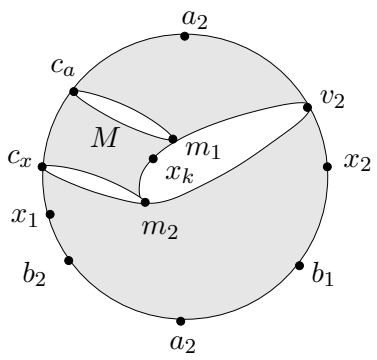
Recall that we have assumed x_k not cofacial with any vertex of $C[x_1, a_1]$. If x_k is cofacial with some vertex $v_2 \in C[a_1, x_2]$, we must have m_2 cofacial with v_2 (Figure 51(a)). Hence there is a separation as desired with $V(O_k \cap N_k) = \{c_1, m_2, v_2\}$.

Let us assume, then, that m_2 is not cofacial with any vertex of $C[a_1, x_2]$. We must have some separation (H', M') of order at most 4 with $\{x_2, a_1, b_1, a_2, b_2\} \subseteq V(H')$, $x_k \in V(M')$, and $|V(H' \cap M') - C[a_1, x_2]| \leq 2$. Label $V(H' \cap M') = \{c'_a, c'_x, m'_1, m'_2\}$ in clockwise order along the boundary of M', with $c'_a, c'_x \in V(C)$, letting $c'_a = c'_x$ or $m'_1 = m'_2$ as appropriate if the separation is of order 3 (Figures 51(b), 51(c)).

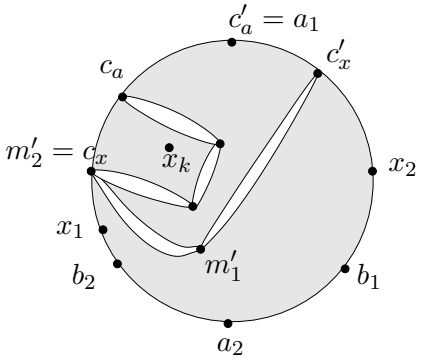
By planarity, we may choose $m'_2 \in C[b_2, c_a]$, and $c'_a = a_1$, and we have the desired separation with $V(O_k, N_k) = \{c'_x, m'_1, m'_2\}$. This proves Claim 1

Assume each separation (O_k, N_k) is chosen with N_k maximal. If $X_3 \subseteq V(N_i)$ for some $i \in \{3, \ldots, n\}$, then we see (Γ, X) satisfies property (R2). So we may assume this is not the case. Hence there are $i, j \in \{3, \ldots, n\}$ such that $x_i \notin V(N_j)$, $x_j \notin V(N_i)$. Consider the separations (O_i, N_i) and (O_j, N_j). Let $V(O_i, N_i) = \{u_1, u_2, u_3\}$ (or $\{u_1, u_2\}$ if the separation is of order 2), with $u_1 \in C[x_1, a_1]$ and $u_2 \in C[a_1, x_2]$. Similarly, let $V(O_j, N_j) = \{v_1, v_2, v_3\}$ (or $\{v_1, v_2\}$ if the separation is of order 2), with $v_1 \in C[x_1, a_1]$ and $v_2 \in C[a_1, x_2]$. From our choice of i and j, we see that neither $N_j \subseteq N_i$ nor $N_i \subseteq N_j$ is true. Hence we may assume that u_1, v_1, a_1, u_2, v_2 appear in this clockwise order along C (with equality possible among pairs in $\{v_1, a_1, u_2\}$).

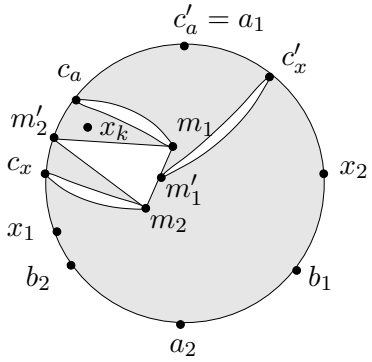
Now from planarity, and maximality of N_i and N_j, there are separations (H_1, M_1), (H_2, M_2), (H_3, M_3) of G such that $V(H_1 \cap M_1) = \{u_1, v_1, u_3\}$, $V(H_2 \cap M_2) = \{u_2, v_2, v_3\}$,



(a) (A1), (A2) combination



(b) (A2), (A2) combination



(c) (A2), (A2) combination

Figure 51: $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, separation (O_k, N_k).

$V(H_3 \cap M_3) = \{v_1, u_2\}$. Observe further that $\{b_1, a_2, b_2, x_1, x_2\} \subseteq V(H_1 \cap H_2 \cap H_3)$, $x_i \in M_1$, and $x_j \in M_2$ (Figure 52).

Claim 2. We may assume $X_3 \subseteq V(M_1 \cup M_2 \cup M_3)$.

Proof of claim 2 Suppose not. So there is some $x_k \in X_3$ with $x_k \notin V(M_1 \cup M_2 \cup M_3)$. Consider the associated separation (O_k, N_k), and let $V(O_k, N_k) = \{w_1, w_2, w_3\}$ (or $\{w_1, w_2\}$ if the separation is of order 2). From planarity we find that one of $(w_1, u_1, v_1, a_1, w_2, u_2, v_2)$, $(w_1, u_1, v_1, a_1, u_2, v_2, w_2)$, or $(u_1, v_1, w_1, a_1, u_2, v_2, w_2)$ appear in clockwise order along C. But then there is separation (H, M) of G with $V(H \cap M) \in \{ \{w_1, w_2\}, \{w_1, u_2\}, \{w_1, u_2, w_3\}, \{v_1, w_2\}, \{v_1, w_2, w_3\}\}$, contradicting the maximality of N_i or N_j. This proves Claim 2.

Claim 3. We may assume $X_3 \cap M_3$ lies on the boundary of M_3.

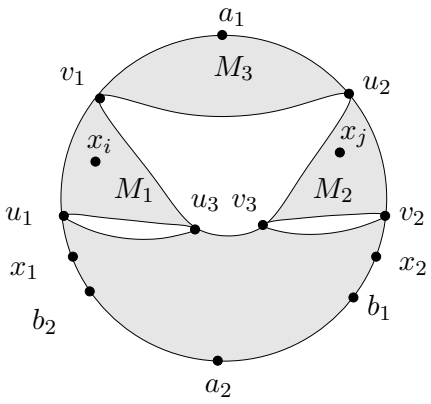


Figure 52: $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, leading to $(\Gamma, X) \in \mathcal{O}_5$.

Proof of Claim 3 Suppose we have some $x_k \in V(M_3)$ which is not incident with the infinite face of M_3. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, and by maximality of M_1 and M_2, we see that M_3 is 2-connected. Let D be the outer cycle of M_3. Again from $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, there is a path P from a_1 to x_k in $M_3 - D[u_2, v_1]$. Now let $P_b = C[b_2, v_1] \cup D[u_2, v_1] \cup C[v_1, b_1]$. Note $x_i \notin P_b$. Suppose there is a vertex set $T \subseteq V(P_b)$ separating x_i from a_2. By planarity, we must have $T = \{t_1, t_2\}$ with $t_1 \in C[b_2, v_1]$, $t_2 \in C[v_1, b_1]$. Hence there is a separation (H, M) of G with $V(H \cap M) = T$, $X_3 \cup \{a_1\} \subseteq V(M)$, demonstrating $(\Gamma, X) \in \mathcal{O}_4$. Hence we may assume no such separating set exists, and we find $x_i \in \mathcal{R}(P)$, so X is feasible, a contradiction.

The separations $(H_1, M_1), (H_2, M_2)$, and (H_3, M_3) now demonstrate $(\Gamma, X) \in \mathcal{O}_5$.

Case 3 $X_1 = \{x_1\}$ and $X_2 = \emptyset$.

We may now assume by symmetry that $|X_3 \cap V(C)| \leq 1$.

(3.1) Suppose $(\Gamma, X') \in \mathcal{O}_3$.

Then $\exists\ i \in \{1, 2\}$ such that for each $x \in X'$ one of the following holds

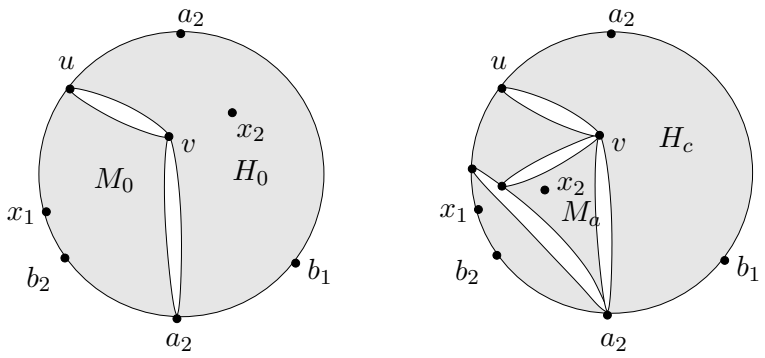
(i) there is a finite face of G incident with both a_i and x, or

(ii) G has a separation (H, K) of order 3 such that $a_i \in V(H \cap K)$, $x \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, or

(iii) $x \in C[b_{3-i}, b_i]$.

If $i = 1$, we find that $x_1 \in C[b_{3-i}, b_i]$, and $(\Gamma, X) \in \mathcal{O}_3$. Hence we may assume $i = 2$. Let $C_a := C[x_1, a_1]$.

Let $\mathcal{M}$ be the collection of all separations (H, M) of G with $V(H \cap M) = \{u, v, a_2, b_2\}$, such that $u \in V(C_a)$, $\{u, v\} \neq \{a_1, b_1\}$, $x_1 \in V(M)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$ (as in Figure 53(a)). Choose $(H_0, M_0) \in \mathcal{M}$ to maximize M_0.



(a) Defining (H_0, M_0) (b) x_2 satisfying (A2)

Figure 53: $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, $(\Gamma, X') \in \mathcal{O}_3$.

From planarity and $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, we find that for any separation $(H, M) \in \mathcal{M}$, $M \subseteq M_0$. If $X' \subseteq V(M_0)$, then (Γ, X) satisfies property (R1). Hence we may assume that $X' \nsubseteq V(M_0)$, so there is some $x_2 \in X'$ with $x_2 \notin V(M_0)$, and for any separation $(H, M) \in \mathcal{M}$, $x_2 \notin V(M)$.

First suppose we have a finite face with boundary D such that $x_2, a_2 \in V(D)$. If x_2 is cofacial with some $u \in V(C_a)$, then $\{u, x_2, a_2\}$ is a 3-cut in G, leading to a 4-separation $(H, M) \in \mathcal{M}$. But $x_2 \in V(M)$, a contradiction. Hence we may assume that x_2 is not cofacial with any vertex of C_a. So (A1) is not satisfied for x_2.

Therefore (A2) must be satisfied, and there is some separation (H_c, M_c) of G of order at most 4 with $\{x_1, a_1, a_2, b_1, b_2\} \subseteq V(H_c)$, $x_2 \in V(M_c)$ and $|V(H_c \cap M_c) \setminus V(C_a)| \leq 2$. Moreover, $x_2 \in V(M_c) \setminus V(H_c)$. Since $a_2 \in V(H_c)$, we must have some vertex $v \in V(H_c \cap M_c) \cap D(x_2, a_2]$. This leads again to a separation $(H, M) \in \mathcal{M}$ with $\{v, a_2\} \subseteq V(H \cap M)$

such that $x_2 \in V(M)$, a contradiction.

Hence we may assume that x_2 and a_2 are not incident with a common finite face.

Now $(\Gamma, X') \in \mathcal{O}_3$ and x_2 is not cofacial with a_2, so there is some separation (H_a, M_a) of G order 3 such that $a_2 \in V(H_a \cap M_a)$, $x_2 \in V(M_a) \setminus V(H_a)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H_a)$. Label $V(H_a \cap M_a) \setminus \{a_2\}$ as $\{v_1, v_2\}$.

If x_2 is cofacial with some vertex $u \in V(C_a)$, then each of v_1, v_2 must also be cofacial with u. Now for $j \in \{1, 2\}$ we have $\{u, v_j, a_2\}$ a 3-cut in G, associated with a 4-separation $(H_j, M_j) \in \mathcal{M}$. But for some $j \in \{1, 2\}$ we find $x_2 \in V(M_j)$, a contradiction. Hence x_2 is not cofacial with any vertex of C_a, and (A1) is not satisfied for x_2.

So (A2) must be satisfied, and there is some separation (H_c, M_c) of G of order at most 4 with $\{x_1, a_1, b_1, a_2, b_2\} \subseteq V(H_c)$, $x_2 \in V(M_c)$ and $|V(H_c \cap M_c) \setminus V(C_a)| \leq 2$. Moreover, $x_2 \in V(M_c) \setminus V(H_c)$. Choose such a separation with M_c maximal. Recalling that $x_2 \in V(M_a) \setminus V(H_a)$, we find that $V(M_c - H_c) \cap V(M_a - H_a) \neq \emptyset$. (See Figure 53(b).) By planarity, and maximality of M_c, we find $(H_c, M_c) \in \mathcal{M}$, a contradiction since $x_2 \in V(M_c)$.

(3.2) Suppose $(\Gamma, X') \in \mathcal{O}_4$ or (Γ, X') satisfies (R2).

Since $X_2 = \emptyset$, and we assume (Γ, X') does not satisfy (R1), we may assume there is some $x_3 \in X' \cap C(b_1, b_2)$. Recalling the assumption for Case 3 that $|X_3 \cap V(C)| \leq 1$, we have $X' \cap C(b_1, b_2) = \{x_3\}$. Let $C'_a = C[x_3, a_2]$ if $x_3 \in C(b_1, a_2)$, and let $C'_a = C[a_2, x_3]\}$ if $x_3 \in C(a_2, b_2)$. Let $X'_3 = X' \setminus \{x_3\}$.

Since $\{x_1, x_3\}$ is infeasible in Γ, there must be no path from b_1 to b_2 disjoint from $C'_a \cup C[x_1, a_1]$. So there is some 2-separation (M, N) of G with $V(M \cap N) = \{u, v\}$, $u \in C[x_1, a_1]$, $v \in C'_a$, and $C[u, v] \subseteq N$. (See Figure 54.) Assume the separation (M, N) is chosen to minimize N

If $(\Gamma, X') \in \mathcal{O}_4$, we have one of the following:

(i) $\exists$ a finite face of G incident with some vertex in $V(C_a)$ and all $x \in X'_3$, or

(ii) G has a separation (H, K) of order 3 with $\{x_3, a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X'_3 \subseteq V(K)$,

and $|V(H \cap K) \setminus V(C_a)| \leq 2$, or

(iii) G has a separation (H, K) of order 4 with $\{x_3, a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3' \subseteq V(K)$, and $|V(H \cap K) \setminus V(C_a)| \leq 2$.

If (Γ, X') satisfies (R2), then G has a separation (H, K) of order at most 4 such that $\{x_3, a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3' \subseteq V(K)$, and $|V(H \cap K) \setminus V(C_a)| \leq 2$. So we have a separation as in (iii), with $|X_3'| \geq 2$. (When $(\Gamma, X') \in \mathcal{O}_4$, the separation of type (iii) has the added condition that $|X_3'| = 1$.)

Suppose first that $x_3 \in C(b_1, a_2)$, so $C[a_2, x_1] \subseteq V(M)$ and $C[a_1, x_3] \subseteq V(N)$ (as in Figure 54(a)). If $X_3' \subseteq V(M)$, we have a separation (M', N') of G with $V(M' \cap N') = \{u, v, a_2, b_2\}$ demonstrating that (Γ, X) satisfies (R2). Similarly, if $X_3' \subseteq V(N)$, we have a separation (M', N') of G with $V(M' \cap N') = \{u, v, a_1, b_1\}$, and again (Γ, X) satisfies (R2). Hence we may assume $X_3' \nsubseteq V(M)$ and $X' \nsubseteq V(N)$.

If (i) above is satisfied, let D be the facial cycle of G including X_3' and some vertex $c_a \in V(C_a)$. Then $D \nsubseteq N$ and $D \nsubseteq M$, so by planarity we have $u, v \in V(D)$, and $(\Gamma, X) \in \mathcal{O}_6$.

Therefore we assume (i) is not satisfied, so G has a separation (H, K) of order at most 4 with $\{x_3, a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3' \subseteq V(K)$, and $|V(H \cap K) \setminus V(C_a)| \leq 2$. Now $K \nsubseteq M$ and $k \nsubseteq N$. Choose $x_k \in X_3'$ with $x_k \in V(N) \setminus V(M)$. By planarity, we must have some vertex $w \in V(H \cap K)$ with $u' \in C[u, a_1]$ cofacial with some vertex $v' \in C[x_3, v]$. This contradicts the choice of (M, N) to minimize N.

Now suppose that $x_3 \in C(a_2, b_2)$, so $C[x_3, x_1] \subseteq V(M)$ and $C[a_1, a_2] \subseteq V(N)$. Let (M', N') be the separation of G with $V(M' \cap N') = \{u, v, b_2\}$, $C[x_3, x_1] \subseteq V(M')$. If $X' \subseteq V(M)$, then (M', N') demonstrates that $(\Gamma, X) \in \mathcal{O}_2$. If all vertices of $X_3' \cap V(N)$ lie on the finite face including u and v, we have $(\Gamma, X) \in \mathcal{O}_7$. Hence we may assume there is some $x_k \in X_3' \cap V(N)$ which is not on the finite face containing u and v.

If (i) above is satisfied, let D be the facial cycle of G including X_3' and some vertex $c_a \in V(C_a)$. Choose $w \in X_3'$ to minimize $D[w, c_a]$ (see Figure 54(b)). If (ii) or (iii) are satisfied, label $V(H \cap K) = \{c_a, c_x, w', w\}$ in clockwise order along the boundary of K, with

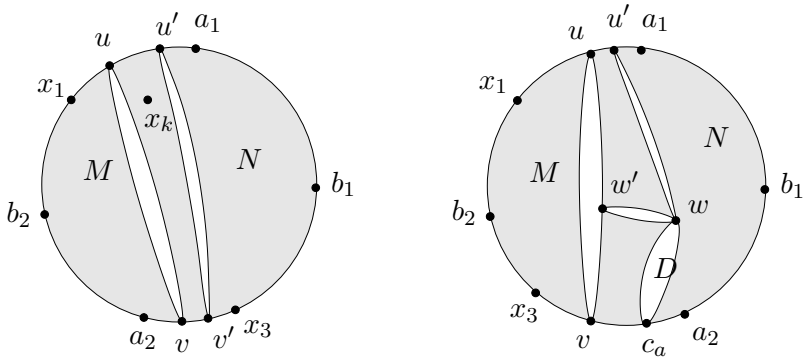


Figure 54: $X_2 = \emptyset$, $(\Gamma, X') \in \mathcal{O}_4$.

$c_a, c_x \in V(C)$, letting $c_a = c_x$ or $w' = w$ as appropriate if the separation is of order 3. Let D be the facial cycle of G including w and c_a, such that $D[w, c_a] \subseteq H$ and $D[c_a, w] \subseteq K$.

Recall that for $x = x_1$ and each $y \in X_3'$ we have either a face satisfying (A1) above, or a separation of G satisfying (A2). In either case, we find that some vertex $u' \in C[u, a_1]$ must be cofacial with w, or with some $v' \in C[a_1, c_a]$. But u' cofacial with v' contradicts our choice of (M, N), so we have u' cofacial with w. We thus have a 4-separation (M'', N'') with $V(M'' \cap N'') = \{u', w, c_a, b_2\}$, $X \subseteq V(M'')$, and $\{a_1, a_2, b_1, b_2\} \subseteq N''$, so (Γ, X) satisfies (R1).

(3.3) Suppose $(\Gamma, X') \in \mathcal{O}_7$.

Let (H, K) be the separation and D the cycle of G satisfying the requirements of $\mathcal{O}_7$. That is, for some $i \in \{1, 2\}$, the following hold

1. $V(H \cap K) = \{u, v, w\}$, $w \in C(b_{3-i}, a_i) \cup C(a_i, b_i)$, and $u, v \in V(D)$
2. $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, $X' \cap V(K) \nsubseteq V(H)$,
3. if $w \in C(b_{3-i}, a_i)$, then $u \in C(w, a_i]$ and $X' \subseteq V(K \cup D \cup C(b_{3-i}, w])$, if $w \in C(a_i, b_i)$, then $u \in C[a_i, w)$ and $X' \subseteq V(K \cup D \cup C[w, b_i))$, and
4. if $w \in \{b_{3-i}, b_i\}$, then $X' \subseteq V(K \cup D)$.

If $X' \subseteq V(K \cup D)$, then we have a cycle D' in G such that $X' \subseteq \overline{Int}(D')$ and $\{a_1, b_1, a_2, b_2\} \cap Int(D') = \emptyset$, and we apply Lemma 5.3. Hence we may assume $X' \nsubseteq V(K \cup D)$, so $X' \cap V(C) \setminus V(K) \neq \emptyset$, and $w \notin \{b_1, b_2\}$ (by 4 above).

Since $X_2 = \emptyset$, we must have some $x_3 \in X' \cap C(b_1, b_2)$, with $x_3 \notin V(K)$. Recalling that $|X_3 \cap V(C)| \leq 1$, we have $X' \cap C(b_1, b_2) = \{x_3\}$ (Figure 55).

Let $X'_3 = X' \setminus \{x_3\}$. If $X'_3 \subseteq V(K)$ or $X'_3 \subseteq V(D)$, we find $(\Gamma, X') \in \mathcal{O}_4$, and apply the arguments of Case (3.2) above. We may assume, then, that $X'_3 \nsubseteq V(K)$ and $X'_3 \nsubseteq V(D)$, so $|X'_3| \geq 2$. Hence $|X'| > 3$, and we may apply Lemma 5.3 to (Γ, X') to find that (Γ, X') satisfies (R2) or $(\Gamma, X') \in \mathcal{O}_4$. We then apply the arguments of Case (3.2) above. □

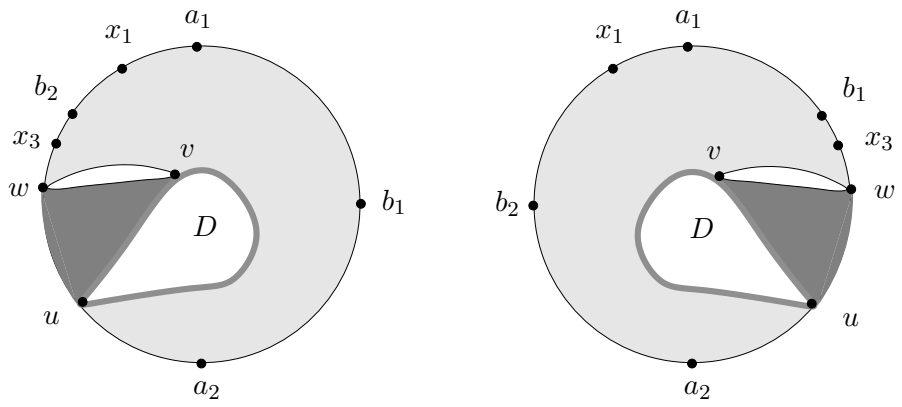


Figure 55: Where $X_2 = \emptyset$, $(\Gamma, X') \in \mathcal{O}_7$.

5.3 Remaining inductive cases

This section considers the remaining cases, where $X \cap V(C) = \emptyset$, and $(\Gamma, X') \notin \mathcal{O}_5 \cup \mathcal{O}_6$. We again use the definitions and lemmas associated with maximizing paths. Our first proposition is a technical observation, which will be useful in combining results on subsets of X.

Proposition 5.5. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 3$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1.*

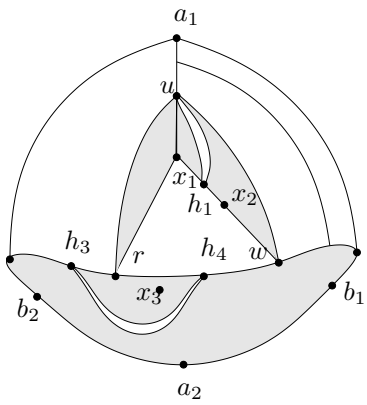


Figure 56: Separation with $X' \not\subseteq V(L_P)$

Let $X' = X \setminus \{x_1\}$. Suppose $X' \cap V(L_P) \neq \emptyset \neq X' \setminus V(L_P)$, and $x_2 \in X' \setminus V(L_P)$. If there is a separation (H, K) of G of order at most 4 with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $(X' \cap V(L_P)) \cup \{x_1\} \subseteq V(K)$, and $x_2 \notin V(K)$, then either $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2$, or (H, K) is of order 4, and (Γ, X) satisfies property (R1).

Proof. Let $Y = X \setminus V(L_P)$. Then by Lemma 3.5, G has a facial cycle D and a 3-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{u, r, w\}$ such that

(a) $u \in V(P)$, $\{r, w\} \subseteq V(D \cap M_P)$, and $D[w, r] = M_P[r, w]$,

(b) $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $Y \subseteq V(G_2)$, and

(c) $Y \subseteq D[r, w]$.

If $X' \subseteq V(D)$, then we have $(\Gamma, X) \in \mathcal{O}_1$, so we may assume there is some $x_3 \in X'$ with $x_3 \notin D[r, w]$. We may further assume, by symmetry, that $x_2 \in D(x_1, w]$. Note from $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G that $u \neq x_1$, so we have $x_1 \notin V(G_1)$.

Suppose there is a separation (H, K) of G of order at most 4 with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $(X' \cap V(L_P) \cup \{x_1\}) \subseteq V(K)$, and $x_2 \notin V(K)$. Letting $K_i = K \cap G_i$ for $i \in \{1, 2\}$, we have a separation (K_1, K_2) of K, with $x_2 \in V(K_1) \setminus V(K_2)$, and $x_1 \in V(K_2) \setminus V(K_1)$. That is, $V(K_1 \cap K_2)$ is a cutset in K. There must be some $h_1, h_2 \in V(H \cap K)$ with $h_1 \in D(x_1, x_2)$ and $h_2 \in V(P)$.

Since $x_3 \notin V(D)$, $|V(H \cap K) \cap V(M_P)| = 2$, so $|V(H \cap K)| = 4$. Let $V(H \cap K) = \{h_1, h_2, h_3, h_4\}$. Now one of h_3, h_4 must be cofacial with h_1, so we may assume $h_4 \in M_P[r, w]$. By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, we have $h_3 \in M_P[b_2, r)$. By planarity $\{h_3, h_4, u, w\}$ is a 4-cut in G, and by Lemma 2.11 we find $(\Gamma, X) \in \mathcal{O}_2$ or satisfies (R1). □

Lemma 5.6. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 3$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to $x_1 \in X$. If $(\Gamma, X \setminus \{x_1\}) \in \mathcal{O}_1$, then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5 \cup \mathcal{O}_7$.*

Proof. Let $X' = X \setminus \{x_1\}$, and label $X' = \{x_2, \ldots, x_n\}$. By definition of $(\Gamma, X') \in \mathcal{O}_1$, all $x \in X'$ are incident with a common finite face, or $X' \subseteq V(C[b_1, b_2])$, or $X' \subseteq V(C[b_2, b_1])$. Since $X \cap V(C) = \emptyset$, we have a finite face incident with all vertices of X'. Let D be the cycle bounding this face, so $X' \subseteq V(D)$. We may assume $x_1 \notin V(D)$; otherwise $(\Gamma, X) \in \mathcal{O}_1$.

Since X is not feasible in Γ, we may assume $X' \cap \mathcal{R}(P) = \emptyset$. So for any $x \in X' \cap \mathcal{A}(P)$, we have either $x \in V(M_P)$ or there is some 2-cut $Z \subseteq V(M_P)$ in L_P separating x from a_2.

If $D \cap P \neq \emptyset$ and $X' \subseteq V(M_P)$, then by Lemma 3.12 we have $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_7$, or satisfies property (R1). So we assume either $D \subseteq L_P$ or $X' \nsubseteq V(L_P)$.

Case 1 Suppose $X' \nsubseteq V(L_P)$.

Let $Y = X \setminus V(L_P)$. From Lemma 3.5, G has a facial cycle F and a separation (G_1, G_2) such that: $V(G_1 \cap G_2) = \{u, r, w\}$ with $r, w \in V(F)$; $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $Y \subseteq V(K)$; $Y \subseteq F[r, w]$. (See Figure 57.) Since $x_1 \notin V(D)$, we have $F \neq D$. For any $y \in Y \setminus \{x_1\}$, we have $y \in V(D \cap F)$. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $|Y \setminus \{x_1\}| = 1$. That is, $|X' \setminus V(L_P)| = 1$. Let $X' \setminus V(L_P) = \{x_2\}$.

Since $|X'| \geq 2$, we have $X' \cap V(L_P) \neq \emptyset$, and hence $V(D \cap L_P) \neq \emptyset$. Let $d_1, d_2 \in V(M_P \cap D)$ be chosen so that $D \cap M_P = M_P[d_1, d_2]$. Assume (G_1, G_2) is chosen to minimize $P[a_1, u]$. From planarity, we find $u \in V(D)$. By symmetry, we may assume $w \in V(D)$, so $D[w, u]$ is a path to the right of P, and $d_1 = w$.

Let $\tilde{X} = X \setminus \{x_2\}$. We have established $D \cap P \neq \emptyset$, and hence $\tilde{X} \setminus \{x_1\} \subseteq V(M_P)$. By Lemma 3.12 we have $(\Gamma, \tilde{X}) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_7$, or satisfying property (R1).

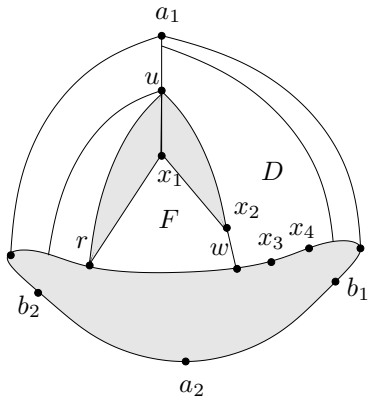


Figure 57: Example of $(\Gamma, X') \in \mathcal{O}_1$ with $X' \not\subseteq V(L_P)$

If $(\Gamma, \tilde{X}) \in \mathcal{O}_1$, we must have $\tilde{X} \subseteq V(F)$ and $|X| = 3$, and hence $(\Gamma, X) \in \mathcal{O}_1$. If $(\Gamma, \tilde{X}) \in \mathcal{O}_2$ or satisfies (R1), we have some separation (H, K) of G of order at most 4, with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $\tilde{X} \subseteq V(K)$. If $x_2 \in V(K)$, then clearly $(\Gamma, X) \in \mathcal{O}_2$ or satisfies (R1). If $x_2 \notin V(K)$, apply Proposition 5.5 to find $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2$, or satisfies (R1).

Suppose $(\Gamma, \tilde{X}) \in \mathcal{O}_3$. Noting that no vertex of P is shares a finite face of G with a_2, we find that $(\Gamma, \tilde{X}) \in \mathcal{O}_3$ with respect to a_1. If $a_1 \in V(D)$, then x_2 is cofacial with a_1, and $(\Gamma, X) \in \mathcal{O}_3$. Otherwise, there must be some 3-separation (H, K) of G with $V(H \cap K) = \{a_1, h_1, h_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, and $x_1 \in V(K)$. Since each of h_1, h_2 is cofacial with a_1, we may assume $h_1 \in M_P[b_2, r]$ and $h_2 \in M_P[d_2, b_1]$. This gives $X \subseteq V(K)$, and $(\Gamma, X) \in \mathcal{O}_2 \cap \mathcal{O}_3$.

Finally, suppose $(\Gamma, \tilde{X}) \in \mathcal{O}_7$. Then there is a 3-separation (H, K) of G with $V(H \cap K) = \{h_1, h_2, h_3\}$ such that $h_1 \in M_P[b_2, r]$, $h_2 \in M_P[d_1, d_2]$, and $h_3 \in P \cap D$, where either $h_1 = b_2$ or $h_1, h_3 \in V(C)$. In this case, $x_2 \in V(K)$, and we have $(\Gamma, X) \in \mathcal{O}_7$.

Case 2 Suppose $X' \subseteq V(L_P)$, and there is some $x_i \in X' \setminus V(M_P)$

Here $D \cap L_P \not\subseteq M_P$, and we find that D is contained in a unique block of L_P. Label this block B_D, and note that $X' \subseteq V(B_D)$. By Lemma 3.7, if there is some cutvertex of L_P separating X' from a_2, then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$. Hence we may assume there is no such cutvertex, and $a_2 \in V(B_D)$. That is, $x_i \in \mathcal{A}(P)$.

Since $x_i \notin V(M_P)$, there is a 2-cut $Z \in V(M_P)$ in L_P separating x_i from a_2. Choose Z as the maximal such cut, and label $Z = \{z_1, z_2\}$ with z_1 left of z_2. Let B_z be the Z-bridge of L_P containing x_i.

We claim that $X' \subseteq V(B_z)$. To see this, suppose not. Then $V(D) \nsubseteq V(B_z)$, so Z separates D, and we must have $Z \subseteq V(D \cap M_P)$. Now if there is some $x_j \in X'$ with $x_j \notin V(B_z)$, we have $x_j \in D[z_2, z_1]$. Since $x_j \notin V(M_P)$, there is some 2-cut $Z' \subseteq V(M_P)$ in L_P separating x_j from a_2. But now Z' also separates $X' \cap V(B_z)$ from a_2, violating the maximality of Z.

We now have $x_i \notin V(M_P)$, and a 2-separation (H, K) of L_P with $V(H \cap K) = Z$, $X \setminus \{x_1\} \subseteq V(K)$, and $\{b_1, a_2, b_2\} \subseteq H$. By Lemma 3.14, either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.

Case 3 Suppose $D \subseteq L_P$ and $X' \subseteq V(M_P)$.

Assume $X' = \{x_2, \ldots, x_n\}$ is labeled in order from left to right along M_P. If $X' = \{x_2, x_3\}$ and x_2x_3 is a single edge, we may choose a different facial cycle D' of G with $X' \subseteq V(D')$ and $D' \nsubseteq L_P$, and apply the arguments of earlier cases. Hence we may assume this is not the case, and $\{x_2, x_n\}$ is a 2-cut in L_P. Let B be the $\{x_2, x_n\}$-bridge of L_P containing $M_P[x_2, x_n]$.

If both x_2 and x_n are cofacial with x_1, we have a 3-cut $\{x_1, x_2, x_n\}$ in G showing $(\Gamma, X) \in \mathcal{O}_2$. We assume this is not the case, so there is some path from $P[a_1, x_1)$ to $M_P(x_2, x_n)$. Let S be a path from $s' \in P[a_1, x_1)$ to $s \in M_P(x_2, x_n)$, chosen to minimize $P[a_1, s']$. Assume by symmetry that S falls to the right of P, and further assume S is chosen so that $M_P[s, x_n]$ is contained in a facial cycle also containing s'. Since we assume X' is not contained in a facial cycle, we have x_2 left of s. (See Figure 58.)

If there is a separation (H, K) of G with $V(H \cap K) = \{s', x_2, x_n\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X \subseteq V(K)$, then $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume there is no such separation, so there is some path T from $P(s', x_1]$ to $P[a_1, s') \cup M_P[b_2, x_2)$. By Lemma 3.2 (every $(P \cup L_P)$-bridge has an attachment in L_P), there is a path T from $t' \in P(s', x_1]$ to $t \in M_P[b_2, x_2)$. Choose such a T to minimize $M_P[b_2, t]$, and subject to this minimizing $P[s', t']$.

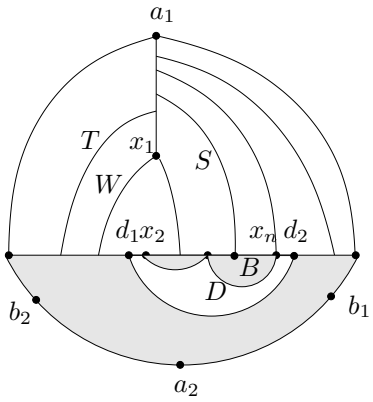


Figure 58: Example of $(\Gamma, X') \in \mathcal{O}_1$ with $X' \subseteq V(M_P)$

Suppose t is cofacial with x_2. Then there is 4-separation (M, N) of G with $V(M \cap N) = \{x_2, x_n, s', t\}$, $X' \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. If $P[t', x_1] \subseteq V(N)$, then by Lemma 2.11 (4-separations in G), $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible.

So we may assume that either t is not cofacial with x_2, or for any such separation, $x_1 \notin V(N)$. Therefore there is some path W from $w' \in P[s', x_1]$ to $w \in M_P(t, x_2)$. We claim that we may choose W such that $w' \neq s'$. To see this, suppose not. Then any path from $P(s', x_1]$ to $M_P[t, s)$ has endpoint t, and $\{s', t\}$ is a 2-cut in G violating $(3, \{a_1, b_1, a_2, b_2\})$-connectivity.

By Corollary 3.13, for any $x_k \in X'$ we have $(\Gamma, \{x_1, x_k\}) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$, or $(\Gamma, \{x_1, x_k\})$ satisfies property (R1). But $(\Gamma, \{x_1, x_n\}) \notin \Theta_1$, so $(\Gamma, \{x_1, x_n\}) \in \Theta_2 \cup \Theta_3$, or satisfies (R1).

(3.1) Suppose $(\Gamma, \{x_1, x_n\}) \in \Theta_2$.

Here G has a 3-separation (H, K) with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $\{x_1, x_n\} \subseteq V(K)$. Let $V(H \cap K) = \{h_1, h_2, h_3\}$. We may assume $h_2 \in M_P[x_n, b_1]$ and $h_3 \in P[a_1, x_1]$. If $h_2 \neq x_n$, we must have $h_1 \in M_P[b_2, x_n]$. If $h_2 = x_n$, then by planarity and since h_1 must be cofacial with h_2, we find $h_1 \in V(D)$, and again $h_1 \in M_P[b_2, x_n]$.

Now h_1 and h_3 share a finite face to the left of P, and h_2 and h_3 share a finite face to the right of P. From the existence of paths S and T, we find $h_2 \in M_P[b_2, t]$, so $M_P[x_2, x_n] \subseteq K$. Hence $X' \in V(K)$ and $(\Gamma, X) \in \mathcal{O}_2$.

(3.2) Suppose $(\Gamma, \{x_1, x_n\}) \in \Theta_3$.

For each $j \in \{1, n\}$, either x_j is cofacial with a_1 or G has a separation (H_j, M_j) of order 3 such that $a_1 \in V(H_j \cap M_j)$, $x_j \in V(M_j)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_j)$. If there is a such a separation (H_n, M_n), we assume $B \nsubseteq M_n$; otherwise $X' \subseteq V(M_n)$, and $(\Gamma, X) \in \mathcal{O}_3$. So there is some vertex $v \in V(H \cap K) \cap V(M_P(x_2, x_n])$. Since v must be cofacial with a_1, this forces $s' = a_1$.

We now have either $s' = a_1$, or x_n, s', and a_1 share a common finite face.

Suppose first that $s' = a_1$. We may assume x_2 is not cofacial with s', or we find $(\Gamma, X) \in \mathcal{O}_3$. So there is a path R from $r' \in P(a_1, x_1]$ to $r \in M_P(x_2, s]$. Choose R such that each vertex of $M_P[r, s]$, is cofacial with $s' = a_1$. From $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, Lemma 3.3, and the existence of paths T, W, and R, we see that any 3-separation (H_1, K_1) must have $M_P[t, r] \subseteq V(K_1)$. If this holds, then each vertex $x_k \in X'$ is either cofacial with a_1 or contained in $V(K_1)$, and $(\Gamma, X) \in \mathcal{O}_3$. We assume, then, that there is no separation (H_1, K_1), and hence x_1 is cofacial with a_1. Therefore, both of R and W are right of P.

By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, $\{t, t'\}$ is not a 2-cut in G, so we find $t' = x_1$. We may choose W with $w' \neq x_1$; otherwise there is 4-separation (M, N) of G with $V(M \cap N) = \{x_2, x_n, a_1, x_1\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and by Lemma 2.11 (4-separations in G), $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible.

Since $x_1 \notin V(C)$, there is a path T_0 from a_1 to $t_0 \in M_P[b_2, t]$. We assume there is no edge $a_1 b_2$, so $t_0 \neq b_2$. Let $Q := T_0 \cup M_P[t_0, t] \cup T$. If there are disjoint paths from b_2 to w and from a_2 to x_2, each disjoint from $M_P[t_0, t] \cup M_P[x_n, b_1]$, then we have $W \cup R \subseteq V(L_Q)$, and $x_2 \in \mathcal{R}(Q)$. Hence X is feasible in Γ, a contradiction. So these paths do not exist, and there must be some separation (M, N) of L_P, of order at most 3, with $\{t_0, b_2, a_2, b_1\} \subseteq V(M)$ and $\{t, w, x_2, x_n\} \subseteq V(N)$. By planarity, we may assume $M_P[x_2, x_n] \subseteq N$, so $X' \subseteq V(N)$.

We may assume there are vertices $m_1, m_2 \in V(M \cap N)$ with $m_1 \in M_P[t_0, t]$ and $m_2 \in M_P[x_n, b_1]$. If $|V(M \cap N)| = 3$, label the third vertex as m_3. By planarity, we have one of the following:

(i) m_1 shares a finite face of L_P with m_2, or

(ii) m_3 shares a face of L_P with each of m_1 and m_2, or

(iii) $m_3 \in M_P(t, w]$ shares a finite face of L_P with m_2.

In case (i), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence $(\Gamma, X) \in \mathcal{O}_2 \cap \mathcal{O}_3$. In case (ii), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, m_3, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence (Γ, X) satisfies (R1). In case (iii), we have m_3 cofacial with x_1, and there is a separation (M', N') of G with $V(M' \cap N') = \{m_2, m_3, x_1, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and again (Γ, X) satisfies (R1).

Now suppose $s' \neq a_1$. Then x_n, s', and a_1 share a common finite face of G. Since $x_n \notin V(C)$, there is a path S_0 from a_1 to $s_0 \in M_P(x_n, b_1]$. We assume there is no edge a_1b_1, so $s_0 \neq b_1$. Let $Q := S_0 \cup M_P[x_n, s_0]$. If there are disjoint paths from a_2 to w and from b_1 to x_2, each disjoint from $M_P[b_2, t] \cup M_P[x_n, s_0]$, then we have $T \cup S \subseteq V(L_Q)$. Path W shows $x_1 \in \mathcal{R}(Q)$, and hence X is feasible in Γ, a contradiction. So these paths do not exist, and there must be some separation (M, N) of L_P, or order at most 3, with $\{b_2, a_2, b_1, s_0\} \subseteq V(M)$ and $\{t, w, x_2, x_n\} \subseteq V(N)$.

We may assume there are vertices $m_1, m_2 \in V(M \cap N)$ with $m_1 \in M_P[b_2, t]$ and $m_2 \in M_P[x_n, s_0]$. If $|V(M \cap N)| = 3$, label the third vertex as m_3. As before, we have one of the following:

(i) m_1 shares a finite face of L_P with m_2, or

(ii) m_3 shares a face of L_P with each of m_1 and m_2, or

(iii) $m_3 \in M_P(t, w]$ shares a finite face of L_P with m_2.

In case (i), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence $(\Gamma, X) \in \mathcal{O}_2 \cap \mathcal{O}_3$. In case (ii), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, m_3, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence (Γ, X) satisfies (R1). In case (iii), we have m_3 cofacial with x_1, and there is a separation (M', N') of G with $V(M' \cap N') = \{m_2, m_3, x_1, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and again (Γ, X) satisfies (R1).

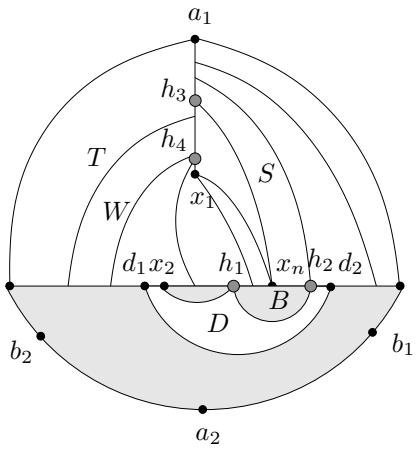


Figure 59: $(\Gamma, X') \in \mathcal{O}_1$, $X' \subseteq V(M_P)$, $(\Gamma, \{x_1, x_2\})$ reducible

(3.3) Suppose $(\Gamma, \{x_1, x_n\})$ satisfies property (R1).

Here G has some separation (H, K) of order 4 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $\{x_1, x_n\} \subseteq V(K)$. Choose such a separation with K maximal. Let $V(H \cap K) = \{h_1, h_2, h_3, h_4\}$. If $V(M_P[x_2, x_n]) \subseteq V(K)$, then (Γ, X) satisfies (R1). So we assume not, and that $h_1 \in M_P(x_2, x_n]$. Further, there is some $h \in V(H \cap K)$ with $h \in M_P[x_n, b_1]$. If we may choose $h_1 \neq x_n$, then we have $h_2 \in M_P[x_n, b_1]$. If $h_1 = x_n$, then by planarity and since h_1 must be cofacial with h_2, we find $h_2 \in V(D)$, and again $h_2 \in M_P[x_n, b_1]$.

Now by planarity and the existence of paths S and T, we have $h_3 \in P[a_1, s']$, sharing a face to the right of P with h_2. (See Figure 59.) Now h_4 is cofacial with each of h_1 and h_3. Since $x_1 \in K$ and $b_2 \in H$, we must have $h_4 \in M_P[b_2, t] \cup T \cup P[t', x_1]$. By the existence of path T, we see that h_1 is not cofacial with any vertex of $M_P[b_2, t)$, so $h_4 \in T \cup P[t', x_1]$. Now by planarity, since h_4 shares a finite face of G with h_1, we find that h_4 also shares a finite face with d_1, and G has a 4-separation (H', K') with $V(H' \cap K') = \{h_1, h_2, h_3, d_1\}$, demonstrating that (Γ, X) satisfies (R1).

□

Lemma 5.7. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 3$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from a_1 to x_1. If $(\Gamma, X \setminus \{x_1\}) \in \mathcal{O}_2$, then (Γ, X) satisfies (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$.*

Proof. Let $X' = X \setminus \{x_1\}$. We have $(\Gamma, X') \in \mathcal{O}_2$, so G has a separation (G_1, G_2) of order 3 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$ and $X' \subseteq V(G_2)$. We may assume the separation (G_1, G_2) is associated with a 3-cut in G; otherwise we have all vertices of X' cofacial, and we apply Lemma 5.6. Choose such a separation to maximize G_2, and let $V(G_1 \cap G_2) = \{g_1, g_2, g_3\}$. We assume $x_1 \notin V(G_2)$, or we have $(\Gamma, X) \in \mathcal{O}_2$.

Case 1 Suppose $X' \setminus V(L_P) \neq \emptyset$.

Let $Y := X \setminus V(L_P)$, and $Y' := X' \setminus V(L_P) = Y \setminus \{x_1\}$. By Lemma 3.5, G has a facial cycle D and a separation (H, K) such that: $V(H \cap K) = \{u, r, w\}$ with $r, w \in V(D)$; $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $Y \subseteq V(K)$; $Y \subseteq D[r, w]$.

If $X' \subseteq D[r, w]$, we have $(\Gamma, X) \in \mathcal{O}_1 \cap \mathcal{O}_2$. So we assume $X' \nsubseteq D[r, w]$, and there is some $x_2 \in X'$ with $x_2 \in V(L_P) \setminus \{r, w\}$. So we must have $|V(G_1 \cap G_2) \cap V(M_P)| \geq 2$. Label $V(G_1 \cap G_2) \cap V(M_P)$ as $\{g_1, g_2\}$.

Since $x_1 \notin V(G_2)$, there is some separation (K_1, K_2) of K with $V(K_1 \cap K_2) \subseteq V(G_1 \cap G_2)$, $\{u, x_1\} \subseteq V(K_1)$, $Y' \subseteq V(K_2)$, and $x_1 \notin V(K_2)$. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $|V(K_1 \cap K_2)| \geq 2$. But this requires (G_1, G_2) to be a separation of order 4, a contradiction.

Case 2 Suppose $X' \subseteq V(L_P)$.

By Lemma 3.7, if there is some cutvertex of L_P separating X' from a_2, then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$. Hence we may assume there is no such cutvertex. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, we must have $X' \subseteq \mathcal{A}(P)$.

Without loss of generality, assume $g_1, g_2 \in V(M_P)$. Suppose g_1 and g_2 do not share a finite face of L_P, so $g_3 \in V(L_P)$. In this case, $M_P[g_1, g_2]$ is contained in a facial boundary of G. Since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, there can be no 2-cut of L_P contained in $M_P[g_1, g_2]$, and we must have $X' \subseteq V(M_P)$; otherwise X is feasible in Γ. By Lemma 3.12, $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_7$, or satisfies property (R1).

We now assume g_1 shares a finite face of L_P with g_2, so $\{g_1, g_2\}$ is a 2-cut in L_P. Let g_1 be left of g_2 along M_P. If there is some $x_k \notin V(M_P)$, then by Lemma 3.14 either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$. We assume, then, that $X' \subseteq V(M_P)$. Label $X' = \{x_2, \ldots, x_n\}$, in order from left to right along M_P.

By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, $\{g_1, g_2\}$ is not a 2-cut in G, so there is some path S from $s' \in P[a_1, x_1]$ to $s \in M_P(g_1, g_2)$. Choose such a path to minimize $P[a_1, s']$. Assume by symmetry that S falls to the right of P, and further assume S is chosen so $M_P[s, g_2]$ is contained in a facial cycle also containing s'. Since we assume X' is not contained in a facial cycle, we have x_2 left of s. Furthermore, since g_3 is cofacial with each of g_1, g_2, we have $g_3 \in \{s', s\}$.

If there is a separation (H, K) of G with $V(H \cap K) = \{s', g_1, g_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X \subseteq V(K)$, then $(\Gamma, X) \in \mathcal{O}_2$. Hence we may assume there is no such separation, so there is some path T from $P(s', x_1]$ to $P[a_1, s') \cup M_P[b_2, g_1)$. By Lemma 3.2 (every $(P \cup L_P)$-bridge has an attachment in L_P), there is a path T from $t' \in P(s', x_1]$ to $t \in M_P[b_2, g_1)$. Choose such a T to minimize $M_P[b_2, t]$, and subject to this minimizing $P[s', t']$.

Suppose t is cofacial with g_1. By our choice of S and Lemma 3.3 (controlling paths within $(P \cup L_P)$-bridges), there is some vertex $v \in P[a_1, s']$ cofacial with both t and g_2. Hence there is 4-separation (M, N) of G with $V(M \cap N) = \{g_1, g_2, v, t\}$, $X \subseteq V(N)$, and $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$. If $P[t', x_1] \subseteq V(N)$, then by Lemma 2.11 (4-separations in G), $(\Gamma, X) \in \mathcal{O}_2$, or (Γ, X) is reducible. So we may assume that either t is not cofacial with g_1, or for any such separation $x_1 \notin V(N)$. Therefore there is some path W from $w' \in P[s', x_1]$ to $w \in M_P(t, g_2)$. We claim that we may choose W such that $w' \neq s'$. To see this, suppose not. Then any path from $P(s', x_1]$ to $M_P[t, s)$ has endpoint t, and $\{s', t\}$ is a 2-cut in G violating $(3, \{a_1, b_1, a_2, b_2\})$-connectivity.

If x_1 shares a finite face of G with each of g_3 and g_1, then we have a 4-separation (G'_1, G'_2) of G with $V(G'_1 \cap G'_2) = \{g_1, g_2, g_3, x_1\}$, and by Lemma 2.11, $(\Gamma, X) \in \mathcal{O}_2$ or satisfies (R1). We assume this is not the case, so either (i) W is right of P, or (ii) $g_3 = s$, and there is a path R to the left of P, from $r' \in P[w', x_1)$ to s. Consider case (ii). Suppose W is chosen to minimize $P[w', r']$. G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, so for any vertex $v \in P[w', r']$, the set $\{v, s\}$ is not a 2-cut in G. Hence there must be a path P' from $p' \in P[a_1, w')$ to $p \in P(r', x_1]$, contained in some $(P \cup L_P)$-bridge and disjoint from $P \cup L_P$. But together with paths W and R, this contradicts Lemma 3.3. Hence case (ii) cannot occur, and we have W right of P. Recalling that $g_3 \in \{s', s\}$ is cofacial with g_1, we have $w \in M_P(t, g_1] \cup \{s\}$.

By Corollary 3.13, for any $x_k \in X'$ we have $(\Gamma, \{x_1, x_k\}) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$, or $(\Gamma, \{x_1, x_k\})$ satisfies property (R1). Note from path W that x_1 is not cofacial with x_n. Therefore $(\Gamma, \{x_1, x_n\}) \notin \Theta_1$, so $(\Gamma, \{x_1, x_n\}) \in \Theta_2 \cup \Theta_3$, or satisfies (R1).

(2.1) Suppose $(\Gamma, \{x_1, x_n\}) \in \Theta_2$.

Here G has a 3-separation (H, K) with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $\{x_1, x_n\} \subseteq V(K)$. If $X' \subseteq V(K)$, then $(\Gamma, X) \in \mathcal{O}_2$, so we assume $X' \nsubseteq V(K)$.

Let $V(H \cap K) = \{h_1, h_2, h_3\}$. Since $X' \nsubseteq V(K)$, we may assume $h_1 \in M_P(x_2, x_n]$. We further assume $h_3 \in P[a_1, x_1]$, and note $V(H \cap K) \cap V(M_P[x_n, b_1]) \neq \emptyset$. By the existence of path S, we may assume $V(H \cap K) \cap V(M_P) \neq \{x_n\}$, so $h_2 \in M_P[x_n, b_1]$. Since h_2 is cofacial with h_3, we have $h_3 \in P[a_1, s']$. But now we have a path from x_1 to b_2 along T, disjoint from $V(H \cap K)$, a contradiction.

(2.2) Suppose $(\Gamma, \{x_1, x_n\})$ satisfies property (R1).

Here G has some separation (H, K) of order 4 such that $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $\{x_1, x_n\} \subseteq V(K)$. Choose such a separation with K maximal. By Lemma 2.8, $V(H \cap K) \cap \{a_1, b_1, a_2, b_2\} \neq \emptyset$ or $|V(H \cap K) \cap V(C)| \geq 2$. If $V(M_P[x_2, x_n]) \subseteq V(K)$, then (Γ, X) satisfies (R1). So we assume not, and there must be a vertex $h_1 \in V(H \cap K)$ with $h_1 \in M_P(x_2, x_n]$.

Let $V(H \cap K) = \{h_1, h_2, h_3, h_4\}$. If $X' \subseteq V(K)$, then (Γ, X) satisfies (R1). So we assume not, and that $h_1 \in M_P(x_2, x_n]$. We further assume $h_3 \in P[a_1, x_1]$, and note $V(H \cap K) \cap V(M_P[x_n, b_1]) \neq \emptyset$. By the existence of path S, we may assume $V(H \cap K) \cap V(M_P) \neq \{x_n\}$, so $h_2 \in M_P[x_n, b_1]$.

Now by planarity and the existence of paths S and T, we have $h_3 \in P[a_1, s']$, sharing a face to the right of P with h_2. Now h_4 is cofacial with each of h_1 and h_3. Since $x_1 \in K$ and $b_2 \in H$, we must have $h_4 \in M_P[b_2, t] \cup T \cup P[t', x_1]$. By the existence of path T, we see that h_1 is not cofacial with any vertex of $M_P[b_2, t)$, so $h_4 \in T \cup P[t', x_1]$. Now by planarity, since h_4 shares a finite face of G with h_1, we find that h_4 also shares a finite face with g_1, and G has a 4-separation (H', K') with $V(H' \cap K') = \{h_1, h_2, h_3, g_1\}$, demonstrating that (Γ, X) satisfies (R1).

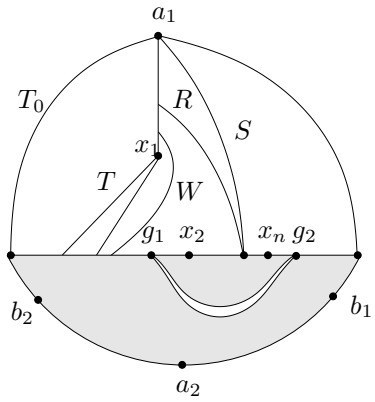


Figure 60: $(\Gamma, X') \in \mathcal{O}_2$

(2.3) Suppose $(\Gamma, \{x_1, x_n\}) \in \Theta_3$.

For each $j \in \{1, n\}$, either x_j is cofacial with a_1 or G has a separation (H_j, K_j) of order 3 such that $a_1 \in V(H_j \cap K_j)$, $x_j \in V(K_j)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H_j)$. If there is a such a separation (H_n, K_n), we assume $X' \nsubseteq V(K_n)$; otherwise $X' \subseteq V(K_n)$, and $(\Gamma, X) \in \mathcal{O}_3$. So there is some vertex $v \in V(H \cap K) \cap V(M_P(x_2, x_n])$. Since v must be cofacial with a_1, this forces $s' = a_1$. We now have either $s' = a_1$, or x_n, s', and a_1 share a common finite face.

Suppose first that $s' = a_1$ (Figure 60). We may assume g_1 is not cofacial with s', or we find $(\Gamma, X) \in \mathcal{O}_3$. But each of g_1, g_2 is cofacial with g_3, so we must have $g_3 = s$, and hence $w \in M_P(t, g_1)$. Furthermore, there is a path R from $r' \in P(a_1, w]$ to s. Note that $g_3 = s$ together with $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G shows that $G_2 \subseteq L_P$, and $G_2 \cap L_P$ is 2-connected.

If $t' \neq x_1$, then the 3-separation (H_1, K_1) exists and must have vertices $h_1, h_2 \in V(H_1 \cap K_1)$, with $h_1 \in M_P[b_2, t]$ and $h_2 \in M_P[s, b_1]$ sharing a finite face. But since $G_2 \cap L_P$ is 2-connected, this forces $h_2 \in M_P[g_2, b_1]$. Hence $X' \subseteq V(K_1)$, and $(\Gamma, X) \in \mathcal{O}_2 \cap \mathcal{O}_3$. Hence we may assume $t' = x_1$.

Since $x_1 \notin V(C)$, there is a path T_0 from a_1 to $t_0 \in M_P[b_2, t]$. We assume there is no edge a_1b_2, so $t_0 \neq b_2$. Let $Q := T_0 \cup M_P[t_0, t] \cup T$. If there are disjoint paths from b_2 to w and from a_2 to x_2, each disjoint from $M_P[t_0, t] \cup M_P[g_2, b_1]$, then we have $W \cup R \subseteq V(L_Q)$, and $x_2 \in \mathcal{R}(Q)$. Hence X is feasible in Γ, a contradiction. So these paths do not exist, and there

must be some separation (M, N) of L_P, or order at most 3, with $\{t_0, b_2, a_2, b_1\} \subseteq V(M)$ and $\{t, w, x_2, x_n\} \subseteq V(N)$.

We may assume there are vertices $m_1, m_2 \in V(M \cap N)$ with $m_1 \in M_P[t_0, t]$ and $m_2 \in M_P[x_n, b_1]$. If $|V(M \cap N)| = 3$, label the third vertex as m_3. By planarity, we have one of the following:

(i) m_1 shares a finite face of L_P with m_2, or

(ii) m_3 shares a face of L_P with each of m_1 and m_2, or

(iii) $m_3 \in M_P(t, w]$ shares a face of L_P with m_2.

In case (i), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, a_1\}$, contradicting our choice of (G_1, G_2) to maximize G_2. In case (ii), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, m_3, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence (Γ, X) satisfies (R1). In case (iii), we have m_3 cofacial with x_1, and there is a separation (M', N') of G with $V(M' \cap N') = \{m_2, m_3, x_1, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and again (Γ, X) satisfies (R1).

Now suppose $s' \neq a_1$, so x_n, s', and a_1 share a common finite face. Since $x_n \notin V(C)$, there is a path S_0 from a_1 to $s_0 \in M_P(g_2, b_1]$. We assume there is no edge a_1b_1, so $s_0 \neq b_1$. Let $Q := S_0 \cup M_P[x_n, s_0]$. If there are disjoint paths from a_2 to w and from b_1 to g_1, each disjoint from $M_P[b_2, t] \cup M_P[x_n, s_0]$, then we have $T \cup S \subseteq V(L_Q)$. Path W shows $x_1 \in \mathcal{R}(Q)$, Hence X is feasible in Γ, a contradiction. So these paths do not exist, and there must be some separation (M, N) of L_P, or order at most 3, with $\{b_2, a_2, b_1, s_0\} \subseteq V(M)$ and $\{t, w, x_2, x_n\} \subseteq V(N)$.

We may assume there are vertices $m_1, m_2 \in V(M \cap N)$ with $m_1 \in M_P[b_2, t]$ and $m_2 \in M_P[x_n, s_0]$. If $|V(M \cap N)| = 3$, label the third vertex as m_3. As before, we have one of the following:

(i) m_1 shares a finite face of L_P with m_2, or

(ii) m_3 shares a face of L_P with each of m_1 and m_2, or

(iii) $m_3 \in M_P(t, w]$ shares a finite face of L_P with m_2.

In case (i), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, a_1\}$, contradicting our choice of (G_1, G_2) to maximize G_2. In case (ii), there is a separation (M', N') of G with $V(M' \cap N') = \{m_1, m_2, m_3, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and hence (Γ, X) satisfies (R1). In case (iii), we have m_3 cofacial with x_1, and there is a separation (M', N') of G with $V(M' \cap N') = \{m_2, m_3, x_1, a_1\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$, $X \subseteq V(N')$, and again (Γ, X) satisfies (R1).

□

Lemma 5.8. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$, with $|X| \geq 3$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. If $(\Gamma, X \setminus \{x_1\}) \in \mathcal{O}_3$, then (Γ, X) satisfies (R1) or $(\Gamma, X) \in \mathcal{O}_3$.*

Proof. Let $X' = X \setminus \{x_1\} = \{x_2, \ldots, x_n\}$. We assume by symmetry that $(\Gamma, X') \in \mathcal{O}_3$ with respect to a_2, so for each $x \in X'$, one of the following holds

(a) there is a finite face of G incident with both a_2 and x, or

(b) G has a separation (H, K) of order 3 such that $a_2 \in V(H \cap K)$, $x \in V(K)$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, or

(c) $x \in C[b_1, b_2]$.

Since $X \cap V(C) = \emptyset$, one of (a) or (b) must hold.

By Lemma 3.1, either $(\Gamma, X) \in \mathcal{O}_3$ with respect to a_1, or $G - V(C[b_1, b_2])$ contains a path P from a_1 to X for which a_2 is not a cutvertex of L_P. We may assume $(\Gamma, X) \notin \mathcal{O}_3$, so we have a path $P \in \mathcal{P}_{(\Gamma, X)}$ from a_1 to X.

If $x \in X'$ is cofacial with a_2, then for any path P' from a_1 to x, we find that a_2 is a cutvertex of $L_{P'}$. Hence $P' \notin \mathcal{P}_{(\Gamma, X)}$. If $x \in X'$ is not cofacial with a_2, then G has a separation (H, K) with $V(H \cap K) = \{h_1, h_2, a_2\}$, $x \in V(K)$, and $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$. Any path P' from a_1 to x must have $V(P') \cap \{h_1, h_2, a_2\} \neq \emptyset$, and again $P' \notin \mathcal{P}_{(\Gamma, X)}$. So we may assume each path in $\mathcal{P}_{(\Gamma, X)}$ is from a_1 to x_1. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path.

Claim 1. $X' \subseteq V(L_P)$.

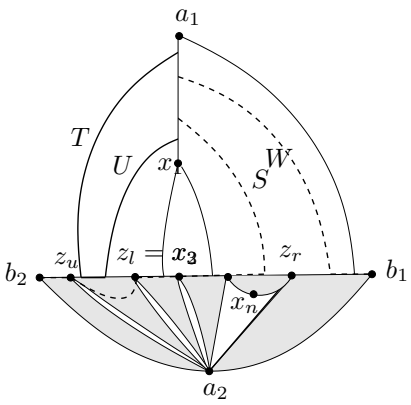


Figure 61: Where $(\Gamma, X') \in \mathcal{O}_3$

Proof of Claim 1 To see this, suppose on the contrary that there is some $x_k \in X' \backslash V(L_P)$. Let $Y = X \backslash V(L_P)$. Then by Lemma 3.5, G has a facial cycle D and a 3-separation (G_1, G_2) such that: $V(G_1 \cap G_2) = \{u, r, w\}$ with $r, w \in V(D)$; $\{a_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $Y \subseteq V(K)$; $Y \subseteq D[r, w]$. Since a_2 is not a cutvertex of L_P, it is not cofacial with x_k. So G has a separation (H, K) meeting condition (b).

There are three internally disjoint paths from x_k to $\{b_1, b_2, a_1\}$; $D[x_k, w] \cup M_P[w, b_1]$, $M_P[b_2, r] \cup D[r, x_k]$, and a path including $P[a_1, u]$, respectively. Hence we must have some $h_1, h_2 \in V(H \cap K)$ with $h_1 \in M_P[b_2, r]$ and $h_2 \in M_P[r, b_1]$, so $V(H \cap K) = \{a_1, h_1, h_2\}$. But this leaves a path from x_k to a_1 disjoint from $V(H \cap K)$, a contradiction. This proves Claim 1.

Now suppose there is some $x_k \notin V(M_P)$. Any separation (H, K) meeting condition (b) must have some $V(H \cap K) = \{h_1, h_2, a_2\}$ with $h_1 \in M_P[b_2, z_k]$ and $h_2 \in M_P[z'_k, b_1]$. But then Z_k is a 2-cut in G, which contradicts the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G. So if $x_k \notin V(M_P)$, (b) cannot be satisfied, hence x_k is cofacial with a_2.

Note that each z_k, z'_k is cofacial with a_2. Let z_l be the leftmost vertex of $\{z_2, \ldots, z_n\}$ and z_r the rightmost vertex of $\{z'_2, \ldots, z'_n\}$. Since each of z_l, z_r is cofacial with a_2, $\{z_l, z_r, a_2\}$ is a 3-cut in L_P. (See Figure 61.)

We claim that we may assume there is some path from P to $M_P(z_l, z_r)$. To see this, suppose first that $X' \nsubseteq V(M_P)$. Then there is some Z_k which is a 2-cut in L_P. Since G

is $(3, \{a_1, b_1, a_2, b_2\})$-connected, Z_k is not a 2-cut in G, and the desired path must exist. Now suppose $X' \subseteq V(M_P)$. Then $X' \subseteq V(M_P[z_l, z_r])$. If $M_P[z_l, z_r]$ is contained in a face boundary of G, then by Lemma 3.12, $(\Gamma, X) \in \mathcal{O}_3$ or (Γ, X) satisfies property (R1). So we may assume $M_P[z_l, z_r]$ is not contained in a face boundary, and again the desired path must exist. Choose a path S from $s' \in V(P)$ to $s \in M_P(z_l, z_r)$ to minimize $P[a_1, s']$. Assume by symmetry that S falls to the right of P, and is chosen to minimize $M_P[s, z_r]$, so z_r is cofacial with s'.

We may assume no vertex $v \in V(P)$ is cofacial with both z_l and z_r; otherwise there is a 4-separation demonstrating that (Γ, X) satisfies (R1). Applying this to x_1, we find $s' \neq x_1$. Applying it to s', we find there is some path U from $u' \in P(s', x_1]$ to $u \in M_P[b_2, z_l)$. Choose U to minimize $M_P[b_2, u]$. If some vertex in $M_P(b_2, u)$ shares a finite face with a_2, let z_u be the rightmost such vertex. Otherwise, let $z_u = b_2$.

Now assume no $v \in V(P)$ is cofacial with both z_u and z_r, or we again find (Γ, X) satisfies (R1). Choose paths T from $t' \in P[a_1, s')$ to $t \in M_P(z_u, u]$ and W from $w' \in P(t', s']$ to $w \in M_P(z_r, b_1)$. Let $Q := T \cup M_P[t, u] \cup U \cup P[u', x_1]$ (Figure 61). By our choice of T, there is a path from z_u to z_l in L_P, disjoint from $M_P[t, u] \cup \{a_2\}$, so $M_P[z_l, s] \cup S \cup W \cup M_P[w, b_1] \subseteq L_Q$. By our choice of S, there is a path from a_2 to x_r along the facial cycle they share, disjoint from $M_P[z_l, s] \cup S \cup W \cup M_P[w, b_1]$, so $x_r \in \mathcal{R}(Q)$, and X is feasible in Γ, a contradiction. □

We now return to using Theorem 1 inductively to complete the final cases. The proof of Lemma 5.9 will not use maximizing paths directly. The choice of a maximizing path in the conditions is included in order to apply previous lemmas.

Lemma 5.9. *Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph with outer cycle C, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| = n \geq 3$, $X \cap V(C) = \emptyset$, and X infeasible in Γ. Let $P \in \mathcal{P}_{(\Gamma, X)}$ be a maximizing path, and assume P is from $\{a_1, a_2\}$ to $x_1 \in X$. If $(\Gamma, X \setminus \{x_1\}) \in \mathcal{O}_7$, and Theorem 1 holds for vertex sets of size $n - 1$, then either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5 \cup \mathcal{O}_7$.*

Proof. Let $X' = X \setminus \{x_1\}$, so $(\Gamma, X') \in \mathcal{O}_7$, as given in Definition 2.9. We assume by symmetry, and since $X \cap V(C) = \emptyset$, that G has a 3-separation (G_1, G_2) and a cycle D

bounding a finite face of G such that the following hold

1. $V(G_1 \cap G_2) = \{u, v, w\}$, $w \in C[b_2, a_1)$, and $u, v \in V(D)$
2. $\{a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $X \cap V(G_2) \not\subseteq V(G_1)$,
3. either $w \in C(b_2, a_1)$ and $u \in C(w, a_1]$, or $w = b_2$,
4. $X \subseteq V(G_2 \cup D)$.

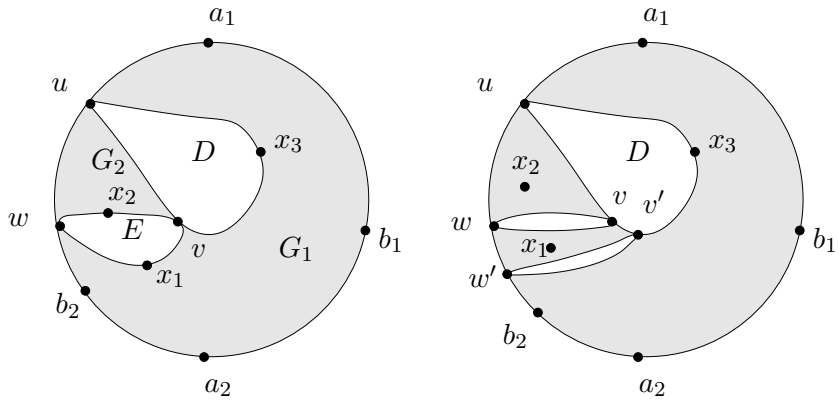
Let $G' = G_2 \cup D$. Let E be the facial cycle of G with $v, w \in V(E)$, so that $E[v, w] \subseteq G_1$ and $E[w, v] \subseteq G_2$.

Note that the subgraph G' may have at most one cutvertex, $w' \in V(G_2)$, with ww' an edge in G_2. To see this, suppose w' is a cutvertex of G'. Since D is 2-connected, we must have w' separating w from $\{u, v\}$ in G'. But since G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, $\{w, w'\}$ is not a 2-cut in G, hence ww' must be a single edge. By a similar argument, if there is a cutvertex w' of G_2, there must be an edge uw' or vw' in G_2. If there is any separation (H, K) of G' of order at most 2 with $x_3 \in V(K)$ and $x_2 \notin V(K)$, we must have either $x_2 \in V(H \cap K)$ or $|V(H \cap K) \cap V(D)| = 2$.

We assume $x_1 \notin V(G')$; otherwise $(\Gamma, X) \in \mathcal{O}_7$. We also assume there is some $x_2 \in V(G_2)$ with $x_2 \notin V(D)$; otherwise $(\Gamma, X') \in \mathcal{O}_1$, and by Lemma 5.6, (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5 \cup \mathcal{O}_7$. Since $X' \cap V(C) = \emptyset$, we have $x_2 \neq w$, so $\{u, v, w\}$ is a 3-cut in G.

We assume some $x_3 \in V(D) \setminus V(G_2)$; otherwise $(\Gamma, X') \in \mathcal{O}_2$, and by Lemma 5.7, (Γ, X) satisfies (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$. We further assume $a_i \notin V(D)$ for $i \in \{1, 2\}$; otherwise $\{a_i, v, w\}$ is a 3-cut in G, and we find $(\Gamma, X') \in \mathcal{O}_3$, and by Lemma 5.8, (Γ, X) satisfies (R1) or $(\Gamma, X) \in \mathcal{O}_3$.

Let $X_2 = X \setminus \{x_2\}$ and $X_3 = X \setminus \{x_3\}$. Since X is infeasible in Γ, we must have X_j infeasible in Γ for each $j \in \{2, 3\}$. We assume Theorem 1 holds for vertex sets of size $n - 1$, so $(\Gamma, X_j) \in \bigcup_{i=1}^{7} \mathcal{O}_i$, or (Γ, X_j) is reducible. Since $X \cap V(C) = \emptyset$, $(\Gamma, X_j) \notin \mathcal{O}_4$, and we may assume (Γ, X_j) does not satisfy (R2). If $(\Gamma, X_j) \in \mathcal{O}_5$, then by Lemma 5.1, (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$. If $(\Gamma, X_j) \in \mathcal{O}_6$, then by Lemma 5.2, either (Γ, X)



(a) $(\Gamma, X_3) \in \mathcal{O}_1$ (b) $(\Gamma, X_3) \in \mathcal{O}_2$

Figure 62: Examples of $(\Gamma, X') \in \mathcal{O}_7$

is reducible or $(\Gamma, X) \in \mathcal{O}_6$. We are left with the cases that $(\Gamma, X_j) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_7$, or satisfies (R1).

Case 1 Suppose $(\Gamma, X_3) \in \mathcal{O}_1$.

Each vertex of X_3 appears on a common facial cycle of G. In particular, x_1 is cofacial with x_2. Since $x_2 \in V(G_2) \setminus V(D)$, $x_2 \neq w$, and by symmetry if $w = b_2$, we may assume the common facial cycle is E (Figure 62(a)). Note that $X \cap (V(D) \setminus \{v\}) = \{x_3\}$.

Suppose we have 4 disjoint paths in G_1: P_1 from a_1 to x_3; O from b_1 to v; P_2 from a_2 to x_1; and $C[b_2, w]$. Then $C[b_2, w] \cup E[w, v] \cup O$ is a path from b_1 to b_2, disjoint from $P_1 \cup P_2$, and we see X is feasible in Γ. Hence no such paths exist, and we must have some separation (H, K) of G of order at most 3 with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $\{x_3, v, x_1, w\} \subseteq V(K)$. By planarity and $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, we find $X \subseteq V(K)$, and $(\Gamma, X) \in \mathcal{O}_2$.

Case 2 Suppose $(\Gamma, X_3) \in \mathcal{O}_2$.

Here there is some 3-separation (H, K) of G with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X_3 \subseteq V(K)$. We may assume $x_3 \notin V(K)$; otherwise $(\Gamma, X) \in \mathcal{O}_2$. Hence we must have $V(H \cap K) = \{u', v', w'\}$ with $u' \in D[u, x_3]$, $v' \in D[x_3, v]$, and by planarity, $w' \in C[b_2, w]$ (Figure 62(b)). So $(\Gamma, X) \in \mathcal{O}_7$.

Case 3 Suppose $(\Gamma, X_3) \in \mathcal{O}_3$.

For some $i \in \{1,2\}$, either x_2 is cofacial with a_i or there is a 3-separation (H_2, K_2) of G with $a_i \in V(H_2 \cap K_2)$, $\{a_1, b_1, a_2, b_2\} \subseteq V(H_2)$ and $x_2 \subseteq V(K_2)$. We may assume x_3 is not cofacial with a_i, and if there is a separation (H_2, K_2), then $x_3 \notin V(K_2)$; otherwise $(\Gamma, X) \in \mathcal{O}_3$.

We may assume x_2 is cofacial with a_i; otherwise, the separation (H_2, K_2) exists, and by planarity, $a_i \in V(D)$, a contradiction. Since $x_2 \notin V(D)$ and $x_2 \neq w$, we may assume $a_i, x_2 \in V(E)$. Since $x_2 \notin V(C)$, we assume E bounds a finite face. If $a_1 \in V(E)$, we have a 2-cut $\{a_1, w\}$ in G, violating $(3, \{a_1, b_1, a_2, b_2\})$-connectivity. Hence we may assume $a_2 \in V(E)$ (Figure 63(a)).

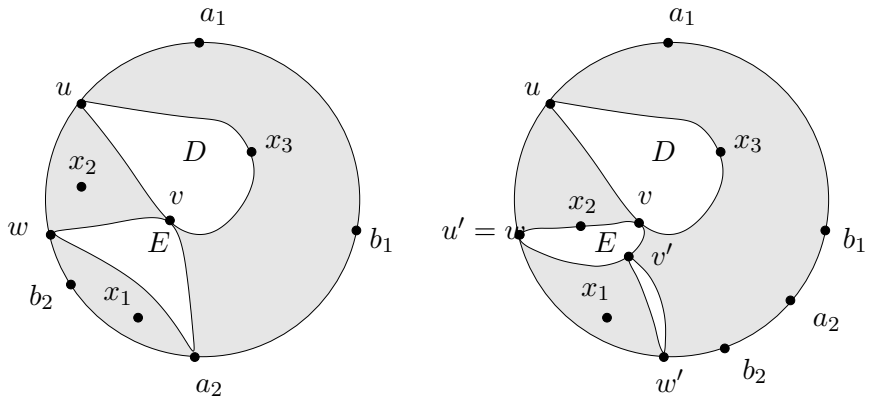
Now $\{a_2, w\}$ is a 2-cut in G, and there is a 4-separation (M, N) of G with $V(M \cap N) = \{w, a_1, b_1, a_2\}$, $\{a_1, b_1, a_2, b_2\} \subseteq V(M)$, and $X' \subseteq V(N)$. If $x_1 \in V(N)$, then (Γ, X) satisfies (R1). So we assume $x_1 \notin V(N)$.

Now $x_1 \notin V(C)$, and G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, so there is a path from a_2 to x_1 in N which is disjoint from $C[b_2, w] \cup E[w, v]$. Suppose there are disjoint paths from a_1 to x_3, and from b_1 to v in $M - a_2$. These paths, together with those identified in N, would show X feasible in Γ, a contradiction. Hence no such paths may exist, and we must have a separation (M_1, M_2) of M, of order at most 2, with $\{a_1, b_1, a_2\} \subseteq V(M_1)$ and $\{x_3, v, a_2\} \subseteq V(M_2)$. But then we have a separation (M', N') of G with $V(M', N') = V(M_1 \cap M_2) \cup \{b_2\}$, with $\{a_1, b_1, a_2, b_2\} \subseteq V(M')$ and $X \subseteq V(N')$, so (Γ, X) satisfies (R1).

Case 4 Suppose $(\Gamma, X_3) \in \mathcal{O}_7$.

There is some 3-separation (H, K) of G and finite facial cycle D' with $V(H \cap K) = \{u', v', w'\}$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H)$, $X \cap V(K) \not\subseteq V(H)$, satisfying Definition 2.9. We may assume $x_3 \notin V(K \cup D')$, or we have $(\Gamma, X) \in \mathcal{O}_7$. We further assume $X_3 \not\subseteq V(D')$ and $X_3 \not\subseteq V(K)$; otherwise we have one of $(\Gamma, X_3) \in \mathcal{O}_1$ or $(\Gamma, X_3) \in \mathcal{O}_2$, and apply the arguments of those cases above.

If $x_2 \in V(K)$, then since $x_3 \notin V(K)$, we must have $|\{u', v', w'\} \cap V(D)| \geq 2$. By planarity, this forces $u', v' \in V(D)$, so $D' = D$, and $(\Gamma, X) \in \mathcal{O}_7$.



(a) $(\Gamma, X_3) \in \mathcal{O}_3$ (b) $(\Gamma, X_3) \in \mathcal{O}_7$

Figure 63: Examples of $(\Gamma, X') \in \mathcal{O}_7$

So we may assume $x_2 \notin V(K)$, and $x_2 \in V(D')$. Hence $u', v' \in V(E)$ (Figure 63(b)). This gives $w \neq b_2$, $w' \in C[b_2, w]$, and $u' = w$. If s' shares a finite face of G with v, we have a 3-cut $\{w', v, u\}$ and cycle D demonstrating $(\Gamma, X) \in \mathcal{O}_7$. We assume this is not the case, so $v' \in E(v, w)$.

Let $N := G_2 \cup K$, so there is a separation (M, N) of G with $V(M, N) = \{u, v, v', w'\}$. Now $x_1 \notin V(C)$, and G is $(3, \{a_1, b_1, a_2, b_2\})$-connected, so there is a path from v' to x_1 in K which is disjoint from $C[w', w]$. Hence there is a path from v' to x_1 in N, disjoint from $C[w', w] \cup E[w, v]$. Suppose there are disjoint paths from a_1 to x_3, from b_1 to v, from a_2 to v', and from b_2 to w' in M. These paths, together with those identified in N, would show X feasible in Γ, a contradiction. Hence no such paths may exist, and we must have a separation (M_1, M_2) of M, of order at most 3, with $\{a_1, b_1, a_2, b_2\} \subseteq V(M_1)$ and $\{x_3, v, v', w'\} \subseteq V(M_2)$. But then $V(M_1 \cap M_2)$ is a 3-cut in G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$.

Case 5 Suppose (Γ, X_3) satisfies (R1).

There is some 4-separation (H, K) of G with $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$, $X_3 \subseteq V(K)$. We assume the separation is chosen with K maximal, so by Lemma 2.8, $V(H \cap K) \cap \{a_1, b_1, a_2, b_2\} \neq \emptyset$ or $|V(H \cap K) \cap V(C)| \geq 2$. Label $V(H \cap K) = \{h_1, h_2, h_3, h_4\}$.

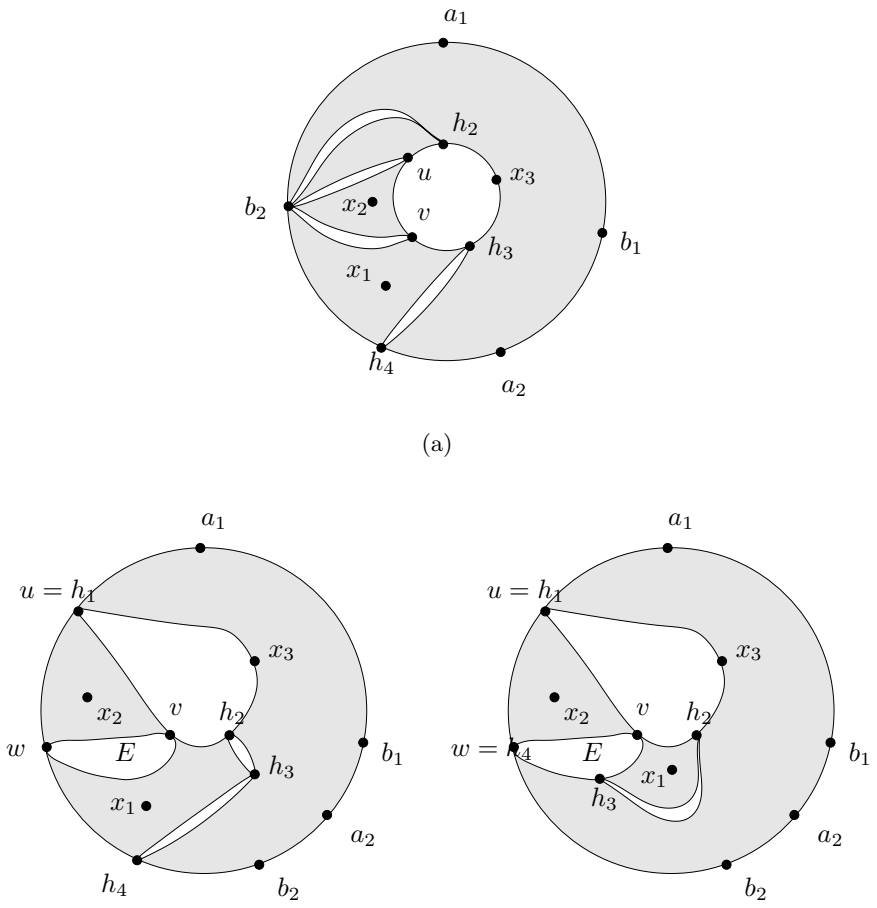


Figure 64: $(\Gamma, X') \in \mathcal{O}_7$ with (Γ, X_3) reducible

We assume $x_3 \notin V(K)$; otherwise (Γ, X) satisfies (R1). So we must have $|V(H \cap K) \cap V(D)| \geq 2$.

(5.1) Suppose $w = b_2$.

We assume by symmetry that $h_1 = b_2$, $h_2 \in D[u, x_3)$, $h_3 \in D(x_3, v]$, and $h_4 \in C[a_2, b_2] \cup E[v, b_2]$ (Figure 64(a)). We assume the 3-cut $\{b_2, h_2, v\}$ does not separate x_1 from x_3; otherwise $(\Gamma, X) \in \mathcal{O}_7$. Similarly assume $h_4 \neq b_2$.

We now have disjoint paths from b_2 to h_3 and from h_4 to x_1 in K. Suppose there are disjoint paths from a_2 to h_4, from b_1 to h_3, and from a_1 to x_3 in $H - b_2$. Together with the paths in K, these would show X feasible in Γ, a contradiction. Hence no such paths exist,

and there must be a separation (H_1, H_2) of H, of order at most 3, with $\{a_1, b_1, a_2, b_2\} \subseteq V(H_1)$ and $\{x_3, h_3, h_4, b_2\} \subseteq V(H_2)$. But then $V(H_1 \cap H_2)$ is a 3-cut in G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$.

(5.2) Suppose $w \neq b_2$.

Here we assume that $h_1 = u$, $h_2 \in D(x_3, v]$, and $h_4 \in C[b_2, w]$ (Figures 64(b) and 64(c)). Now there is a path from h_3 to x_1 in K, disjoint from $C[h_4, w] \cup E[w, v] \cup D[h_2, v]$. Suppose there are disjoint paths in H from b_2 to h_4, from a_2 to h_3, from b_1 to h_2, and from a_1 to x_3. Together with the paths in K, these show X feasible in Γ, a contradiction. Hence no such paths exist, and there must be a separation (H_1, H_2) of H, of order at most 3, with $\{a_1, b_1, a_2, b_2\} \subseteq V(H_1)$ and $\{x_3, h_2, h_3, h_4\} \subseteq V(H_2)$. But then $V(H_1 \cap H_2)$ is a 3-cut in G demonstrating that $(\Gamma, X) \in \mathcal{O}_2$.

□

5.4 Proof of Theorem 1

Let $\Gamma = (G, a_1, b_1, a_2, b_2)$ be a disk graph, and let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ with $|X| \geq 2$. We wish to prove that one of the following holds:

(C1) X is feasible in Γ,

(C2) (Γ, X) is reducible, or

(C3) $(\Gamma, X) \in \bigcup_{i=1}^{7} \mathcal{O}_i$.

We assume $|X| > 2$; otherwise apply Theorem 2. We suppose that X is infeasible in Γ, and show that (Γ, X) satisfies one of (C2) or (C3). This is done inductively, assuming that for any $Y \subseteq X$ with $Y \neq X$, (Γ, Y) satisfies one of (C2) or (C3).

Suppose first that $X \cap V(C) \neq \emptyset$. By Lemma 5.4, either (Γ, X) is reducible, or $(\Gamma, X) \in \bigcup_{i=1}^{7} \mathcal{O}_i$. So we may assume $X \cap V(C) = \emptyset$.

Recall that $\mathcal{P}$ is defined as the collection of all paths P from $a_i \in \{a_1, a_2\}$ to $x_j \in \{x_1, x_2\}$ in $G - (C[b_i, b_{3-i}] \cup \{x_{3-j}\})$ such that a_{3-i} is not a cutvertex of L_P. From Lemma 3.1, either $\mathcal{P} \neq \emptyset$ or $(\Gamma, X) \in \mathcal{O}_3$, so we may assume that there is a maximizing path $P \in \mathcal{P}$,

from a_1 to $x_1 \in X$. Define $X' = X \setminus \{x_1\}$, so (Γ, X') is reducible, or $(\Gamma, X) \in \bigcup_{i=1}^{7} \mathcal{O}_i$. Since $X' \cap V(C) = \emptyset$, $(\Gamma, X') \notin \mathcal{O}_4$ and (Γ, X') does not satisfy property (R2).

If $(\Gamma, X') \in \mathcal{O}_1$, then by Lemma 5.7 either (Γ, X) satisfies (R1) or $(\Gamma, X) \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_5$. If $(\Gamma, X') \in \mathcal{O}_2$, Lemma 5.7 gives (Γ, X) reducible or $(\Gamma, X) \in \bigcup_{i=1}^{7} \mathcal{O}_i$. If $(\Gamma, X') \in \mathcal{O}_3$, Lemma 5.8 gives (Γ, X) reducible or $(\Gamma, X) \in \mathcal{O}_3$. If $(\Gamma, X') \in \mathcal{O}_5$, we see from Lemma 5.1 that either (Γ, X) satisfies property (R1) or $(\Gamma, X) \in \mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$. If $(\Gamma, X') \in \mathcal{O}_6$, Lemma 5.2 gives (Γ, X) reducible or $(\Gamma, X) \in \mathcal{O}_6$.

We may now assume (Γ, X') satisfies (R1), and $(\Gamma, X') \notin \bigcup_{i=1}^{7} \mathcal{O}_i$. From the definition of (R1), G has some separation (H, K) of order 4 such that

(i) $\{a_1, b_1, a_2, b_2\} \subseteq V(H)$ and $X \subseteq V(K)$,

(ii) the vertices in $V(H \cap K)$ can be labeled as a'_1, b'_1, a'_2, b'_2 such that

  (a) a'_i is cofacial with b'_j for each $i, j \in \{1, 2\}$,

  (b) any four disjoint paths from $\{a_1, b_1, a_2, b_2\}$ to $\{a'_1, b'_1, a'_2, b'_2\}$ join a_i to a'_i and b_i to b'_i respectively $(i = 1, 2)$

Choose such a separation with K maximal. We may assume $x_1 \notin V(K)$; otherwise (Γ, X) satisfies (R1). We may further assume there is no 3-separation (G_1, G_2) of G with $\{a_1, b_1, a_2, b_3\} \subseteq V(G_1)$ and $K \subseteq V(G_2)$; otherwise $(\Gamma, X') \in \mathcal{O}_2$.

Case 1 Suppose $\{a_1, a_2\} \cap V(H \cap K) \neq \emptyset$.

Assume by symmetry that $a_1 = a'_1$.

Let E_1 be the facial cycle of G shared by b'_1 and a'_2, and let E_2 be the facial cycle of G shared by b'_2 and a'_2. Note by the maximality of K that each of E_1, E_2 bounds a finite face (Figure 65(b)). By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and our assumption that $(\Gamma, X') \notin \mathcal{O}_3$, we have $a_1 \notin V(E_i)$ for $i \in \{1, 2\}$. By $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G, $b'_2 \notin V(E_1)$ and $b'_1 \notin V(E_2)$.

(1.1) Suppose one of the following holds (Figure 65(a)):

(i) there is a finite face of G incident with both a_1 and x_1, or

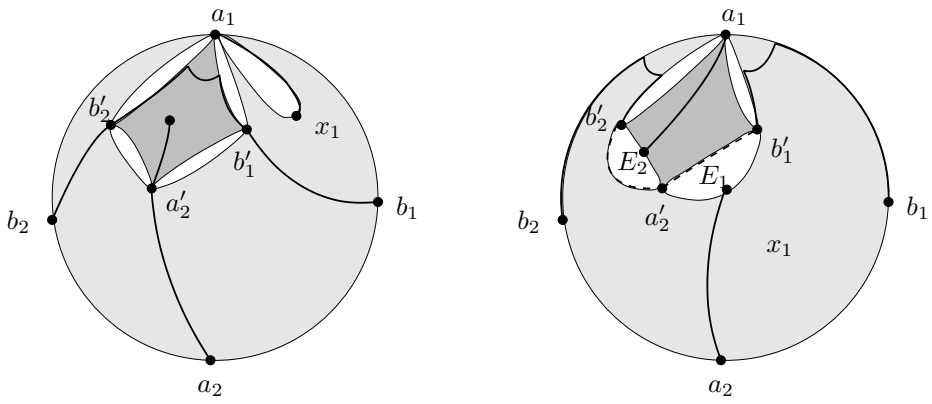


Figure 65: (Γ, X') satisfies (R1), $a'_1 = a_1$

(ii) G has a separation (H', K') of order 3 such that $a_1 \in V(H' \cap K')$, $x_1 \in V(K')$, $\{a_1, a_2, b_1, b_2\} \subseteq V(H')$.

We first claim that we may assume there are 4 disjoint paths from a_1 to x_1, b_1 to b'_1, a_2 to a'_2, and b_2 to b'_2 in G. Otherwise, there is some separation (G_1, G_2) of G of order at most 3 with $\{x_1, b_1, a_2, b_2\} \subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $V(K) \neq V(G_2)$. Letting $G'_2 = G_2 \cup \{a_1\}$, we see that (G_1, G'_2) is a separation of G. By the maximality of K, this separation (G_1, G'_2) is not of order 4. Hence it is of order at most 3, and $(\Gamma, X) \in \mathcal{O}_3$, a contradiction.

We may assume that $\{a_1, a'_2\}$ is not a 2-cut in K; otherwise, we have 3-cuts $\{a_1, a'_2, b'_1\}$ and $\{a_1, a'_2, b'_2\}$ in G demonstrating that $(\Gamma, X) \in \mathcal{O}_3$. Hence there is some path from b'_1 to b'_2 in $K - \{a_1, a_2\}$. Let Q be such a path, chosen so that each vertex of Q is cofacial with a_1. We claim that we may assume there is some $x_k \in X'$ for which there is a path Q' from a'_2 to x_k in $K - Q$. If not, then for each vertex $x \in X'$, either $x \in V(Q)$, and hence x is cofacial with a_1, or there is a 2-cut $T \subseteq V(Q)$ in $K - a_1$ separating x from $\{b'_1, a'_2, b'_2\}$. But this again gives $(\Gamma, X) \in \mathcal{O}_3$.

Now the four disjoint paths in H together with Q and Q' demonstrate that X is feasible in Γ, a contradiction.

(1.2) Suppose the conditions of (1.1) do not hold.

Since $x_1 \notin V(K)$, and by our choice of K, $x_1 \notin V(E_1) \cap V(E_2)$. If $x_1 \in V(E_1 \cup E_2)$, assume by symmetry that $x_1 \in V(E_1)$.

If there is some 2-cut T in H separating x_1 from b_1, a_2, b_2, we note by $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G and our choice of K that $T \cap V(E_1 \cup E_2) \neq \emptyset$ and $T \nsubseteq V(E_i)$ for $i \in \{1, 2\}$. In this case, we assume by symmetry that $T \cap V(E_1) \neq \emptyset$.

We claim there are disjoint paths P_1 from b_1 to b_1', Q from a_2 to x_1, and P_2 from b_2 to a_2' in $H - \{a_1, b_2'\}$. Otherwise, there is some 2-cut T' of $H - \{a_1, b_2'\}$ separating $\{b_1, a_2, b_2\}$ from $\{b_1', x_1, a_2'\}$. So either $T' \cup \{a_1\}$ is a 3-cut in G, demonstrating that $(\Gamma, X) \in \mathcal{O}_3$, or $T' \cup \{a_1, b_2'\}$ is a 4-cut in G, demonstrating that (Γ, X) satisfies (R1).

So we have the path $Q \in \mathcal{P}_{(\Gamma, X)}$ from a_2 to x_1, with $P_1 \cup P_2 \cup E_1[a_2', b_1'] \subseteq L_Q$. We may assume there is some $x_2 \in X'$ with $x_2 \notin V(E_1)$; otherwise $(\Gamma, X') \in \mathcal{O}_1$. Since X is infeasible in Γ, there is no path from a_1 to x_2 in $K - V(E_1)$. So there must be some 2-cut $T' \subseteq V(E_1)$ separating x_2 from a_2 in K. But then T' is a 2-cut in G, and this contradicts the $(3, \{a_1, b_1, a_2, b_2\})$-connectivity of G.

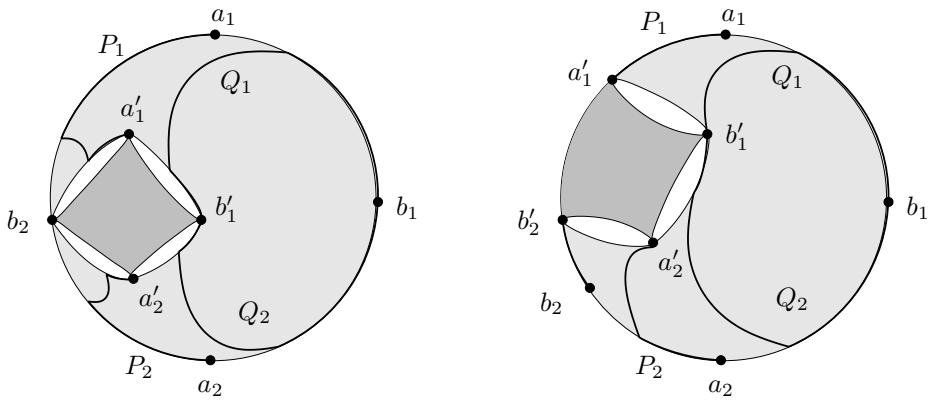
Case 2 Suppose $\{a_1, a_2\} \cap V(H \cap K) \neq \emptyset$.

We assume by symmetry that $b_2' \in C[b_2, a_1]$, and either $b_2' = b_2$ or $a_1' \in C(b_2', a_1)$. By our choice of (H, K) with K maximal, we may find the following internally disjoint paths in G: P_1 from a_1 to a_1'; P_2 from a_2 to a_2'; Q_1 and Q_2 each from b_1 to b_1'; and $C[b_2, b_2']$, as shown in Figure 66. Choose such paths so that each vertex of $P_1 \cup P_2$ is cofacial with some vertex of $C[b_2, b_2']$, each vertex of Q_1 is cofacial with some vertex of P_1, and each vertex of Q_2 is cofacial with some vertex of P_2.

(2.1) Suppose $x_1 \in V(P_i)$, for some $i \in \{1, 2\}$.

Note that if $a_1' \in V(C)$, we have $P_1 = C[a_1', a_1]$. Since $x_1 \notin V(C)$, we may assume either $i = 2$ or $a_1' \notin V(C)$.

Let E_1 be the finite facial cycle of G shared by a_i' and b_1', and let E_2 be the finite facial cycle shared by a_i' and b_2'. If $x_1 \in V(E_j)$, there is some $x_2 \in X'$ with $x_2 \notin V(E_j)$; otherwise $(\Gamma, X) \in \mathcal{O}_1$. Note that $\{b_1', b_2'\}$ does not separate a_{3-i}' from X in K; otherwise $(\Gamma, X') \in \mathcal{O}_3$.



(a) $b'_2 = b_2$ (b) $a'_1 \in C(b'_2, a_1)$

Figure 66: Disjoint paths in H when $\{a_1, a_2\} \cap V(K) = \emptyset$

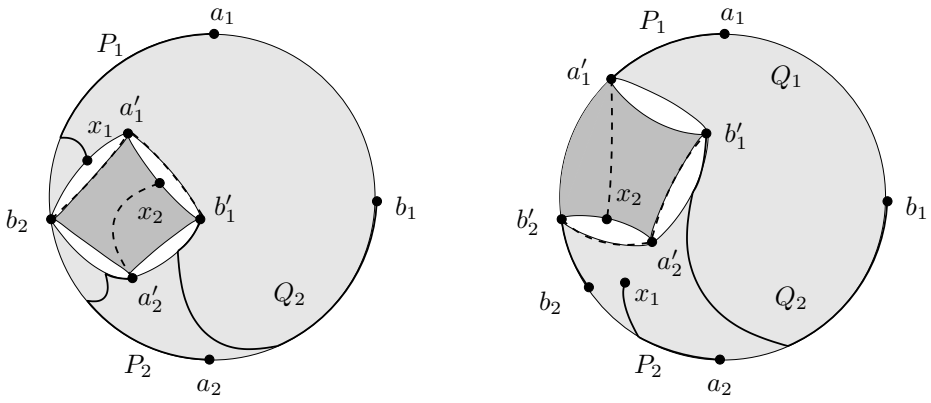
So we find disjoint paths P_3 from a'_{3-i} to x_2 and Q_3 from b'_2 to b'_1 in $K \cup E_1 \cup E_2$, each disjoint from P_i (Figure 67). Together with $P_i[a_i, x_1]$, P_{3-i}, and Q_1 (or Q_2), these show X is feasible in Γ, a contradiction.

(2.2) Suppose $x_1 \notin V(Q_{3-i})$ for some $i \in \{1, 2\}$.

We may choose a path P' from a_1 to x_1 in G, disjoint from $P_{3-j} \cup Q_{3-j} \cup C[b_2, b'_2]$.

As in the previous case, let E_1 be the finite facial cycle of G shared by a'_i and b'_1, and let E_2 be the facial cycle shared by a'_i and b'_2. If $x_1 \in V(E_j)$, there is some $x_2 \in X'$ with $x_2 \notin V(E_j)$; otherwise $(\Gamma, X) \in \mathcal{O}_1$. Note that $\{b'_1, b'_2\}$ does not separate a'_{3-i} from X in K; otherwise $(\Gamma, X') \in \mathcal{O}_3$. So we find disjoint paths P_3 from a'_{3-i} to x_2 and Q_3 from b'_2 to b'_1 in $K \cup E_1 \cup E_2$, each disjoint from P' (Figure 68). Together with P', P_{3-i}, and Q_{3-i}, these show X is feasible in Γ, a contradiction.

□



(a) $x_1 \in V(P_1)$ (b) $x_1 \in V(P_2)$

Figure 67: When $x_1 \in V(P_i)$

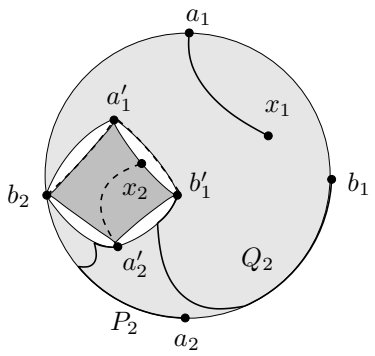


Figure 68: When $x_1 \in V(Q_1)$

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