# Limit theorems for linear random fields with innovations in the domain of attraction of a stable law

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**Abbreviated Title:** Limit theorems for linear random fields

## Abstract

In this paper we study the convergence in distribution and the local limit theorem for the partial sums of linear random fields with i.i.d. innovations that have infinite second moment and belong to the domain of attraction of a stable law with index $0 < \alpha \leq 2$ under the condition that the innovations are centered if $1 < \alpha \leq 2$ and are symmetric if $\alpha = 1$. We establish these two types of limit theorems as long as the linear random fields are well-defined, the coefficients are either absolutely summable or not absolutely summable.

**Keywords:** Linear random fields, domain of attraction of a stable law, local limit theorem, weak convergence

## 1 Introduction

For a sequence of random variables $\{\xi_i\}_{i \in \mathbb{Z}^d}$ indexed by vectors in $\mathbb{Z}^d$, many statistical procedures produce estimators of the type

$$S_n = \sum_{i \in \mathbb{Z}^d} b_{ni} \xi_i, \tag{1}$$

where $\{b_{ni}\}_{i\in\mathbb{Z}^d}$ are real numbers. To give an example, let us consider the nonlinear regression model

$$Y(x) = g(x) + \eta(x),$$

where $g(x)$ is an unknown function and $\eta(x)$ is the random noise. If we fix the design points $\{x_i\}_{i\in\Gamma_n^d}$, where $\Gamma_n^d$ is a sequence of finite regions of $\mathbb{Z}^d$, we get

$$Y_i = Y(x_i) = g(x_i) + \eta(x_i) = g(x_i) + \eta_i.$$

A nonparametric kernel estimator of $g(x)$ is defined as

$$\hat{g}_n(x) = \sum_{i\in\Gamma_n^d} \omega_{n,i}(x)Y_i \tag{2}$$

with

$$\omega_{n,i}(x) = \frac{K((x_i - x)/h_n)}{\sum_{j\in\Gamma_n^d} K((x_j - x)/h_n)},$$

where $K$ is a kernel function and $h_n$ is a sequence of bandwidths which goes to zero as $n \to \infty$. Define $g_n(x) = \sum_{i\in\Gamma_n^d} \omega_{n,i}(x)g(x_i)$; then it is clear that, $\hat{g}_n(x) - g_n(x)$ is of the type (1). Moreover, if the noise $\{\eta_i\}_{i\in\mathbb{Z}^d}$ is given by a linear random field $\eta_i = \sum_{j\in\mathbb{Z}^d} a_j\xi_{i-j}$, under the conditions that the innovations are independent and identically distributed (i.i.d.) random variables with a bounded second moment, the estimator (2) has been well-studied in the literature, see, e.g., Tran (1990), Hallin, Lu and Tran (2004), El Machkoui (2007), El Machkouri and Stoica (2010), and Beknazaryan, Sang and Xiao (2019).

As we shall see later, another important example of statistics of type (1) appears in the study of the partial sums

$$S_n = \sum_{i\in\Gamma_n^d} X_i$$

of the versatile linear random field $\{X_i\}_{i\in\mathbb{Z}^d}$, expressed as a linear combination

$$X_j = \sum_{i\in\mathbb{Z}^d} a_i\xi_{j-i} \tag{3}$$

with real coefficients $\{a_i\}_{i\in\mathbb{Z}^d}$, where $\Gamma_n^d$ is a sequence of finite regions of $\mathbb{Z}^d$.

Under the condition $\mathbb{E}\,\xi_0^2 < \infty$, the linear random fields in the form of (3) have been intensively studied in the literature. Among the last decade contributions, we would like to mention first that Mallik and Woodroofe (2011) studied the CLT for the partial sums of linear random fields over rectangular regions. Lahiri and Robinson (2016) established a CLT for the sums over dilated regions. Sang and Xiao (2018) extended the exact moderate and large deviation limit theorems studied in Peligrad et al. (2014) from linear processes to linear random fields. Beknazaryan, Sang and Xiao (2019) established the Cramér type moderate deviation for the partial sums of linear random fields. Recently, Fortune, Peligrad and Sang (2021) proved a local limit theorem under some regularity conditions for the innovations and the sampling regions. The authors of the above research considered both short range and long range dependence cases for linear random fields with $\mathbb{E}\,\xi_0^2 < \infty$. Koul, Mimoto and Surgailis (2016) studied the goodness-of-fit test for marginal distribution of linear random fields with long range dependance. For additional work in studying the limit theorems of linear random fields under the condition that the innovation has finite second moment, see, for example, the review in Lahiri and Robinson (2016) and Sang and Xiao (2019) and the references therein.

On the other hand, a rather small number of papers treat the asymptotics of linear random fields when the innovations do not have finite second moment, in particular, when the innovations belong to the domain of attraction of a stable law with index $0 < \alpha \le 2$. In the one-dimensional case, regarding the model (1), Shukri (1976) studied the stable limit theorem under some specified conditions on the weights $\{b_{ni}\}_{i\in\mathbb{Z}}$. Astrauskas (1983) studied the limit theorem and the functional limit theorem for linear processes in (3) in the case where the coefficients have some specific form. Davis and Resnick (1985) established a limit theorem for the partial sums of linear processes with $0 < \alpha < 2$ when the coefficients are at least absolutely summable. Balan, Jakubowski and Louhichi (2016) studied the functional limit theorem for linear processes

when the coefficients are absolutely summable. McElroy and Politis (2003) studied the limit theorem for the partial sums of linear random fields over one rectangle under the condition $1 < \alpha < 2$, the coefficients are summable and the width of each side of the rectangle goes to infinity.

Our paper will deal with two types of results: convergence in distribution and local limit theorems for the models (1) and (3) with innovation in the domain of attraction of a stable law with index $0 < \alpha \leq 2$. A local limit theorem is a more delicate limit result than the corresponding convergence in distribution result. More specifically, based on a convergence in distribution theorem in Shukri (1976), for the model (1) we obtain a general local limit theorem that is applicable to the situation when the innovation's characteristic function might not be integrable.

The model (3) is very difficult to study and, as far as we know, concerning the local limit theorem, there is no completed work in the literature even for linear processes (i.e. $d = 1$ in our setting). As a matter of fact, as mentioned above, the convergence in distribution has not been studied in its full generality and only partial results have been developed for the convergence in distribution. For the model (3) we provide both convergence in distribution and local limit theorem for random fields. In the case that the coefficients are not absolutely summable, the domain of summation is a union of rectangles and our condition involves the rate of convergence to infinite of the number of rectangles. Otherwise we only require that the normalizer goes to infinity.

We refer to Petrov (1975), Dolgopyat (2016) and the references therein for a review on local limit theorem of partial sums of independent random variables. It is remarkable that, in a recent paper, Fortune, Peligrad and Sang (2021) studied the local limit theorem for linear random fields in the form of (3) under the conditions that the innovations have mean zero and finite second moment.

The paper is organized as follows. We first introduce some preliminaries and notations in Section 2. To prepare for the local limit theorems we present convergence in distribution results for random fields in Section 3. Next, Section 4 contains our local limit theorems. We discuss some examples in Section 5. Section 6 is dedicated to the proofs.

# 2 Notations and preliminaries

Throughout the paper we shall use the following notations. We use $\iota$ to denote $\sqrt{-1}$, the imaginary unit. By $\varphi_X(v)$, we denote the characteristic function of the random variable $X$, i.e.,

$$\varphi_X(v) = \mathbb{E}\left(\exp(\iota v X)\right).$$

The notation $\Rightarrow$ is used for convergence in distribution and also for convergence in probability to a constant. For sequences of positive constants $a_n$ and $b_n$, the notation $a_n \ll b_n$ means that $a_n/b_n$ is bounded; by $a_n \asymp b_n$ we mean that the ratios $a_n/b_n$ are bounded from below and above by positive and finite constants. $f(r) \sim g(r)$ means that $\lim f(r)/g(r) = 1$ as $r \to \infty$ or $r \to 0$ depending on the context. We denote the cardinality of a set $\Gamma$ by $|\Gamma|$. $c_\alpha$, $C$ and $C_i, 1 \leq i \leq 4$ denote positive constants, which are independent of $x$, $t$ or $n$ and may change values from line to line. The constant $c_\alpha$ may depend on $\alpha$.

Let $\{\xi_i\}_{i \in \mathbb{Z}^d}$ be i.i.d. random variables in the domain of attraction of a stable law with index $0 < \alpha \leq 2$ and $\mathbb{E}\xi_0^2 = \infty$. They satisfy the following condition

$$\lim_{x \to \infty} \frac{\mathbb{P}(|\xi_0| > x)}{x^{-\alpha} L_1(x)} = 1, \tag{4}$$

where $L_1(x)$ is a slowly varying function at infinity. When $0 < \alpha < 2$,

$$\frac{\mathbb{P}(\xi_0 > x)}{\mathbb{P}(|\xi_0| > x)} \to c^+ \quad \text{and} \quad \frac{\mathbb{P}(\xi_0 < -x)}{\mathbb{P}(|\xi_0| > x)} \to c^- \qquad \text{as } x \to \infty. \tag{5}$$

Here $0 \leq c^+ \leq 1$ and $c^+ + c^- = 1$.

By (8.5) of Feller (1971), page 313, for $0 < \alpha \leq 2$,

$$\lim_{x \to \infty} \frac{x^2 \mathbb{P}(|\xi_0| > x)}{\mathbb{E}\left(\xi_0^2 I(|\xi_0| \leq x)\right)} = \frac{2 - \alpha}{\alpha}. \tag{6}$$

Combining the two relations (4) and (6) we obtain

$$\lim_{x\to\infty} \frac{x^{2-\alpha}L_1(x)}{\mathbb{E}\left(\xi_0^2 I(|\xi_0| \le x)\right)} = \frac{2-\alpha}{\alpha}.$$

Let

$$b = \inf\left\{x \ge 1 : \mathbb{E}\left(\xi_0^2 I(|\xi_0| \le x)\right) > 1\right\}.$$

For $0 < \alpha < 2$, we define

$$L(x) = \frac{2-\alpha}{\alpha}x^{\alpha-2}\mathbb{E}\left(\xi_0^2 I(|\xi_0| \le x)\right) \quad \text{for } x \ge b,$$

$$\text{and} \quad L(x) = \frac{2-\alpha}{\alpha}b^{\alpha-2}\mathbb{E}\left(\xi_0^2 I(|\xi_0| \le b)\right) \quad \text{for } 0 < x < b. \tag{7}$$

Also for $\alpha = 2$, we define

$$L(x) = \mathbb{E}\left(\xi_0^2 I(|\xi_0| \le x)\right) \quad \text{for } x \ge b,$$

$$\text{and} \quad L(x) = \mathbb{E}\left(\xi_0^2 I(|\xi_0| \le b)\right) \quad \text{for } 0 < x < b. \tag{8}$$

**Remark 2.1** *By this definition, for all $0 < \alpha \le 2$, the function $L(x)$ is slowly varying at infinity, the function $x^{2-\alpha}L(x)$ is non-decreasing, continuous from the right and has left-hand limits, and $\lim_{x\to\infty} x^{2-\alpha}L(x) = \infty$. In addition, $\lim_{x\to\infty} L_1(x)/L(x) = 1$, $L(x) \ge (\max\{b,x\})^{\alpha-2}(2-\alpha)/\alpha$ for $0 < \alpha < 2$ and in the case $\alpha = 2$, $L(x) \ge 1$, $\lim_{x\to\infty} L_1(x)/L(x) = 0$ and $L(x) = (1 + o(1)) \int_0^x t^{-1}L_1(t)dt$ as $x \to \infty$. See page 313 of Feller (1971) and page 83 of Ibragimov and Linnik (1971).*

For the main results of this paper, we assume the following condition:

**Condition $A$** : $\mathbb{E}\,\xi_0 = 0$ if $1 < \alpha \le 2$ and the innovation $\xi_0$ has symmetric distribution if $\alpha = 1$.

We apply the following convention throughout the paper.

**Convention:** For $x = 0$, $p > 0$, $|x|^p L(1/|x|) = 0$, because $L(y)$ is slowly varying at infinity.

Under Condition $A$, the characteristic function $\varphi_\xi(t)$ of the innovations has the form

$$\varphi_\xi(t) = \exp\left\{-c_\alpha|t|^\alpha L(1/|t|)\left(1 - \iota\beta\tau(\alpha, t)\right)\right\} \tag{9}$$

for $t$ in the neighborhood of zero, where $c_\alpha > 0$,

$$\tau(\alpha, t) = \begin{cases} \text{sgn}\,t \tan\left(\pi\alpha/2\right), & \text{if } \alpha \ne 1, \\ 0, & \text{if } \alpha = 1, \end{cases}$$

and $\beta = c^+ - c^-$ if $0 < \alpha < 2$. Notice that $\tau(\alpha, t) = 0$ if $\alpha = 2$. See, for example, Ibragimov and Linnik (1971, Theorem 2.6.5) for the case $\alpha \ne 1$, Aaronson and Denker (1998, Theorem 2) for $\alpha = 1$.

We shall introduce some conditions on the innovations for the local limit theorem. Recall that a random variable $X$ does not have a lattice distribution if and only if $|\varphi_X(t)| < 1$ for all $t \ne 0$. On the other hand, the Cramér condition means that $\limsup_{|t|\to\infty} |\varphi_X(t)| < 1$. It should be mentioned that $X$ has a non-lattice distribution whenever it satisfies the Cramér condition. The Cramér condition was also called strongly non-lattice in Stone (1965).

For comparison purpose, let us recall some known results for linear processes. For i.i.d. random variables $\{\xi_i\}_{i=1}^\infty$, which satisfy conditions (4), (5) and Condition $A$, let $\sigma_n$ be a sequence of positive numbers such that

$$n\mathbb{P}(|\xi_1| > \sigma_n x) \to x^{-\alpha} \quad \text{as } n \to \infty \quad \text{for all} \quad x > 0 \text{ and } 0 < \alpha < 2.$$

The normalizing constant $\sigma_n$ can also be defined as $\inf\{x : \mathbb{P}(|\xi_1| > x) \le n^{-1}\}$. In general, up to a constant factor $((2-\alpha)/\alpha$ if $0 < \alpha < 2$, see Feller, 1971), $\sigma_n^2$ can be chosen to satisfy

$$\frac{n}{\sigma_n^2}\mathbb{E}\,\xi_1^2 I(|\xi_1| \le \sigma_n) \to 1, \quad \text{as } n \to \infty, 0 < \alpha \le 2.$$

Then, by Feller (1971, p. 580), we have the following limit theorem

$$\sigma_n^{-1} \sum_{i=1}^{n} \xi_i \Rightarrow S \tag{10}$$

as $n \to \infty$, where $S$ is an $\alpha$-stable random variable with characteristic function

$$\mathbb{E}\left(\exp(\iota t S)\right) = \exp\left\{-c_\alpha |t|^\alpha \left(1 - \iota \beta \tau(\alpha, t)\right)\right\} \tag{11}$$

with the same $c_\alpha$, $\beta$ and $\tau(\alpha, t)$ as in (9). In the sequel, we will denote the distribution function of $S$ by $G(\cdot)$.

Under the conditions (4) and (5), for one-sided linear processes $X_j = \sum_{i=0}^{\infty} a_i \xi_{j-i}$, $S_n = \sum_{j=1}^{n} X_j$, Davis and Resnick (1985) assumed that the real coefficients satisfy

$$\sum_{i=0}^{\infty} |a_i|^\delta < \infty \quad \text{for some} \quad \delta < \alpha, \ \delta \leq 1,$$

and proved that

$$\sigma_n^{-1}\left(S_n - nC_n \sum_{i=0}^{\infty} a_i\right) \Rightarrow \left(\sum_{i=0}^{\infty} a_i\right) \tilde{S}$$

as $n \to \infty$, where $C_n = 0$ if $0 < \alpha \leq 1$ and $C_n = \mathbb{E}\left[\xi_1 I(|\xi_1| \leq \sigma_n)\right]$ if $1 < \alpha < 2$, and where $\tilde{S}$ is a stable random variable with index $\alpha$. If $\sum_{i=0}^{\infty} a_i = 0$, then $\sigma_n^{-1} S_n \Rightarrow 0$. We remark that if, in addition, Condition **A** holds, then $\tilde{S}$ is the random variable $S$ in (10).

## 3    Convergence in distribution for random fields

We mention that we can easily extend the results in Shukri (1976) to triangular arrays of random fields.

Let $\{c_{ni}\}_{i \in \mathbb{Z}^d}$ be real numbers. We shall introduce two conditions:

**Condition $(A_1)$**: $\sum_{i \in \mathbb{Z}^d} |c_{ni}|^\alpha L(1/|c_{ni}|) \to 1$ as $n \to \infty$, where $L(\cdot)$ is defined in (7) or (8)

and

**Condition $(A_2)$**: $\rho_n = \sup_i |c_{ni}| \to 0$ as $n \to \infty$.

We consider a sequence $\{S_n\}$ of random variables with the form $S_n = \sum_{i \in \mathbb{Z}^d} c_{ni} \xi_i$. This model is a specialized case of model (1) if we impose additional conditions on the coefficients $\{c_{ni}\}$ such as Condition $(A_1)$ or $(A_2)$. With a similar proof as in Theorem 1, Corollary 1 and Corollary 2 in Shukri (1976), we formulate the following general result and mention one corollary.

**Theorem 3.1** *For $0 < \alpha \leq 2$, assume that innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ are i.i.d. random variables in the domain of attraction of an $\alpha$-stable law. The innovations satisfy Condition **A** and have characteristic function (9). The coefficients $\{c_{ni}\}$ satisfy Condition $(A_2)$ and*

$$\lim_{n \to \infty} \sum_{i \in \mathbb{Z}^d, c_{ni} > 0} |c_{ni}|^\alpha L(1/|c_{ni}|) = c \tag{12}$$

$$\lim_{n \to \infty} \sum_{i \in \mathbb{Z}^d, c_{ni} < 0} |c_{ni}|^\alpha L(1/|c_{ni}|) = 1 - c, \tag{13}$$

*for some $0 \leq c \leq 1$. Then the sequence $S_n = \sum_{i \in \mathbb{Z}^d} c_{ni} \xi_i$ converges weakly to an $\alpha$-stable random variable with the form $c^{1/\alpha} S' - (1-c)^{1/\alpha} S''$, where $S'$ and $S''$ are independent $\alpha$-stable random variables that have distribution function $G(\cdot)$ and characteristic function (11).*

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Notice that the probability distribution function of the limiting random variable in Theorem 3.1 can be represented as $G\big(c^{-1/\alpha}\ \cdot\ \big) * \big(1 - G(-(1-c)^{-1/\alpha}\ \cdot)\big)$ for $0 < c < 1$; it is $G(x)$ for $c = 1$ and $1 - G(-x)$ for $c = 0$. The following is a direct consequence of Theorem 3.1.

**Corollary 3.1** *For $0 < \alpha \leq 2$, assume that innovations $\{\xi_i\}_{i\in\mathbb{Z}^d}$ are i.i.d. random variables in the domain of attraction of an $\alpha$-stable law. The innovations satisfy Condition **A** and have characteristic function (9). The coefficients $\{c_{ni}\}$ satisfy Conditions $(A_1)$ and $(A_2)$. If either the innovations have symmetric distribution or $c_{ni} \geq 0$ for all $n$ and $i$, then $S_n$ converges weakly to a random variable that has distribution function $G(\cdot)$ and characteristic function (11).*

Next, for the sake of applications we shall first study the normalized form of these results and provide more flexible sufficient conditions on the real coefficients. These results will be especially useful for analyzing the partial sums of linear random fields $\{X_j\}_{j\in\mathbb{Z}^d}$ defined by (3), namely

$$X_j = \sum_{i\in\mathbb{Z}^d} a_i \xi_{j-i}.$$

By the three-series theorem, the linear random field in (3) converges almost surely if and only if

$$\sum_{i\in\mathbb{Z}^d} |a_i|^\alpha L(1/|a_i|) < \infty. \tag{14}$$

For the one-dimensional case $d = 1$, this statement can be found in Astrauskas (1983) and in Proposition 5.4 in Balan, Jakubowski and Louhichi (2016) for the case $0 < \alpha < 2$ and in the proof of Theorem 2.1 in Peligrad and Sang (2012) for the case $\alpha = 2$. The proof for the general $d$-dimensional linear random fields is similar. With the same argument, the linear random field of the form (1) converges almost surely if and only if for any $n$

$$\sum_{i\in\mathbb{Z}^d} |b_{ni}|^\alpha L(1/|b_{ni}|) < \infty. \tag{15}$$

For all the results in this paper, we assume condition (14) for linear random field in (3) and condition (15) for linear random field in (1).

Let us normalize the linear random fields in (1). Recall the definition of $L(x)$ in (7) and (8). For $0 < \alpha \leq 2$ define

$$B_n = \inf\left\{ x \geq 1 : \sum_{i\in\mathbb{Z}^d} (|b_{ni}|/x)^\alpha L(x/|b_{ni}|) \leq 1 \right\}. \tag{16}$$

By the definition of $L(x)$, it can be shown that $B_n$ is a sequence of positive numbers such that

$$\sum_{i\in\mathbb{Z}^d} (|b_{ni}|/B_n)^\alpha L(B_n/|b_{ni}|) = 1. \tag{17}$$

See the proof in Lemma 6.1. In the case $\alpha = 2$, and $\mathbb{E}\xi_0^2 = \infty$, Peligrad and Sang (2013) used the same definition for the normalizer $B_n$ for studying a triangular array of martingale differences.

For each $0 < \alpha \leq 2$, with $c_{ni} = b_{ni}/B_n$, Condition $(A_1)$ always holds because of (17). Under the assumption $B_n \to \infty$ and $\{b_{ni}\}$ are bounded uniformly on $n$ and $i$, Condition $(A_2)$ holds for $c_{ni} = b_{ni}/B_n$ and therefore Theorem 3.1 and Corollary 3.1 hold for linear random fields (1).

If $\limsup_n \sup_{i\in\mathbb{Z}^d} |b_{ni}| = \infty$, in Lemma 6.2, we shall provide sufficient conditions for Condition $(A_2)$, which are easier to verify. We first introduce the following notations, as in Fortune, Peligrad and Sang (2021); see also Mallik and Woodroofe (2011). For a countable collection of real numbers $\{b_i, i \in \mathbb{Z}^d\}$, where $i = (i_1, ..., i_d)$, we denote an increment in the direction $k$ by

$$\Delta_k b_{i_1,...,i_k,...,i_d} = b_{i_1,...,i_k,...,i_d} - b_{i_1,...,i_k-1,...,i_d}$$

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and their composition is denoted by $\Delta$ :

$$\Delta b_i = \Delta_1 \circ \Delta_2 \circ ... \circ \Delta_d b_i. \tag{18}$$

For a collection of sequences $\{b_{ni}, i \in \mathbb{Z}^d\}$, define $\Delta b_{ni}$ by (18) with $b_i$ replaced by $b_{ni}$ and denote

$$\Delta_n = \sup_{i \in \mathbb{Z}^d} |\Delta b_{ni}|.$$

Define

$$\rho_n = \sup_{i \in \mathbb{Z}^d} |b_{ni}|/B_n.$$

By the first inequality in Lemma 6.2, $\rho_n \to 0$ as $n \to \infty$ if $\Delta_n = o(B_n)$. Together with the above results and analysis, we have the following limit theorem for the linear model (1).

**Theorem 3.2** *For the linear model (1) with i.i.d. innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ in the domain of attraction of an $\alpha$-stable law $(0 < \alpha \leq 2)$, assume that the innovations satisfy Condition $\boldsymbol{A}$ and have characteristic function (9), the $B_n$ defined in (16) satisfies $B_n \to \infty$ as $n \to \infty$. In the case that $\limsup_{n \to \infty} \sup_{i \in \mathbb{Z}^d} |b_{ni}| = \infty$, we additionally require that $\Delta_n = o(B_n)$.*

1. *If conditions (12) and (13) hold for $c_{ni} = b_{ni}/B_n$, then $S_n/B_n$ converges weakly to an $\alpha$-stable random variable with the form $c^{1/\alpha}S' - (1-c)^{1/\alpha}S''$, where $S'$ and $S''$ are independent $\alpha$-stable random variables that have distribution function $G(\cdot)$ and characteristic function (11).*

2. *If either the innovations have symmetric distribution or the coefficients $b_{ni} \geq 0$ for all $n$ and $i$, then $S_n/B_n$ converges weakly to an $\alpha$-stable random variable with distribution function $G(\cdot)$.*

Now we study the linear random fields (3). Let $\Gamma_n^d$ be a sequence of finite subsets of $\mathbb{Z}^d$, and define the sum

$$S_n = \sum_{j \in \Gamma_n^d} X_j, \tag{19}$$

where $X_j$ is a linear random field given in (3). With the notation

$$b_{ni} = \sum_{j \in \Gamma_n^d} a_{j-i}, \tag{20}$$

the sum $S_n$ can be expressed as an infinite linear combination of the innovations as in form (1). We still have the definition of $B_n$ as in (16).

Similarly, by the analysis for linear model (1) with $c_{ni} = b_{ni}/B_n$, Condition $(A_1)$ always holds because of (17). In the case that $0 < \alpha < 1$, the existence of (3) implies that $\sum_{i \in \mathbb{Z}^d} |a_i|$ is finite and then $\{b_{ni}\}$ are bounded uniformly on $n$ and $i$. Then Condition $(A_2)$ holds for $c_{ni} = b_{ni}/B_n$ if we assume $B_n \to \infty$. Therefore with the condition $B_n \to \infty$, Theorem 3.1 and Corollary 3.1 hold for the partial sums of linear random fields (3) in (19) if $0 < \alpha < 1$ or $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$. We remark that, in order to have the convergence theorems in these two cases, except assuming $|\Gamma_n^d| \to \infty$ as $n \to \infty$ (which is due to the requirement $B_n \to \infty$), we do not impose any other restriction on the shape of the regions $\Gamma_n^d$.

If $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, we require the index sets to be of the form

$$\Gamma_n^d = \bigcup_{k=1}^{J_n} \Gamma_n^d(k), \tag{21}$$

where $\{\Gamma_n^d(k)\}_{k=1}^{J_n}$ is a pairwise disjoint family of discrete rectangles in $\mathbb{Z}^d$ and $\Gamma_n^d(k)$ has the form

$$\prod_{\ell=1}^{d} [\underline{n}_\ell(k), \overline{n}_\ell(k)] \cap \mathbb{Z}^d$$

with $\underline{n}_\ell(k) \leq \overline{n}_\ell(k)$, where $1 \leq \ell \leq d$, $1 \leq k \leq J_n$.

By the second inequality in Lemma 6.2, notice that $J_n = o(B_n^q)$ implies $\rho_n \to 0$ as $n \to \infty$. Hence if $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, under the condition $J_n = o(B_n^q)$, the generalization of Theorem 1, Corollary 1 and Corollary 2 in Shukri (1976), i.e., Theorem 3.1 and Corollary 3.1 hold.

In summary, we have the following limit theorem for the partial sums of linear random fields (3).

**Theorem 3.3** *For the linear random field (3) with i.i.d. innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ in the domain of attraction of an $\alpha$-stable law ($0 < \alpha \leq 2$), let $S_n$ be the partial sum defined in (19) and $B_n$ be defined as in (16). Assume that the innovations satisfy Condition **A** and have characteristic function (9), and $B_n \to \infty$ as $n \to \infty$. In the case that $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, we additionally require that the sets $\Gamma_n^d$ are constructed as a disjoint union of $J_n$ discrete rectangles as in (21), where $J_n = o(B_n^q)$, $1/p + 1/q = 1$, for some $p > \alpha$ if $1 \leq \alpha < 2$, and $p = 2$ if $\alpha = 2$. Otherwise no such restriction is required.*

1. *If conditions (12) and (13) hold for $c_{ni} = b_{ni}/B_n$, where $\{b_{ni}\}$ are defined by (20), then $S_n/B_n$ converges weakly to an $\alpha$-stable random variable with the form $c^{1/\alpha} S' - (1-c)^{1/\alpha} S''$, where $S'$ and $S''$ are independent $\alpha$-stable random variables that have distribution function $G(\cdot)$ and characteristic function (11).*

2. *If either the innovations have symmetric distribution or the coefficients $a_i \geq 0$, for all $i \in \mathbb{Z}^d$, then $S_n/B_n$ converges weakly to an $\alpha$-stable random variable with distribution function $G(\cdot)$.*

**Remark 3.1** *In Theorem 3.3, for $1 \leq \alpha < 2$, we can take $p > \alpha$ but arbitrarily close to $\alpha$. Then $q > 2$ can be arbitrarily close to $\alpha/(\alpha - 1)$. If $\alpha = 1$, the value of $q$ can be taken arbitrarily large.*

Let us compare our results with some available results in the literature.

In Theorem 2.1 of McElroy and Politis (2003), the authors only obtained the convergence theorem for the partial sums of linear random field over one rectangle under the conditions: $1 < \alpha < 2$, the coefficients $\{a_i\}$ are summable and $\min_i n_i \to \infty$, where $n_i$ is the size of the rectangle in the $i$-th dimension. In the present paper, we do not have such a restriction on the regions of the sums provided $0 < \alpha < 1$, or $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$. The only condition for the weak convergence is $B_n \to \infty$ as $n \to \infty$. If $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, we obtain the limit theorems for the sums over a finite number $J_n$ of non-empty pairwise mutually exclusive rectangles, as long as $J_n = o(B_n^q)$ for some $q \geq 2$ that is related to $\alpha$. For our limit results, we do not require the coefficients to be summable if the existence of the linear random field itself does not implies it, as in the case $0 < \alpha < 1$.

The result in Theorem 3.3 is new even in one-dimensional case. Davis and Resnick (1985) studied the limit theorem for the partial sums $S_n = \sum_{j=1}^n X_j$ of one-sided linear process under the condition the coefficients are summable. In Theorem 3.3, we do not require the coefficients to be summable. The sum is not necessarily of the form $S_n = \sum_{j=1}^n X_j$. If $1 \leq \alpha \leq 2$ and the coefficients are not summable, we require that $J_n = o(B_n^q)$. Otherwise, we only require that $B_n \to \infty$ as $n \to \infty$.

# 4 Local limit theorems

## 4.1 General local limit theorem

First we prove a general local limit theorem based on characteristic functions. Let $\{S_n\}$ be a sequence of random variables and $\{B_n\}$ be a sequence of positive numbers.

**Theorem 4.1** *Assume that*

$$\frac{S_n}{B_n} \Rightarrow \mathcal{L}, \ \mathcal{L} \text{ has an integrable characteristic function and } B_n \to \infty. \tag{22}$$

*In addition, suppose that for each $D > 0$*

$$\lim_{T \to \infty} \ \limsup_{n \to \infty} \int_{T < |t| \leq DB_n} \left| \mathbb{E} \exp\left( \iota t \frac{S_n}{B_n} \right) \right| dt = 0. \tag{23}$$

*Then for any continuous function $m$ on $\mathbb{R}$ with compact support,*

$$\lim_{n \to \infty} \sup_{u \in \mathbb{R}} \left| B_n \mathbb{E} m(S_n + u) - f_{\mathcal{L}}\left(-\frac{u}{B_n}\right) \int m(t) \lambda(dt) \right| = 0, \qquad (24)$$

*where $\lambda$ is the Lebesgue measure and $f_{\mathcal{L}}$ is the density of $\mathcal{L}$.*

**Remark 4.1** *Since $\mathcal{L}$ has an integrable characteristic function, by Billingsley (1995) page 371 we know that the limiting distribution has a density.*

**Remark 4.2** *By decomposing the integral in (23) into two parts, on $\{T \leq |u| \leq \delta B_n\}$ and $\{\delta B_n \leq |u| \leq D B_n\}$, and changing the variable in the second integral, we easily argue that in order to prove this theorem, it is enough to show that for each $D$ fixed there is $0 < \delta < D$ such that*

$$(D_1) \quad \lim_{T \to \infty} \lim_{n \to \infty} \sup \int_{T < |t| \leq B_n \delta} \left| \mathbb{E} \exp\left(\iota t \frac{S_n}{B_n}\right) \right| dt = 0$$

*and*

$$(D_2) \quad \lim_{n \to \infty} B_n \int_{\delta < |t| \leq D} |\mathbb{E} \exp(\iota t S_n)| dt = 0.$$

By taking $u = 0$ in (24), as in the proof of the classical Portmanteau weak convergence theorem (Theorem 25.8 in Billingsley, 1995), we obtain:

**Corollary 4.1** *Under the conditions of Theorem 4.1 we also have for all $a < b$,*

$$\lim_{n \to \infty} B_n \mathbb{P}(a < S_n \leq b) = f_{\mathcal{L}}(0)(b - a).$$

## 4.2 Local limit theorem for linear random fields

We first consider the normalized form of the general linear random field (1). This field has the form $S_n = \sum_{i \in \mathbb{Z}^d} b_{ni} \xi_i$ where $\{b_{ni}\}$ is a general sequence of constants and does not necessarily has the form (20). We define $B_n$ as in (16).

Recall that $\rho_n = \sup_{i \in \mathbb{Z}^d} |b_{ni}|/B_n$. We have the following local limit theorem for model (1).

**Theorem 4.2** *Assume that the linear random field (1) has i.i.d. innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ in the domain of attraction of a stable law with index $0 < \alpha \leq 2$, $B_n \to \infty$ and $\rho_n \to 0$ as $n \to \infty$. The innovations satisfy Condition $\mathbf{A}$ and have a non-lattice distribution. If $\limsup_{n \to \infty} \sup_{i \in \mathbb{Z}^d} |b_{ni}| = \infty$, we further assume that the innovations satisfy the Cramér condition. Otherwise these additional assumptions are not required. Then, under the conditions for each of the two cases in Theorem 3.2, for any continuous function $m$ on $\mathbb{R}$ with compact support, (24) holds with $\mathcal{L}$ being the limiting $\alpha$-stable random variable in Theorem 3.2.*

Notice that if $\limsup_{n \to \infty} \sup_{i \in \mathbb{Z}^d} |b_{ni}| < \infty$, $\rho_n \to 0$ automatically since $B_n \to \infty$. In the case that $\limsup_{n \to \infty} \sup_{i \in \mathbb{Z}^d} |b_{ni}| = \infty$, by Lemma 6.2, we can replace the condition $\rho_n \to 0$ by $\Delta_n = o(B_n)$.

As an application of Theorem 4.2, we derive the following local limit theorems for the linear random fields in (3) with the short range dependence and long range dependence, respectively.

**Theorem 4.3** *For the linear random field (3) with i.i.d. innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ in the domain of attraction of an $\alpha$-stable law $(0 < \alpha \leq 2)$, let $S_n$ be the partial sum defined in (19) and $B_n$ be defined as in (16). Assume that the innovations satisfy Condition $\mathbf{A}$ and have a non-lattice distribution, $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$ and $B_n \to \infty$ as $n \to \infty$. Then, under the conditions for each of the two cases in Theorem 3.3, for any continuous function $m$ on $\mathbb{R}$ with compact support, (24) holds with $\mathcal{L}$ being the limiting $\alpha$-stable random variable in Theorem 3.3.*

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The next theorem deals with the case when the sum of the coefficients are divergent. In the case where $0 < \alpha < 1$, the existence of the linear random fields implies $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$, a case which we have already treated it in the above theorem. Recall that $\xi$ has a non-lattice distribution whenever it satisfies the Cramér condition.

**Theorem 4.4** *For the linear random field (3) with i.i.d. innovations $\{\xi_i\}_{i \in \mathbb{Z}^d}$ in the domain of attraction of an $\alpha$-stable law ($1 \leq \alpha \leq 2$), let $S_n$ be the partial sum defined in (19) and $B_n$ be defined as in (16). Assume that the innovations satisfy Condition **A** and the Cramér condition, $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$ and $B_n \to \infty$ as $n \to \infty$. The sum $S_n$ is over $J_n$ non-empty pairwise mutually exclusive rectangles in $\mathbb{Z}^d$ with $J_n = o(B_n^q)$, $1/p + 1/q = 1$, for some $p > \alpha$ if $1 \leq \alpha < 2$, and $p = 2$ of $\alpha = 2$. Then, under the conditions for each of the two cases in Theorem 3.3, for any function $m$ on $\mathbb{R}$, which is continuous and has compact support, (24) holds with $\mathcal{L}$ being the limiting $\alpha$-stable random variable in Theorem 3.3.*

**Remark 4.3** *The two theorems give us local limit theorems for linear random fields with innovations in the domain of attraction of a stable law under some regularity conditions for the innovations and the coefficients as long as the linear random fields are well defined. In particular, if $1 \leq \alpha \leq 2$ and the coefficients are not absolutely summable, the local limit theorem holds for the sums over a sequence of regions $\Gamma_n^d$ which is a disjoint union of discrete rectangles. In application it allows us to have disjoint discrete rectangles as spatial sampling regions, and the number of these disjoint rectangles may increase as the sample size increases. In particular we may have $(\prod_{k=1}^d [\underline{n}_k, \overline{n}_k]) \cap \mathbb{Z}^d$ where $\underline{n}_k = \overline{n}_k$ for some k's. So we may have a single point region if the equality holds for all k's. The local limit theorem is new also for $d = 1$. Furthermore, we have the freedom to take the sum over $J_n$ blocks of random variables as long as it satisfies the condition $J_n = o(B_n^q)$ for $q \geq 2$.*

In fact, with the method of proof as in Theorem 4.2 and Theorem 4.3, we can improve the local limit theorem Theorem 2.2 in Fortune, Peligrad and Sang (2021) for linear random fields when the innovations have finite variance. Condition (9) there can be improved to $J_n = o(B_n^2)$. For completeness, we state the local limit theorem in this case below.

**Theorem 4.5** *For the linear random field defined in (3), assume that the innovations $\xi_i$ are i.i.d. random variables with mean zero ($\mathbb{E}\,\xi_i = 0$), finite variance ($\mathbb{E}\,\xi_i^2 = \sigma_\xi^2$), and non-lattice distribution and the collection $\{a_i : i \in \mathbb{Z}^d\}$ of real coefficients satisfies*

$$\sum_{i \in \mathbb{Z}^d} a_i^2 < \infty.$$

*Let $S_n$ be defined as in (19) or its equivalent form (1) and assume that $B_n = \mathrm{Var}(S_n) = \sigma_\xi^2 \sum_{i \in \mathbb{Z}^d} b_{ni}^2 \to \infty$. In the case $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, the field has long range dependence, we additionally assume that the innovations satisfy the Cramér condition and that the sets $\Gamma_n^d$ are constructed as a disjoint union of $J_n$ discrete rectangles with $J_n = o(B_n^2)$. Under these conditions, for any function $m$ on $\mathbb{R}$ which is continuous and has compact support,*

$$\lim_{n \to \infty} \sup_{u \in \mathbb{R}} \left| \sqrt{2\pi} B_n \mathbb{E}\, m(S_n + u) - [\exp(-u^2/2B_n^2)] \int m(x)\lambda(dx) \right| = 0,$$

*where $\lambda$ is the Lebesgue measure.*

## 5   Examples

In this section, we provide some examples to illustrate applications of the main results of this paper.

**Example 5.1** (*Doubly geometric spatial autoregressive models*) Suppose that $d = 2$ and $\{\xi_{i_1,i_2}, (i_1, i_2) \in \mathbb{Z}^2\}$ are i.i.d. random variables in the domain of attraction of a stable law with index $0 < \alpha \leq 2$. For any $i = (i_1, i_2) \in \mathbb{Z}^2$, let

$$X_i = X_{i_1,i_2} = \theta X_{i_1-1,i_2} + \rho X_{i_1,i_2-1} - \theta\rho X_{i_1-1,i_2-1} + \xi_{i_1,i_2}, \tag{25}$$

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where $|\theta| < 1$ and $|\rho| < 1$. By using the back-shift operators $\mathbf{B}_1$ and $\mathbf{B}_2$ respectively in the horizontal and vertical directions defined by

$$\mathbf{B}_1 \xi_{i_1,i_2} = \xi_{i_1-1,i_2}, \quad \mathbf{B}_2 \xi_{i_1,i_2} = \xi_{i_1,i_2-1},$$

we have

$$X_{i_1,i_2} = (1 - \theta\mathbf{B}_1)^{-1}(1 - \rho\mathbf{B}_2)^{-1}\xi_{i_1,i_2}$$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \theta^{j_1} \rho^{j_2} \xi_{i_1-j_1,i_2-j_2}.$$

Thus, $\{X_i, i \in \mathbb{Z}^2\}$ is a linear random field with coefficients $a_{j_1,j_2} = \theta^{j_1}\rho^{j_2}$ $(j_1, j_2 \geq 0)$ which are summable. When the innovations have finite second moments, this model was introduced by Martin (1979) as an analogue in the spatial setting to the classical AR(1) time series. For more information on this and more general stationary spatial MA and ARMA models with second moments and their statistical inference one can refer to Tjøstheim (1978, 1983), Guyon (1995), Gaetan and Guyon (2010), Beran et al. (2013).

Extension of the spatial model (25) to the setting of heavy-tailed innovations can be justified by the convergence criterion as in (14).

For any integer $n \geq 1$, let $\Gamma_n^2 = [0, n]^2 \cap \mathbb{Z}^2$. Then for every $i = (i_1, i_2) \in \mathbb{Z}^2$,

$$b_{ni} = \sum_{(j_1,j_2)\in\Gamma_n^2} a_{j_1-i_1,j_2-i_2}$$

$$= \begin{cases} \sum_{j_1=i_1\vee 0}^{n} \sum_{j_2=i_2\vee 0}^{n} \theta^{j_1-i_1}\rho^{j_2-i_2}, & \text{if } i_1 \leq n, i_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{\theta^{(i_1\vee 0)-i_1}(1-\theta^{n+1})}{1-\theta} \frac{\rho^{(i_2\vee 0)-i_2}(1-\rho^{n+1})}{1-\rho}, & \text{if } i_1 \leq n, i_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

In the above, $i_1 \vee 0 = \max\{i_1, 0\}$. Obviously, the coefficients $\{b_{ni}\}$ are uniformly bounded in $n$ and $i = (i_1, i_2)$. To determine the size of $B_n$, we assume $L(\cdot) \equiv 1$ for simplicity. Then

$$B_n = \left( \sum_{i\in\mathbb{Z}^2} |b_{ni}|^\alpha \right)^{1/\alpha}$$

$$= \frac{1}{(1-\theta)(1-\rho)} \left( \sum_{i_1=-\infty}^{n} \sum_{i_2=-\infty}^{n} \left[\theta^{(i_1\vee 0)-i_1}(1-\theta^{n+1})\right]^\alpha \left[\rho^{(i_2\vee 0)-i_2}(1-\rho^{n+1})\right]^\alpha \right)^{1/\alpha}.$$

We break the summation $\sum_{i_1=-\infty}^{n}$ into $\sum_{i_1=-\infty}^{-1} + \sum_{i_1=0}^{n}$. Notice that

$$\sum_{i_1=-\infty}^{-1} \left[\theta^{(i_1\vee 0)-i_1}(1-\theta^{n+1})\right]^\alpha = \frac{\theta^\alpha}{1-\theta^\alpha}(1-\theta^{n+1})^\alpha$$

and the second sum is

$$\sum_{i_1=0}^{n} (1-\theta^{n+1})^\alpha = (n+1)(1-\theta^{n+1})^\alpha.$$

Hence, we derive $B_n \sim (1-\theta)^{-1}(1-\rho)^{-1}n^{2/\alpha}$ as $n \to \infty$.

**Example 5.2** (*Spatial fractional ARIMA models*) For $d = 2$ and constants $\beta_1, \beta_2 \in (0, 1)$, as a special case of the following spatial fractional ARIMA model:

$$\Phi(\mathbf{B}_1, \mathbf{B}_2)X_i = \Psi(\mathbf{B}_1, \mathbf{B}_2)(1 - \mathbf{B}_1)^{-\beta_1}(1 - \mathbf{B}_2)^{-\beta_2}\xi_i, \ i = (i_1, i_2) \in \mathbb{Z}^2, \tag{26}$$

where $\Phi$ and $\Psi$ are polynomials of two variables, we take $\Phi \equiv 1$ and $\Psi \equiv 1$. Then

$$X_i = X_{i_1,i_2} = \left( \sum_{j_1=0}^{\infty} \frac{\Gamma(j_1+\beta_1)}{\Gamma(\beta_1)j_1!} \right)\left( \sum_{j_2=0}^{\infty} \frac{\Gamma(j_2+\beta_2)}{\Gamma(\beta_2)j_2!} \right)\xi_{i_1,i_2}$$

$$= \sum_{(j_1,j_2)\in\mathbb{Z}_+^2} a_{j_1,j_2}\xi_{i_1-j_1,i_2-j_2}, \tag{27}$$

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where $\Gamma(\cdot)$ is the Gamma function, $\mathbb{Z}_+^2$ is the set of pairs of non-negative integers and the coefficients

$$a_{j_1,j_2} = \frac{\Gamma(j_1+\beta_1)\Gamma(j_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)j_1!j_2!}, \quad (j_1,j_2) \in \mathbb{Z}_+^2.$$

It follows from Stirling's formula that $a_{j_1,j_2}$ satisfy

$$a_{j_1,j_2} \sim Cj_1^{\beta_1-1}j_2^{\beta_2-1} \quad \text{as } j_1, j_2 \to \infty.$$

The linear random field $\{X_i, i \in \mathbb{Z}^2\}$ in (26) is an extension of of the fractional ARIMA time series with i.i.d. stable innovations considered by Kokoszka and Taqqu (1995) and Kokoszka (1996). A similar time series with finite second moment was considered by Dedecker, Merlevède and Peligrad (2011), among others. One of the important properties of fractional ARIMA time series is long-range dependence; see, e.g., Beran, et al. (2013) for more information. However, studies of fractional ARIMA random fields have not been well developed.

In order for (27) to be well defined, we assume $\beta_1, \beta_2 \in (0,1)$ satisfy

$$(1-\beta_k)\alpha > 1, \text{ for } k = 1,2,$$

which also requires $\alpha \in (1,2]$.

For any integer $n \geq 1$, we consider the index set $\Gamma_n^2 = [0,n]^2 \cap \mathbb{Z}^2$. Then, under the conditions stated above, for every $i \in \mathbb{Z}^2$,

$$\begin{aligned}
b_{ni} &= \sum_{(j_1,j_2)\in\Gamma_n^2} a_{j_1-i_1,j_2-i_2} \\
&\sim \begin{cases} C\sum_{i_1\vee 0\leq j_1\leq n, i_2\vee 0\leq j_2\leq n}(j_1-i_1)^{\beta_1-1}(j_2-i_2)^{\beta_2-1}, & \text{if } i_1\leq n, i_2\leq n, \\ 0, & \text{otherwise} \end{cases} \\
&\sim \begin{cases} \frac{C}{\beta_1\beta_2}\big[(n-i_1)^{\beta_1}-(-i_1)_+^{\beta_1}\big]\big[(n-i_2)^{\beta_2}-(-i_2)_+^{\beta_2}\big], & \text{if } i_1\leq n, i_2\leq n, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

as $n \to \infty$. In the above, $(-i_1)_+ = 0$ if $i_1 \geq 0$ and $(-i_1)_+ = |i_1|$ if $i_1 < 0$.

Furthermore, under the additional assumption $L(\cdot) \equiv 1$, we can verify that

$$\begin{aligned}
B_n &= \left(\sum_{i\in\mathbb{Z}^2}|b_{ni}|^\alpha\right)^{1/\alpha} \\
&\sim \frac{C}{\beta_1\beta_2}\left(\sum_{i_1\leq n}\sum_{i_2\leq n}\big[(n-i_1)^{\beta_1}-(-i_1)_+^{\beta_1}\big]^\alpha\big[(n-i_2)^{\beta_2}-(-i_2)_+^{\beta_2}\big]^\alpha\right)^{1/\alpha} \\
&\sim C_0\, n^{\beta_1+\beta_2+\frac{2}{\alpha}},
\end{aligned}$$

as $n \to \infty$, where the constant $C_0$ is given by

$$C_0 = \frac{C}{\beta_1\beta_2}\left(\int_{-\infty}^1\big[(1-x)^{\beta_1}-(-x)_+^{\beta_1}\big]^\alpha dx \cdot \int_{-\infty}^1\big[(1-x)^{\beta_2}-(-x)_+^{\beta_2}\big]^\alpha dx\right)^{1/\alpha},$$

where $x_+ = \max\{x,0\}$ for all $x \in \mathbb{R}$.

The models in the first two examples satisfy the conditions for the main results in Section 3 and 4 and then we have the convergence in distribution and local limit theorem. It is also clear that the two models can be readily extended to the case of $d > 2$.

**Example 5.3** (*Linear random fields with isotropic coefficients*) For any $i \in \mathbb{Z}^d$, let $a_i = \|i\|^{-\beta}$ where $\beta > 0$ is a constant and $\|\cdot\|$ denotes the Euclidean norm. Obviously, we have that

(i) $\beta \leq d \Longleftrightarrow \sum_{i \in \mathbb{Z}^d} |a_i| = \infty$,

(ii) $\alpha\beta > d \Longrightarrow \sum_{i \in \mathbb{Z}^d} |a_i|^\alpha L(1/|a_i|) < \infty$,

and $b_{ni} = \sum_{j \in \Gamma_n^d} a_{j-i} = \sum_{j \in \Gamma_n^d} \|j - i\|^{-\beta}$.

In order to determine $B_n$, we assume $L(\cdot) \equiv 1$ (for simplicity) and

$$\Gamma_n^d = \big\{ k = (k_1, \dots, k_d) \in \mathbb{Z}^d : |k_l| \leq c_l n, \, l = 1, 2, \dots, d \big\},$$

where $c_l > 0$ $(l = 1, \dots, d)$ are constants. Denote by $I = \prod_{l=1}^d [-c_l, c_l]$ the corresponding rectangle. Then we have

$$
\begin{aligned}
B_n &= \left( \sum_{i \in \mathbb{Z}^d} |b_{ni}|^\alpha \right)^{1/\alpha} = \left( \sum_{i \in \mathbb{Z}^d} \Big| \sum_{j \in \Gamma_n^d} \|j - i\|^{-\beta} \Big|^\alpha \right)^{1/\alpha} \\
&= n^{d-\beta} \cdot \left( \sum_{i \in \mathbb{Z}^d} \Big| \sum_{j/n \in I} \|j/n - i/n\|^{-\beta} \cdot n^{-d} \Big|^\alpha \cdot n^{-d} \cdot n^d \right)^{1/\alpha} \\
&\sim n^{d-\beta+\frac{d}{\alpha}} \cdot \left( \int_{\mathbb{R}^d} \Big| \int_I \|x - y\|^{-\beta} dx \Big|^\alpha dy \right)^{1/\alpha}.
\end{aligned}
\tag{28}
$$

By splitting the integral in $y$ over $J = \prod_{l=1}^d [-2c_l, \, 2c_l]$ and $\mathbb{R}^d \backslash J$, we can verify that

$$\int_{\mathbb{R}^d} \Big| \int_I \|x - y\|^{-\beta} dx \Big|^\alpha dy < \infty \Longleftrightarrow \beta < d \ \text{ and } \ \alpha\beta > d.$$

Therefore, when $\beta < d < \alpha\beta$, we have $B_n \sim C_1 n^{(1+\alpha^{-1})d-\beta}$ which shows that the effect of "long memory" index $\beta$ on the normalizing constants $B_n$. It follows from the first two lines in (28) that $\sup_i |b_{ni}| \sim C_2 n^{d-\beta}$.

**Example 5.4** (*Linear random fields with anisotropic coefficients*) For any $i \in \mathbb{Z}^d$, let $a_i = \big( \sum_{l=1}^d |i_l|^{\beta_l} \big)^{-\gamma}$, where $\beta_1, \dots, \beta_d$ and $\gamma$ are positive constants. An important feature of this model is that, if $\|i\| \to \infty$ along the $l$th direction of $\mathbb{Z}^d$, then the rate of $a_i \to 0$ is $|i_l|^{-\beta_l \gamma}$. This class of linear random fields is an extension of the model in Damarackas and Paulauskas (2017), where linear random fields with symmetric $\alpha$-stable innovations were considered.

It is elementary to verify the following statements:

$$\sum_{i \in \mathbb{Z}^d} |a_i| = \infty \Longleftrightarrow \int_{[1,\infty)^d} \Big( \sum_{l=1}^d |x_l|^{\beta_l} \Big)^{-\gamma} dx_1 \dots dx_d = \infty \Longleftrightarrow \gamma \leq \sum_{l=1}^d \beta_l^{-1}$$

and

$$\alpha\gamma > \sum_{l=1}^d \beta_l^{-1} \Longrightarrow \sum_{i \in \mathbb{Z}^d} \Big( \sum_{l=1}^d |i_l|^{\beta_l} \Big)^{-\alpha\gamma} L\Big( \big( \sum_{l=1}^d |i_l|^{\beta_l} \big)^\gamma \Big) < \infty.$$

In the following, we consider a family of index sets $\Gamma_n^d$ which are formed by rectangular lattice points that matches the anisotropy in $\{a_i, i \in \mathbb{Z}^d\}$ and determine the rates of $B_n \to \infty$. For simplicity we assume again that $L(\cdot) \equiv 1$.

For any integer $n \geq 1$, let $\Gamma_n^d \subset \mathbb{Z}^d$ be the set of rectangular lattice points defined by

$$\Gamma_n^d = \Big\{ k \in \mathbb{Z}^d : |k_l| \leq n^{1/\beta_l} \text{ for } l = 1, \dots, d \Big\}.$$

Notice that $|\Gamma_n^d| \asymp \prod_{l=1}^d n^{1/\beta_l} = n^Q$, where $Q = \sum_{l=1}^d \beta_l^{-1}$. Moreover, the volume of a rectangle of side lengths $n^{-1/\beta_l}$ $(l = 1, \dots, d)$ is $n^{-Q}$.

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Now we determine the rate of $B_n \to \infty$. Since

$$b_{ni} = \sum_{j \in \Gamma_n^d} a_{j-i} = \sum_{j \in \Gamma_n^d} \left( \sum_{l=1}^{d} |j_l - i_l|^{\beta_l} \right)^{-\gamma},$$

we have

$$
\begin{aligned}
B_n &= \left( \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \Gamma_n^d} \left( \sum_{l=1}^{d} |j_l - i_l|^{\beta_l} \right)^{-\gamma} \right|^\alpha \right)^{1/\alpha} \\
&= \left( \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \Gamma_n^d} \left( \sum_{l=1}^{d} \left| \frac{j_l}{n^{1/\beta_l}} - \frac{i_l}{n^{1/\beta_l}} \right|^{\beta_l} \right)^{-\gamma} \cdot n^{-\gamma} \cdot n^{-Q} \cdot n^Q \right|^\alpha \right)^{1/\alpha} \\
&= n^{Q-\gamma+\frac{Q}{\alpha}} \cdot \left( \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \Gamma_n^d} \left( \sum_{l=1}^{d} \left| \frac{j_l}{n^{1/\beta_l}} - \frac{i_l}{n^{1/\beta_l}} \right|^{\beta_l} \right)^{-\gamma} \cdot n^{-Q} \right|^\alpha \cdot n^{-Q} \right)^{1/\alpha} \\
&\sim n^{Q-\gamma+\frac{Q}{\alpha}} \left( \int_{\mathbb{R}^d} \left| \int_I \left( \sum_{l=1}^{d} |x_l - y_l|^{\beta_l} \right)^{-\gamma} dx \right|^\alpha dy \right)^{1/\alpha}
\end{aligned}
\tag{29}
$$

as $n \to \infty$, in the last line, $I = [-1, 1]^d$. For the last integral in (29), it can be verified that

$$\int_{\mathbb{R}^d} \left| \int_I \left( \sum_{l=1}^{d} |x_l - y_l|^{\beta_l} \right)^{-\gamma} dx \right|^\alpha dy < \infty \iff \alpha\gamma > \sum_{l=1}^{d} \beta_l^{-1} > \gamma.$$

Hence, under the condition $\alpha\gamma > \sum_{l=1}^{d} \beta_l^{-1} > \gamma$, we have $B_n \sim C_3 n^{(1+\alpha^{-1})Q-\gamma}$ which also shows that the effect of "long memory" on the normalizing constants $B_n$. From the first two lines in (29), we can derive that $\sup_i |b_{ni}| \sim C_4 n^{Q-\gamma}$.

# 6 Proofs

**Proof of Theorem 3.1.**

The proof is same as the proof of Theorem 1 in Shukri (1976). For readers convenience and completeness, we provide the proof below. We can select $\theta$ with $|\theta| < \pi\alpha/2$ $(0 < \alpha < 1)$, $\theta = 0$ $(\alpha = 1)$ and $|\theta| < (2-\pi)\alpha/2$ $(1 < \alpha \le 2)$ and adjust the constant $c_\alpha > 0$, then rewrite the characteristic function in (9) as

$$\varphi_\xi(t) = \exp\left\{ -c_\alpha |t|^\alpha L(1/|t|) \exp((\operatorname{sgn} t)\iota\theta) \right\}$$

for $t$ in the neighborhood of zero. This is because $|\beta| \le 1$ and

$$\exp((\operatorname{sgn} t)\iota\theta) = \cos\theta(1 + \iota\tan\theta\operatorname{sgn} t).$$

Then the characteristic function of $S_n$ can be written as

$$
\begin{aligned}
\varphi_{S_n}(t) &= \prod_{i \in \mathbb{Z}^d} \varphi_\xi(t c_{ni}) \\
&= \exp\left\{ -c_\alpha |t|^\alpha \sum_{i \in \mathbb{Z}^d} |c_{ni}|^\alpha L(1/|t c_{ni}|) \exp((\operatorname{sgn} t)(\operatorname{sgn} c_{ni})\iota\theta) \right\} \\
&= \exp\left\{ -c_\alpha |t|^\alpha \sum_{i \in \mathbb{Z}^d} |c_{ni}|^\alpha L(1/|c_{ni}|) \frac{L(1/|t c_{ni}|)}{L(1/|c_{ni}|)} \exp((\operatorname{sgn} t)(\operatorname{sgn} c_{ni})\iota\theta) \right\}.
\end{aligned}
$$

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By (12), (13), Condition $(A_2)$ and the definition of slowly varying function at infinity, for any $0 < \varepsilon < 1$, we can find $N \in \mathbb{N}$, such that for all $n > N$, $c(1-\varepsilon) < \sum_{i \in \mathbb{Z}^d, c_{ni} > 0} |c_{ni}|^\alpha L(1/|c_{ni}|) < c(1+\varepsilon)$, $(1-c)(1-\varepsilon) < \sum_{i \in \mathbb{Z}^d, c_{ni} < 0} |c_{ni}|^\alpha L(1/|c_{ni}|) < (1-c)(1+\varepsilon)$, and for all $n > N$ and all $i$, we have $1 - \varepsilon < L(1/|tc_{ni}|)/L(1/|c_{ni}|) < 1 + \varepsilon$. Since $0 < \varepsilon < 1$ is arbitrary, we have

$$\lim_{n \to \infty} \varphi_{S_n}(t) = \exp\left\{-c_\alpha c |t|^\alpha \exp((\mathrm{sgn}t)\iota\theta)\right\} \exp\left\{-c_\alpha(1-c)|t|^\alpha \exp(-(\mathrm{sgn}t)\iota\theta)\right\},$$

and then

$$S_n \Rightarrow c^{1/\alpha} S' - (1-c)^{1/\alpha} S'',$$

where $S'$ and $S''$ are independent $\alpha$-stable random variables that have the same distribution as $S$ in (10) and the characteristic function as in (11). ∎

**Lemma 6.1** *The quantity $B_n$ defined in (16) satisfies (17).*

**Proof.** By the definition of $B_n$, for $0 < \varepsilon < 1$

$$\sum_{i \in \mathbb{Z}^d} (|b_{ni}|/(B_n - \varepsilon))^\alpha L((B_n - \varepsilon)/|b_{ni}|) > 1,$$

and because $x^{2-\alpha} L(x)$ is increasing

$$\sum_{i \in \mathbb{Z}^d} (|b_{ni}|/(B_n - \varepsilon))^\alpha L((B_n - \varepsilon)/|b_{ni}|)$$
$$= \sum_{i \in \mathbb{Z}^d} (|b_{ni}|/(B_n - \varepsilon))^2 \left[((B_n - \varepsilon)/|b_{ni}|)^{2-\alpha} L((B_n - \varepsilon)/|b_{ni}|)\right]$$
$$\leq \sum_{i \in \mathbb{Z}^d} (|b_{ni}|/(B_n - \varepsilon))^2 (B_n/|b_{ni}|)^{2-\alpha} L(B_n/|b_{ni}|)$$
$$= (B_n - \varepsilon)^{-2} \sum_{i \in \mathbb{Z}^d} (|b_{ni}|)^2 (B_n/|b_{ni}|)^{2-\alpha} L(B_n/|b_{ni}|).$$

By letting $\varepsilon \to 0$ we obtain

$$\sum_{i \in \mathbb{Z}^d} (|b_{ni}|/B_n)^\alpha L(B_n/|b_{ni}|) \geq 1.$$

On the other hand, by the definition of $B_n$ there is a sequence $x_m = x_m(n)$, which is decreasing to $B_n$ such that

$$\sum_{i \in \mathbb{Z}^d} (|b_{ni}|/x_m)^\alpha L(x_m/|b_{ni}|) \leq 1.$$

By the Fatou's lemma and the right continuity of $L$, by passing with $m \to \infty$

$$\sum_{i \in \mathbb{Z}^d} (|b_{ni}|/B_n)^\alpha L(B_n/|b_{ni}|) \leq \liminf_{m \to \infty} \sum_{i \in \mathbb{Z}^d} (|b_{ni}|/x_m)^\alpha L(x_m/|b_{ni}|) \leq 1.$$

This proves (17). ∎

In the stable case, we have the following lemma paralleling to Proposition 2 of Mallik and Woodroofe (2011). It is required in the proof of Theorem 3.2 and Theorem 3.3.

**Lemma 6.2** *For any $p > \max\{1, \alpha\}$ if $0 < \alpha < 2$, and $p = 2$ if $\alpha = 2$ and $\mathbb{E}\xi_0^2 = \infty$, take $q$ with $1/p + 1/q = 1$. For the linear random field in the form of (1), we have*

$$\rho_n \ll 2^d[(\Delta_n/B_n)^{1/(dp+1)} + \Delta_n/B_n]. \tag{30}$$

*For the linear random field in the form of (3), if the sum is over $\Gamma_n^d$, which has the form (21), then*

$$\Delta_n \ll 2^d J_n^{1/q} \tag{31}$$

*and hence*

$$\rho_n \ll 2^d (J_n^{1/q}/B_n)^{1/(dp+1)} + 2^{2d} J_n^{1/q}/B_n. \tag{32}$$

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**Proof.** For the ease on notation, we only prove the results for the case $d = 2$. In this case, by (18),

$$\Delta b_{i,j} = b_{i,j} - b_{i,j-1} - b_{i-1,j} + b_{i-1,j-1}.$$

Suppose that $\sup_{i \in \mathbb{Z}^2} |b_{ni}| = \sup_{r,s \in \mathbb{Z}} |b_{r,s}|$ occurs at $|b_{r_0,s_0}|$. Following the line in Mallik and Woodroofe (2011), we have

$$b_{r_0+r,s_0+s} - b_{r_0,s_0+s} - b_{r_0+r,s_0} + b_{r_0,s_0} = \sum_{u=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} \Delta b_{u,v} \tag{33}$$

for $r, s \geq 1$. For $m \geq 1$, let

$$Q_m = \sum_{r=1}^{m} \sum_{s=1}^{m} \sum_{u=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} |\Delta b_{u,v}|. \tag{34}$$

Putting (33) and (34) together, we have

$$m^2 |b_{r_0,s_0}| \leq \sum_{r=1}^{m} \sum_{s=1}^{m} (|b_{r_0+r,s_0+s}| + |b_{r_0,s_0+s}| + |b_{r_0+r,s_0}|) + Q_m. \tag{35}$$

First we consider the case $\alpha = 2$. By Remark 2.1 and Cauchy-Schwarz inequality, we have

$$\sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| \leq m \left( \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}|^2 \right)^{1/2}$$

$$\leq mB_n \left( \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^2 L(B_n/|b_{r,s}|) \right)^{1/2} = mB_n.$$

In the case $0 < \alpha < 2$, for $0 < \epsilon < A$ to be decided later, we write

$$\sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| = \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| I(B_n/|b_{r_0+r,s_0+s}| > A)$$

$$+ \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| I(B_n/|b_{r_0+r,s_0+s}| < \epsilon)$$

$$+ \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| I(\epsilon \leq B_n/|b_{r_0+r,s_0+s}| \leq A)$$

$$= I + II + III.$$

By Hölder's inequality, applied with $p > \max\{1, \alpha\}$ we have for $q$ satisfying $1/p + 1/q = 1$

$$I \leq m^{2/q} \left( \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}|^p I(B_n/|b_{r_0+r,s_0+s}| > A) \right)^{1/p}$$

$$\leq m^{2/q} B_n \left( \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^p I(B_n/|b_{r,s}| > A) \right)^{1/p}.$$

By the properties of slowly varying functions, there is $A > 0$ such that uniformly in $r$, $s$ and $n$

$$(B_n/|b_{r,s}|)^{\alpha-p} I(B_n/|b_{r,s}| > A) \leq L(B_n/|b_{r,s}|).$$

By (17),

$$I \leq m^{2/q} B_n \left( \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^\alpha L(B_n/|b_{r,s}|) \right)^{1/p} = m^{2/q} B_n.$$

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To treat $II$, note that by the definition of $L(x)$ in (7), $L(x)$ is a constant for $x < b$. Therefore there exists a sufficiently small $\epsilon > 0$, for all $x < \epsilon$ we have

$$x^{-\alpha} L(x) > 1.$$

Then, if $B_n/|b_{r,s}| < \epsilon$, we have $(|b_{r,s}|/B_n)^{\alpha} L(B_n/|b_{r,s}|) > 1$. Recall that $B_n$ satisfies (17). Clearly for this selection of $\epsilon$, we have

$$II = \sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| I(B_n/|b_{r_0+r,s_0+s}| < \epsilon) = 0.$$

To treat $III$, by the definition of $L(x)$ in (7), we choose $A > b$ and have

$$L(x) \geq A^{\alpha-2}(2-\alpha)/\alpha := M_A > 0$$

for $\epsilon \leq x \leq A$. So

$$L(B_n/|b_{r,s}|) > M_A \text{ for } \epsilon \leq B_n/|b_{r,s}| \leq A.$$

We apply again Hölder's inequality,

$$III \leq m^{2/q} B_n \left( \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^p I(\epsilon \leq B_n/|b_{r,s}| \leq A) \right)^{1/p}$$

$$\leq m^{2/q} B_n \left( \frac{1}{M_A} \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^p L(B_n/|b_{r,s}|) I(\epsilon \leq B_n/|b_{r,s}| \leq A) \right)^{1/p}$$

$$\leq m^{2/q} B_n \left( \frac{\epsilon^{\alpha-p}}{M_A} \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (|b_{r,s}|/B_n)^{\alpha} L(B_n/|b_{r,s}|) \right)^{1/p} \ll m^{2/q} B_n.$$

By the above estimates of $I$, $II$ and $III$ we obtain

$$\sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0+s}| \ll m^{2/q} B_n.$$

Similarly,

$$\sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0,s_0+s}| \ll m^{1+1/q} B_n,$$

and

$$\sum_{r=1}^{m} \sum_{s=1}^{m} |b_{r_0+r,s_0}| \ll m^{1+1/q} B_n.$$

So

$$m^2 |b_{r_0,s_0}| \ll m^{1+1/q} B_n + Q_m.$$

By (35) and the above considerations we have

$$\rho_n = \frac{|b_{r_0,s_0}|}{B_n} \ll m^{\frac{1}{q}-1} + \frac{Q_m}{m^2 B_n}.$$

Notice that in (34),

$$Q_m = \sum_{r=r_0+1}^{r_0+m} \sum_{s=s_0+1}^{s_0+m} (r_0 + m + 1 - r)(s_0 + m + 1 - s)|\Delta b_{r,s}| \leq m^4 \sup_{r,s} |\Delta b_{r,s}|.$$

Then, recalling our notation $\Delta_n = \sup_{r,s} |\Delta b_{r,s}|$, we obtain

$$\rho_n \ll m^{\frac{1}{q}-1} + \frac{m^2 \Delta_n}{B_n} = m^{-\frac{1}{p}} + \frac{m^2 \Delta_n}{B_n}.$$

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Denoting now by $\lceil x \rceil$ the smallest integer that exceeds $x$ and letting $m = \left\lceil (B_n/\Delta_n)^{p/(2p+1)} \right\rceil$ and $M = (B_n/\Delta_n)^{p/(2p+1)}$, we have

$$\rho_n \ll (\Delta_n/B_n)^{1/(2p+1)} + 2(M^2+1)\Delta_n/B_n = 3(\Delta_n/B_n)^{1/(2p+1)} + 2\Delta_n/B_n.$$

In the $d$ dimensional case,

$$\rho_n \ll m^{-\frac{1}{p}} + \frac{m^d \Delta_n}{B_n}.$$

Let $m = \left\lceil (B_n/\Delta_n)^{p/(dp+1)} \right\rceil$ and $M = (B_n/\Delta_n)^{p/(dp+1)}$, we have

$$\begin{aligned}
\rho_n &\ll (\Delta_n/B_n)^{1/(dp+1)} + 2^{d-1}(M^d+1)\Delta_n/B_n \\
&= (2^{d-1}+1)(\Delta_n/B_n)^{1/(dp+1)} + 2^{d-1}\Delta_n/B_n \\
&\ll 2^d[(\Delta_n/B_n)^{1/(dp+1)} + \Delta_n/B_n].
\end{aligned}$$

This proves (30).

Now we prove (31) for linear random field defined in (3). By the linearity of $\Delta$ and the definition of $\Gamma_n^2$, we have

$$\begin{aligned}
\Delta b_{r,s} = \Delta\left[ \sum_{(u,v)\in\Gamma_n^2} a_{u-r,v-s} \right] &= \Delta\left[ \sum_{k=1}^{J_n} \sum_{(u,v)\in\Gamma_n^2(k)} a_{u-r,v-s} \right] \\
&= \sum_{k=1}^{J_n} \Delta\left[ \sum_{(u,v)\in\Gamma_n^2(k)} a_{u-r,v-s} \right].
\end{aligned}$$

For each $k \in \{1, 2, ..., J_n\}$, we have

$$\Delta\left[ \sum_{(u,v)\in\Gamma_n^2(k)} a_{u-r,v-s} \right]$$
$$= a_{\underline{n}_1(k)-r,\underline{n}_2(k)-s} - a_{(\overline{n}_1(k)+1)-r,\underline{n}_2(k)-s} + a_{(\overline{n}_1(k)+1)-r,(\overline{n}_2(k)+1)-s} - a_{\underline{n}_1(k)-r,(\overline{n}_2(k)+1)-s}.$$

See Fortune, Peligrad and Sang (2021) for the detailed calculation of the above formula. Hence, by Hölder's inequality,

$$\begin{aligned}
|\Delta b_{r,s}| &= \left| \sum_{k=1}^{J_n} (a_{\underline{n}_1(k)-r,\underline{n}_2(k)-s} - a_{(\overline{n}_1(k)+1)-r,\underline{n}_2(k)-s} + a_{(\overline{n}_1(k)+1)-r,(\overline{n}_2(k)+1)-s} - a_{\underline{n}_1(k)-r,(\overline{n}_2(k)+1)-s}) \right| \\
&\leq 4J_n^{1/q}\|a\|_p \text{ for all } r,s,
\end{aligned}$$

where $\|a\|_p = (\sum_{i\in\mathbb{Z}^d} |a_i|^p)^{1/p}$. Therefore $\Delta_n \ll 4J_n^{1/q}$. This proves (31) for $d = 2$.

Similarly, in the $d$ dimensional case, we have

$$\Delta_n \ll 2^d J_n^{1/q},$$

hence (31) holds. Finally, it is clear that (32) follows from (30) and (31). This completes the proof. ∎

Now we prove the main theorems in Section 4.

**Proof of Theorem 4.1.**

The proof is based on the study of the characteristic function of the sum $S_n$. According to Lemma 4.5 and arguments in Section VI.4 in Hennion and Hervé (2001) (see also Theorem 10.7 in Breiman (1992) and Section 10.4 there), it suffices to prove (24) for all continuous complex valued functions $g$ defined on $\mathbb{R}$, $|g| \in L^1(\mathbb{R})$ such that the transformation

$$\hat{g}(t) = \int_{\mathbb{R}} e^{-\iota t x} g(x) dx$$

has compact support contained in some finite interval $[-M, M]$.

The inversion formula gives:

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\iota s x} \hat{g}(s) \ ds.$$

Therefore, employing a change of variables and taking the expected value

$$\mathbb{E}\left[g(S_n + u)\right] = \frac{1}{2\pi B_n} \int \hat{g}\left(\frac{t}{B_n}\right) \ \varphi_{S_n}\left(\frac{t}{B_n}\right) \ \exp\left(\frac{\iota t u}{B_n}\right) \ dt,$$

where we used the notation

$$\varphi_X(v) = \mathbb{E}\left(\exp(\iota v X)\right).$$

By the Fourier inversion formula we also have

$$f_{\mathcal{L}}(-u) = \frac{1}{2\pi} \int \varphi_{\mathcal{L}}(t) \ \exp(\iota t u) \ dt$$

and then

$$f_{\mathcal{L}}\left(-\frac{u}{B_n}\right) = \frac{1}{2\pi} \int \varphi_{\mathcal{L}}(t) \ \exp\left(\frac{\iota t u}{B_n}\right) \ dt.$$

Therefore

$$2\pi \left[ B_n \mathbb{E}\left[g(S_n + u)\right] - f_{\mathcal{L}}\left(-\frac{u}{B_n}\right) \int_{\mathbb{R}} g(x) dx \right]$$
$$= \int \hat{g}\left(\frac{t}{B_n}\right) \ \varphi_{S_n}\left(\frac{t}{B_n}\right) \ \exp\left(\frac{\iota t u}{B_n}\right) \ dt - \int \varphi_{\mathcal{L}}(t) \exp\left(\frac{\iota t u}{B_n}\right) \ dt \int_{\mathbb{R}} g(x) dx.$$

So

$$2\pi \left| B_n \mathbb{E}\left[g(S_n + u)\right] - f_{\mathcal{L}}\left(-\frac{u}{B_n}\right) \int_{\mathbb{R}} g(x) dx \right|$$
$$\leq \int \left| \hat{g}\left(\frac{t}{B_n}\right) \ \varphi_{S_n}\left(\frac{t}{B_n}\right) - \varphi_{\mathcal{L}}(t) \int_{\mathbb{R}} g(x) dx \right| dt.$$

By adding and subtracting $\hat{g}(\frac{t}{B_n})\varphi_{\mathcal{L}}(t)$, using the triangle inequality and taking into account that $\hat{g}$ vanishes outside $[-M, M]$ we bound the last term by

$$\int_{|t| \leq M B_n} \left| \hat{g}\left(\frac{t}{B_n}\right) \ \varphi_{S_n}\left(\frac{t}{B_n}\right) - \hat{g}\left(\frac{t}{B_n}\right) \varphi_{\mathcal{L}}(t) \right| dt$$
$$+ \int \left| \hat{g}\left(\frac{t}{B_n}\right) \varphi_{\mathcal{L}}(t) - \varphi_{\mathcal{L}}(t) \int_{\mathbb{R}} g(x) dx \right| dt = I_n + II_n.$$

$$I_n \leq ||\hat{g}||_\infty \left( \int_{|t| \leq T} |\varphi_{\frac{S_n}{B_n}}(t) - \varphi_{\mathcal{L}}(t)| dt + \int_{T < |t| \leq M B_n} |\varphi_{\frac{S_n}{B_n}}(t) - \varphi_{\mathcal{L}}(t)| dt \right)$$
$$\leq ||\hat{g}||_\infty \left( \int_{|t| \leq T} |\varphi_{\frac{S_n}{B_n}}(t) - \varphi_{\mathcal{L}}(t)| dt + \int_{T < |t| \leq M B_n} |\varphi_{\frac{S_n}{B_n}}(t)| dt + \int_{T < |t|} |\varphi_{\mathcal{L}}(t)| dt \right).$$

Because $S_n/B_n \Rightarrow \mathcal{L}$, and $\varphi_{\mathcal{L}}(t)$ is integrable, $I_n$ converges to 0 provided condition (23) is satisfied. The second term

$$II_n \leq \int \left| \hat{g}\left(\frac{t}{B_n}\right) - \int_{\mathbb{R}} g(x) dx \right| \cdot |\varphi_{\mathcal{L}}(t)| dt$$

converges to 0 because $\hat{g}$ is continuous and bounded, $g$ and $\varphi_{\mathcal{L}}$ are integrable and $B_n \to \infty$. We obtain

$$\sup_u \left| B_n \mathbb{E}\left[g(S_n + u)\right] - f_{\mathcal{L}}\left(-\frac{u}{B_n}\right) \int_{\mathbb{R}} g(x)dx \right| \to 0, \text{ as } n \to \infty$$

provided

$$\lim_{T \to \infty} \limsup_n \int_{T < |t| < M B_n} |\varphi_{\frac{S_n}{B_n}}(t)| dt \to 0.$$

∎

**Proof of Theorem 4.2.**

Recall that $\varphi_{S_n}(\cdot)$ is the characteristic function of $S_n$ and $\varphi_{\mathcal{L}}$ is the characteristic function of $\mathcal{L}$. We have $\varphi_{S_n}(t/B_n) = \prod_{i \in \mathbb{Z}^d} \varphi_\xi(tb_{ni}/B_n)$. Let $\gamma_n = \sup_i |b_{ni}|$.

By Theorem 3.2, we have the weak convergence result (22) in Theorem 4.1. Notice that if $\{b_{ni}\}$ are uniformly bounded in $n$ and $i$, then obviously $\rho_n = \gamma_n/B_n \to 0$ as $n \to \infty$. In the case where $\limsup_n \sup_{i \in \mathbb{Z}^d} |b_{ni}| = \infty$, we impose that $\rho_n \to 0$.

Now we verify condition (23) in Theorem 4.1. A part of the argument is similar to the proof of Theorem 2 in Shukri (1976). In (1), we renumber $\{b_{ni}\}$ into a sequence $\{b'_{k,n}\}_{k=1}^\infty$ such that $\gamma_n = |b'_{1,n}| \geq |b'_{2,n}| \geq \cdots$. Since (17), we can select a number $k_n \in \mathbb{N}$ such that

$$k_n = \inf\left\{l : \sum_{k=1}^l (|b'_{k,n}|/B_n)^\alpha L(B_n/|b'_{k,n}|) \geq 1/2\right\}.$$

Then

$$\frac{1}{2} \leq \sum_{k=1}^{k_n} (|b'_{k,n}|/B_n)^\alpha L(B_n/|b'_{k,n}|) < \frac{1}{2} + (\gamma_n/B_n)^{\alpha/2}.$$

Since $\gamma_n/B_n \to 0$ as $n \to \infty$, by the properties of slowly varying functions, there exists a constant $C > 0$, such that for sufficiently large $n$,

$$\sum_{k=1}^{k_n} (|b'_{k,n}|/B_n)^\alpha L(B_n/|b'_{k,n}|) \leq C(\gamma_n/B_n)^\alpha L(B_n/\gamma_n)k_n.$$

Hence, for $n$ large enough

$$k_n \geq \frac{1}{2C(\gamma_n/B_n)^\alpha L(B_n/\gamma_n)}. \tag{36}$$

By (9), for $0 < \alpha \leq 2$, there exists $\epsilon > 0$ such that for all $|x| \leq \epsilon$, we have

$$|\varphi_\xi(x)| = \exp\{-c_\alpha |x|^\alpha L(1/|x|)\}.$$

Since $|tb'_{k,n}/B_n| \leq \epsilon$ for $|t| \leq \epsilon B_n/|b'_{k_n,n}|$, $k > k_n$, we obtain from the above equality

$$|\varphi_{S_n}(t/B_n)| \leq \prod_{k=k_n+1}^\infty |\varphi_\xi(tb'_{k,n}/B_n)| \tag{37}$$

$$= \exp\left\{-c_\alpha |t|^\alpha \sum_{k=k_n+1}^\infty |b'_{k,n}/B_n|^\alpha L(B_n/|tb'_{k,n}|)\right\}.$$

Now, by Lemma 5.1 in Davydov (1974), without restricting the generality we shall assume that $L$ also satisfies the following condition

$$\sup_{x>0} \frac{L(x)}{L(ux)} \leq u^{-\alpha/2} \text{ for all } 0 < u \leq 1. \tag{38}$$

Therefore, by applying (38) with $u = |T/t|$ we obtain

$$L\left(\frac{B_n}{|tb'_{k,n}|}\right) = L\left(\frac{T}{|t|}\frac{B_n}{|b'_{k,n}|T}\right) \geq \left|\frac{T}{t}\right|^{\alpha/2} L\left(\frac{B_n}{|b'_{k,n}|T}\right) \text{ for } |t| > T.$$

By combining inequality (37) with the last inequality we obtain for $T < |t| \leq \epsilon B_n / |b'_{k_n,n}|$, and for $n$ and $T$ sufficiently large

$$|\varphi_{S_n}(t/B_n)| \leq \exp \left\{ -c_\alpha |t|^\alpha \sum_{k=k_n+1}^\infty |b'_{k,n}/B_n|^\alpha L(B_n/|tb'_{k,n}|) \right\}$$

$$\leq \exp \left\{ -c_\alpha T^{\alpha/2} |t|^{\alpha/2} \sum_{k=k_n+1}^\infty |b'_{k,n}/B_n|^\alpha L(B_n/|b'_{k,n}T|) \right\}$$

$$\leq \exp \left\{ -c_\alpha |t|^{\alpha/2} \sum_{k=k_n+1}^\infty |b'_{k,n}/B_n|^\alpha L(B_n/|b'_{k,n}|) \right\}.$$

In the last step we used the properties of slowly varying functions. Notice that $B_n/|b'_{k,n}| \to \infty$ uniformly for all $k$ as $n \to \infty$.

By (17) and the selection of $k_n$, we have

$$\sum_{k=k_n+1}^\infty |b'_{k,n}/B_n|^\alpha L(B_n/|b'_{k,n}|) \geq \frac{1}{2} - \left( \frac{\gamma_n}{B_n} \right)^{\alpha/2}$$

and, as a consequence

$$|\varphi_{S_n}(t/B_n)| \leq \exp \left\{ -c_\alpha |t|^{\alpha/2} \right\}.$$

Then we have found $\epsilon > 0$ such that

$$\lim_{n \to \infty} \int_{T < |t| < \epsilon B_n/|b'_{k_n,n}|} |\varphi_{S_n}(t/B_n)| dt$$

$$\leq \int_{|t|>T} \exp \left\{ -c_\alpha |t|^{\alpha/2} \right\} dt,$$

which goes to 0 as $T \to \infty$. If $\epsilon B_n/|b'_{k_n,n}| \geq DB_n$, then condition (23) holds and the proof is complete. Otherwise, it remains to analyze the term

$$G_n = \int_{\epsilon B_n/|b'_{k_n,n}| < |t| \leq DB_n} |\varphi_{S_n}(t/B_n)| dt.$$

We shall consider two cases. In the case (i) that $|b_{ni}|$ is uniformly bounded by a constant $b_0 > 0$ for all $i \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, notice that $\epsilon \leq \epsilon |b'_{k,n}|/|b'_{k_n,n}| \leq t|b'_{k,n}|/B_n \leq Db_0$ for $\epsilon B_n/|b'_{k_n,n}| < |t| \leq DB_n$ and $1 \leq k \leq k_n$. Since the innovations have a non-lattice distribution and $\varphi_\xi$ is continuous, there exists $0 < r < 1$ such that $|\varphi_\xi(tb'_{k,n}/B_n)| \leq r$ for all $\epsilon B_n/|b'_{k_n,n}| < |t| \leq DB_n$ and $1 \leq k \leq k_n$. Therefore

$$G_n \leq \int_{\epsilon B_n/|b'_{k_n,n}| < |t| \leq DB_n} \prod_{k=1}^{k_n} \left| \varphi_\xi(tb'_{k,n}/B_n) \right| dt \qquad (39)$$

$$\leq 2DB_n r^{k_n}.$$

By (36),

$$k_n \geq \frac{1}{2C(\gamma_n/B_n)^{\alpha-\delta}} > \frac{B_n^{\alpha-\delta}}{2Cb_0^{\alpha-\delta}}$$

for some $0 < \delta < \alpha$ and $n$ is sufficiently large. Then we have $\lim_{n\to\infty} B_n r^{k_n} = 0$ and $G_n \to 0$ as $n \to \infty$.

In the case (ii) where $\limsup_n \sup_{i \in \mathbb{Z}^d} |b_{ni}| = \infty$ we shall use the Cramér condition. By Lemma 5.2 in Fortune, Peligrad and Sang (2021) we know that under Cramér condition, for every $\epsilon > 0$ there exists $r > 0$ such that

$$|\varphi_\xi(t)| \leq r \text{ for all } |t| > \epsilon.$$

Since $\epsilon |b'_{k,n}|/|b'_{k_n,n}| \geq \epsilon$ for $|t| > \epsilon B_n/|b'_{k_n,n}|$ and $1 \leq k \leq k_n$, $|\varphi_\xi(tb'_{k,n}/B_n)| \leq r$ for all $|t| > \epsilon B_n/|b'_{k_n,n}|$ and $1 \leq k \leq k_n$. Therefore (39) still holds and we also have

$$\lim_{n \to \infty} G_n = 0.$$

This completes the proof. ∎

**Proof of Theorem 4.3 and Theorem 4.4.**

In the case that $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$, $\{b_{ni}\}$ are uniformly bounded on $i$ and $n$. Hence Theorem 4.3 holds by the first part of Theorem 4.2. The proof of Theorem 4.4 with $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$ is similar to the proof of the second case in Theorem 4.2. Notice that condition $J_n = o(B_n^q)$ for $q \geq 2$ implies the weak convergence in Theorem 3.3. ∎

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