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Lower bounds for moments of $\zeta'(\rho)$

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Abstract

Assuming the generalized Riemann hypothesis for Dirichlet L-functions, we establish lower bounds for the $2k$th moments of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$ for every positive integer k. Our proof is based upon a method of Rudnick and Soundararajan that provides analogous bounds for moments of L-functions at the central point, averaged over families.

1. *Introduction*

Let $\zeta(s)$ denote the Riemann zeta-function. In this article we are interested in obtaining lower bounds for moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \tag{1.1}$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, we let the function

$$N(T) = \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.2}$$

denote the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity.

Independently, Gonek [**6**] and Hejhal [**8**] have conjectured that $J_k(T) \asymp (\log T)^{k(k+2)}$ for each $k \in \mathbb{R}$. By modeling the Riemann zeta-function and its derivative using characteristic polynomials of random matrices, Hughes, Keating, and O'Connell [**9**] have refined this conjecture to state that $J_k(T) \sim C_k(\log T)^{k(k+2)}$ for a precise constant C_k when $k \in \mathbb{C}$ and $\Re k > -3/2$. However, we no longer believe that $J_k(T) \ll (\log T)^{k(k+2)}$ when $\Re k < -3/2$. This is since we expect that there exist infinitely many zeros ρ such that $|\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon}$ for each $\varepsilon > 0$.

Results that are consistent with these conjectures are only known to hold for a few small values of k. See, for instance, the results of Gonek [**4**] for the case $k = 1$ and the second author [**11**] for the case $k = 2$. Also, Gonek [**6**] obtained a lower bound in the case $k = -1$. Our main result is to obtain a lower bound for $J_k(T)$ for each positive integer k of the order of magnitude that was conjectured by Gonek and Hejhal.

Theorem 1.1. *Let $k \in \mathbb{N}$ and assume the generalized Riemann hypothesis* (GRH) *for Dirichlet L-functions. Then, for sufficiently large T, we have*

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \gg_k (\log T)^{k(k+2)}.$$

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Under the assumption of the Riemann hypothesis, the first author [**10**] has recently shown that $J_k(T) \ll_{k,\varepsilon} (\log T)^{k(k+2)+\varepsilon}$ for $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. When combined with Theorem 1.1, this result lends strong support for the conjecture of Gonek and Hejhal, mentioned above, in the case where k is a positive integer.

Theorem 1.1 can be used to exhibit large values of $\zeta'(\rho)$. For example, assuming GRH, it immediately follows that for each $A > 0$ the inequality $\left|\zeta'(\rho)\right| \geq (\log|\gamma|)^A$ is satisfied for infinitely many zeros ρ of $\zeta(s)$.

Our proof of Theorem 1.1 is based upon a method of Rudnick & Soundararajan [**12**, **13**]. It is likely that our proof can be adapted to prove a lower bound for $J_k(T)$ of the conjectured order of magnitude for all rational $k \geq 1$ in a manner analogous to that suggested in [**12**].

Let $k \in \mathbb{N}$ and define, for $\xi \geq 1$, the function $\mathcal{A}_\xi(s) = \sum_{n\leq\xi} n^{-s}$. Assuming GRH, we will estimate

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\overline{\mathcal{A}_\xi(\rho)}^k \quad \text{and} \quad \Sigma_2 = \sum_{0<\gamma\leq T} \left|\mathcal{A}_\xi(\rho)\right|^{2k} \tag{1.3}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Hölder's inequality implies that

$$\sum_{0<\gamma\leq T} \left|\zeta'(\rho)\right|^{2k} \geq \frac{\left|\Sigma_1\right|^{2k}}{\left(\Sigma_2\right)^{2k-1}},$$

so Theorem 1.1 will follow from the estimates

$$\Sigma_1 \gg T(\log T)^{k^2+2} \quad \text{and} \quad \Sigma_2 \ll T(\log T)^{k^2+1}. \tag{1.4}$$

The length of the Dirichlet polynomial $\mathcal{A}_\xi(s)$ turns out to be important in our argument. We will eventually chose $\xi = T^{\frac{\vartheta}{k}}$ where $0 < \vartheta \leq \frac{1}{4}$ is described in Lemma 4.2, below.

It is convenient to express the sums Σ_1 and Σ_2 slightly differently. Assuming the Riemann hypothesis, $1-\rho = \bar{\rho}$ for any non-trivial zero ρ of $\zeta(s)$. Thus, $\overline{\mathcal{A}_\xi(\rho)} = \mathcal{A}_\xi(1-\rho)$. This allows us to re-write the sums in (1.3) as

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\mathcal{A}_\xi(1-\rho)^k \quad \text{and} \quad \Sigma_2 = \sum_{0<\gamma\leq T} \mathcal{A}_\xi(\rho)^k\mathcal{A}_\xi(1-\rho)^k. \tag{1.5}$$

It is with these representations of Σ_1 and Σ_2 that we establish the bounds in (1.4).

2. *Conventions*

In this article we shall use the convention that ε denotes an arbitrarily small positive constant which may vary from line to line. For functions $f(x)$ and $g(x)$ we shall interchangeably use the notations $f(x) = O(g(x))$, $f(x) \ll g(x)$, or $g(x) \gg f(x)$ to mean there exists $M > 0$ such that $|f(x)| \leq M|g(x)|$ for all x sufficiently large. The constants implied in our big-O, $\ll$, and $\gg$ estimates are allowed to depend on k and ε.

3. *Some preliminary estimates*

For each real number $\xi \geq 1$ and each $k \in \mathbb{N}$, we define the arithmetic sequence of real numbers $\tau_k(n;\xi)$ by

$$\mathcal{A}_\xi(s)^k = \Big(\sum_{n\leq\xi} \frac{1}{n^s}\Big)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n;\xi)}{n^s}. \tag{3.1}$$

The function $\tau_k(n;\xi)$ is a truncated approximation to the arithmetic function $\tau_k(n)$, the kth iterated divisor function, which is defined by

$$\zeta^k(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n^s}\Big)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} \tag{3.2}$$

for $\Re s > 1$. We use $\tau(n) = \tau_2(n)$ to denote the divisor function.

We require a few estimates for sums involving the functions $\tau_k(n)$ and $\tau_k(n;\xi)$ in order to establish the bounds for Σ_1 and Σ_2 in (1.4). We use repeatedly that, for $x \geq 3$ and $k, \ell \in \mathbb{N}$,

$$\sum_{n\leq x} \frac{\tau_k(n)\tau_\ell(n)}{n} \asymp_{k,\ell} (\log x)^{k\ell} \tag{3.3}$$

where the implied constants depend on k and ℓ. These bounds are well known.

From (3.1) and (3.2) we notice that $\tau_k(n;\xi)$ is non-negative and $\tau_k(n;\xi) \leq \tau_k(n)$ with equality holding when $n \leq \xi$. In particular, choosing $k = \ell$ in (3.3) we find that, for $\xi \geq 3$,

$$(\log \xi)^{k^2} \ll_k \sum_{n\leq\xi} \frac{\tau_k(n)^2}{n} \leq \sum_{n\leq\xi^k} \frac{\tau_k(n;\xi)^2}{n} \leq \sum_{n\leq\xi^k} \frac{\tau_k(n)^2}{n} \ll_k (\log \xi)^{k^2}. \tag{3.4}$$

Also, we will need the following inequality concerning the arithmetic function $\tau_k(n;\xi)$.

LEMMA 3.1. *For $k, m, n \in \mathbb{N}$ and $\xi \geq 1$, we have $\tau_k(mn;\xi) \leq \tau_k(m;\xi)\tau_k(n;\xi)$.*

Proof. Let k be a fixed positive integer and $\xi \geq 1$. For each $n \in \mathbb{N}$, let

$$S_n = \big\{(d_1, d_2, \ldots, d_k) \ : \ d_1 d_2 \cdots d_k = n, \ d_i \leq \xi \ \text{ for } i = 1, \ldots, k\big\}.$$

In order to prove the lemma, it suffices to exhibit an injection from the set S_{mn} into $S_m \times S_n$. Supposing that $mn = d_1 d_2 \cdots d_k$ where each $d_i \leq \xi$, we see that $(d_1, d_2, \ldots, d_k) \in S_{mn}$. For each $i = 1, \ldots, k$, we can decompose each d_i as e_i times f_i where $m = e_1 e_2 \cdots e_k$ and $n = f_1 f_2 \cdots f_k$ as follows. Define e_1 to be the unique divisor of d_1 such that $d_1 = e_1 f_1$, $e_1 | m$, and $(m/e_1, f_1) = 1$. Next, define e_2 to be the unique divisor of d_2 such that $d_2 = e_2 f_2, e_2 | m/e_1$, and $(m/(e_1 e_2), f_2) = 1$. In general, e_i is defined (for $i = 2, \ldots, k$) as the unique divisor of d_i such that $d_i = e_i f_i, e_i | m/(e_1 e_2 \cdots e_{i-1})$, and $(m/(e_1 e_2 \cdots e_i), f_i) = 1$. This construction guarantees that $m = e_1 e_2 \cdots e_k$ and, hence, that $n = f_1 f_2 \cdots f_k$. Moreover, because $d_i \leq \xi$ for each $i = 1, \ldots, k$ we also see that $e_i \leq \xi$ and $f_i \leq \xi$ for each $i = 1, \ldots, k$. Therefore, this construction gives a map from the k-tuple

$$(d_1, d_2, \ldots, d_k) \in S_{mn} \mapsto (e_1, e_2, \ldots, e_k) \times (f_1, f_2, \ldots, f_k) \in S_m \times S_n.$$

Since $d_i = e_i f_i$ for each $i = 1, \ldots, k$, one can readily verify that this map is an injection. This proves the lemma. □

4. *A Lower Bound for Σ_1*

In order to establish a lower bound for Σ_1, we require a mean-value estimate for sums of the form

$$S(X, Y; T) = \sum_{0<\gamma\leq T} \zeta'(\rho) X(\rho) Y(1-\rho)$$

where

$$X(s) = \sum_{n\leq N} \frac{x_n}{n^s} \qquad \text{and} \qquad Y(s) = \sum_{n\leq N} \frac{y_n}{n^s}$$

are Dirichlet polynomials. We provide a proof of Lemma 4.1, stated below, in section 6 of this article. Before stating the formula for $S(X,Y;T)$, we first introduce some notation. For T large, we let $\mathscr{L} = \log \frac{T}{2\pi}$ and $N = T^{\vartheta}$ where $\vartheta > 0$. The functions $\mu(\cdot)$ and $\Lambda(\cdot)$ are used to denote the usual arithmetic functions of Möbius and von Mangoldt given by the generating series, for $\Re s > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{and} \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

respectively. Also, we define the function $\Lambda_2(\cdot)$ by

$$\Lambda_2(n) = (\mu * \log^2)(n) \tag{4.1}$$

for each $n \in \mathbb{N}$. Then, the following formula for $S(X,Y;T)$ holds.

LEMMA 4.1. *Assume the GRH. Let $j, \ell \in \mathbb{N}$ and set $\xi = N^{1/\max(j,\ell)}$. We define*

$$x_n = \tau_j(n;\xi) \quad \text{and} \quad y_n = \tau_\ell(n;\xi). \tag{4.2}$$

Then for $0 \le \vartheta < \frac{1}{2}$, $\varepsilon > 0$, and sufficiently large T we have

$$\begin{aligned} S(X,Y;T) = \frac{T}{2\pi} \sum_{mn \le N} \frac{x_m y_{mn}}{mn} \Big(\frac{1}{2}\mathcal{P}_2(\mathscr{L}) - \mathcal{P}_1(\mathscr{L}) \log n - \frac{1}{2}(\log n)^2 + (\Lambda * \log)(n) \Big) \\ + \frac{T}{2\pi} \sum_{\substack{a,b \le N \\ (a,b)=1}} \frac{r(a;b)}{ab} \sum_{g \le \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{x_{ag} y_{bg}}{g} + O\big(T \exp(-c\sqrt{\log T}) + T^{\vartheta + \frac{1}{2} + \varepsilon}\big) \end{aligned}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are monic polynomials of degrees 1 and 2, respectively, and for $a, b \in \mathbb{N}$

$$r(a;b) \ll \Lambda_2(a) + (\log T)\Lambda(a). \tag{4.3}$$

The proof of this lemma will be provided in the last section of the article. For the remainder of this section, we assume the truth of Lemma 4.1 and derive from it our desired lower bound for Σ_1. We henceforth assume that $0 < \vartheta \le \frac{1}{4}$. Letting $j = k-1$ and $\ell = k$, we find that $X(s) = \mathcal{A}_\xi(s)^{k-1}$ and $Y(s) = \mathcal{A}_\xi(s)^k$. Therefore, by Lemma 4.1, we have

$$\begin{aligned} \Sigma_1 &= \frac{T}{2\pi} \sum_{\substack{mn \le \xi^k \\ m \le \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \Big(\frac{1}{2}\mathcal{P}_2(\mathscr{L}) - \mathcal{P}_1(\mathscr{L}) \log n - \frac{1}{2}(\log n)^2 + (\Lambda * \log)(n) \Big) \\ &\quad + \frac{T}{2\pi} \sum_{\substack{a,b \le \xi^k \\ (a,b)=1}} \frac{r(a;b)}{ab} \sum_{g \le \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{\tau_{k-1}(ag;\xi)\tau_k(bg;\xi)}{g} + O(T) \\ &= \mathcal{S}_{11} + \mathcal{S}_{12} + O(T), \end{aligned}$$

say. Here $N = T^{\vartheta} = \xi^k$ and so $\xi = T^{\vartheta/k} \le T^{1/4k}$. We will derive a lower bound for Σ_1 by establishing the following two results.

LEMMA 4.2. *There exists a ϑ satisfying $0 < \vartheta \le \frac{1}{4}$ such that $|\mathcal{S}_{12}| \le \frac{1}{2}\mathcal{S}_{11}$ when T is sufficiently large.*

LEMMA 4.3. *For any fixed ϑ satisfying $0 < \vartheta \le \frac{1}{4}$, we have $\mathcal{S}_{11} \gg T(\log T)^{k^2+2}$.*

Remark: Note that by combining the results of the above two lemmas, it follows that

$$\Sigma_1 = \mathcal{S}_{11} + \mathcal{S}_{12} + O(T) \geq \tfrac{1}{2} \cdot \mathcal{S}_{11} + O(T) \gg T(\log T)^{k^2+2}$$

for the value of ϑ in Lemma 4.2. This establishes the estimate for Σ_1 stated in (1.4).

Proof of Lemma 4.2. By Lemma 3.1, we have $\tau_k(mn;\xi) \leq \tau_k(m;\xi)\tau_k(n;\xi)$ for $k,m,n \in \mathbb{N}$. Using this inequality along with the bound for the arithmetic function $r(a,b)$ given in (4.3), we find that $\mathcal{S}_{12} \ll T\left(\mathcal{S}_{13} + \mathcal{S}_{14}\right)$ where

$$\mathcal{S}_{13} = \sum_{bg \leq N} \frac{\tau_{k-1}(g;\xi)\tau_k(bg;\xi)}{bg} \sum_{a \leq N} \frac{\Lambda_2(a)\tau_{k-1}(a;\xi)}{a}$$

and

$$\mathcal{S}_{14} = \log T \sum_{bg \leq N} \frac{\tau_{k-1}(g;\xi)\tau_k(bg;\xi)}{bg} \sum_{a \leq N} \frac{\Lambda(a)\tau_{k-1}(a;\xi)}{a}.$$

Next, note that Λ_2 is supported on those integers a with at most two distinct prime factors and, moreover, that $\Lambda_2(p^j) \ll j(\log p)^2$ and $\Lambda_2(p^j q^k) \ll (\log p)(\log q)$ for prime powers p^j and q^k. Since $\tau_{k-1}(a;\xi) \leq \tau_{k-1}(a)$ it follows that

$$\mathcal{S}_{13} \ll \log^2 N \sum_{bg \leq N} \frac{\tau_{k-1}(g;\xi)\tau_k(bg;\xi)}{bg}.$$

A similar argument establishes that

$$\begin{aligned}\mathcal{S}_{14} &\ll \log T \sum_{bg \leq N} \frac{\tau_{k-1}(g;\xi)\tau_k(bg;\xi)}{bg} \sum_{a \leq N} \frac{\Lambda(a)\tau_{k-1}(a)}{a} \\ &\ll \log T \log N \sum_{bg \leq N} \frac{\tau_{k-1}(g;\xi)\tau_k(bg;\xi)}{bg}.\end{aligned}$$

Since $N \leq T$, it follows from the above estimates that

$$\begin{aligned}\mathcal{S}_{12} &\ll T \log T \log N \sum_{mn \leq N} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \\ &\leq (A\vartheta) T \mathscr{L}^2 \sum_{\substack{mn \leq \xi^k \\ m \leq \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn}\end{aligned}$$

for some constant $A > 0$ and T sufficiently large.

We now turn our attention to $\mathcal{S}_{11}$. We claim that

$$\Big(\frac{1}{2}\mathcal{P}_2(\mathscr{L}) - \mathcal{P}_1(\mathscr{L}) \log n - \frac{1}{2}(\log n)^2 + (\Lambda * \log)(n)\Big) \geq \frac{1}{8}\mathscr{L}^2 \tag{4.4}$$

when T is sufficiently large. To see why, notice that since $\mathcal{P}_2$ is monic, $\mathcal{P}_2(\mathscr{L}) = \big(1+o(1)\big)\mathscr{L}^2$. Also since $n \leq \xi^k \leq T^{1/4}$ and $\mathcal{P}_1$ is monic, we have

$$\mathcal{P}_1(\mathscr{L}) \log n + \frac{1}{2}(\log n)^2 \leq \Big(\frac{1}{4} + \frac{1}{2 \cdot 4^2} + o(1)\Big)\mathscr{L}^2.$$

Therefore, since $(\Lambda * \log)(n) \geq 0$, these estimates together imply that the estimate in (4.4) holds when T is sufficiently large. Now, using (4.4), it follows that for any fixed ϑ satisfying $0 < \vartheta \leq \frac{1}{4}$ we have

$$\mathcal{S}_{11} \geq \frac{T}{16\pi}\mathscr{L}^2 \sum_{\substack{mn \leq \xi^k \\ m \leq \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \tag{4.5}$$

for T sufficiently large. Hence, choosing $\vartheta = \min\left(\frac{1}{32A\pi}, \frac{1}{4}\right)$, we establish Lemma 4.2. $\square$

Proof of Lemma 4.3. In order to prove the lemma, we will estimate the sum on right-hand side of (4.5). Letting $mn = \ell$, we see that

$$\sum_{\substack{mn\le\xi^k \\ m\le\xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} = \sum_{\ell\le\xi^k} \frac{\tau_k(\ell;\xi)}{\ell}\left(\sum_{\substack{m|\ell \\ m\le\xi^{k-1}}} \tau_{k-1}(m;\xi)\right). \tag{4.6}$$

But, noticing that

$$\sum_{\substack{m|\ell \\ m\le\xi^{k-1}}} \tau_{k-1}(m;\xi) \ge \tau_k(\ell;\xi), \tag{4.7}$$

we can conclude from (3.4), (4.6), and (4.7) that

$$\sum_{\substack{mn\le\xi^k \\ m\le\xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \ge \sum_{\ell\le\xi} \frac{\tau_k(\ell;\xi)^2}{\ell} \gg (\log T)^{k^2}$$

which, when combined with (4.5), implies that $\mathcal{S}_{11} \gg T(\log T)^{k^2+2}$ for any fixed $0 < \vartheta \le \frac{1}{4}$. $\square$

5. *An Upper Bound for* Σ_2

Throughout this section, we assume that $\xi = T^{\frac{\vartheta}{k}}$ where $0 < \vartheta \le \frac{1}{4}$ is the number described in Lemma 4.2. Assuming the Riemann hypothesis, we interchange the sums in (1.5) and find that

$$\Sigma_2 = N(T)\sum_{n\le\xi^k} \frac{\tau_k(n;\xi)^2}{n} + 2\Re \sum_{m\le\xi^k}\sum_{m<n\le\xi^k} \frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n} \sum_{0<\gamma\le T}\left(\frac{n}{m}\right)^{\rho} \tag{5.1}$$

where $N(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to a height T. Using (1.2) and (3.4), it follows that

$$N(T)\sum_{n\le\xi^k} \frac{\tau_k(n;\xi)^2}{n} \ll T(\log T)^{k^2+1}. \tag{5.2}$$

In order to bound the second sum on the right-hand side of (5.1), we require the following version of the Landau-Gonek explicit formula. (See Gonek [**5**, **7**].)

LEMMA 5.1. *Let $x, T > 1$ and let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then*

$$\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O\big(x\log(2xT)\log\log(3x)\big)$$
$$+ O\Big(\log x \min\Big(T, \frac{x}{\langle x\rangle}\Big)\Big) + O\Big(\log(2T)\min\Big(T, \frac{1}{\log x}\Big)\Big)$$

where $\langle x\rangle$ denotes the distance from x to the closest prime power other than x itself and $\Lambda(x) = \log p$ if x is a positive integral power of a prime p and $\Lambda(x) = 0$ otherwise.

Applying the lemma, we find that

$$\begin{aligned}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n}\sum_{0<\gamma\le T}\Big(\frac{n}{m}\Big)^{\rho} &= -\frac{T}{2\pi}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)\Lambda(\frac{n}{m})}{n}\\ &\quad+O\left(\mathscr{L}\log\mathscr{L}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\right)\\ &\quad+O\left(\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\frac{\log\frac{n}{m}}{\langle\frac{n}{m}\rangle}\right)\\ &\quad+O\left(\mathscr{L}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n\log\frac{n}{m}}\right)\\ &=\mathcal{S}_{21}+\mathcal{S}_{22}+\mathcal{S}_{23}+\mathcal{S}_{24},\end{aligned}$$

say. Since we only require an upper bound for Σ_2 (which, by definition, is clearly positive), we can ignore the contribution from $\mathcal{S}_{21}$ because $\tau_k(m;\xi)\tau_k(n;\xi)\Lambda(\frac{n}{m})\ge 0$ for each $m,n\in\mathbb{N}$ and so the contribution of $\mathcal{S}_{21}$ to Σ_2 is clearly negative.

To estimate $\mathcal{S}_{22}$, we note that $\tau_k(n;\xi)\le\tau_k(n)\ll_\varepsilon n^\varepsilon$ which implies $\mathcal{S}_{22}\ll T^{1/4+\varepsilon}$. Turning to $\mathcal{S}_{23}$, we write n as $qm+\ell$ with $-\frac{m}{2}<\ell\le\frac{m}{2}$ and find that

$$\mathcal{S}_{23}\ll T^\varepsilon\sum_{m\le\xi^k}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}\sum_{-\frac{m}{2}<\ell\le\frac{m}{2}}\frac{1}{\langle q+\frac{\ell}{m}\rangle}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Notice that $\langle q+\frac{\ell}{m}\rangle=\frac{|\ell|}{m}$ if q is a prime power and $\ell\ne0$, otherwise $\langle q+\frac{\ell}{m}\rangle\ge\frac{1}{2}$. Hence,

$$\begin{aligned}\mathcal{S}_{23}&\ll T^\varepsilon\Big(\sum_{m\le\xi^k}\frac{1}{m}\sum_{\substack{q\le\lfloor\frac{\xi^k}{m}\rfloor+1\\ \Lambda(q)\ne0}}\sum_{1\le\ell\le\frac{m}{2}}\frac{m}{\ell}+\sum_{m\le\xi^k}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}\sum_{1\le\ell\le\frac{m}{2}}1\Big)\\ &\ll T^\varepsilon\Big(\sum_{m\le\xi^k}\sum_{q\le\lfloor\frac{\xi^k}{m}\rfloor+1}1\Big)\ll T^{1/4+\varepsilon}\ .\end{aligned}$$

It remains to consider $\mathcal{S}_{24}$. For integers $1\le m<n\le\xi^k$, let $n=m+\ell$. Then

$$\log\frac{n}{m}=-\log\big(1-\frac{\ell}{m}\big)>\frac{\ell}{m}.$$

Consequently,

$$\mathcal{S}_{24}\ll T^\varepsilon\sum_{m\le\xi^k}\sum_{1\le\ell\le\xi^k}\frac{1}{(m+\ell)\frac{\ell}{m}}\ll T^{2\varepsilon}\xi^k=T^{1/4+2\varepsilon}. \tag{5.3}$$

Combining (5.2) with our estimates for $\mathcal{S}_{22}$, $\mathcal{S}_{23}$, and $\mathcal{S}_{24}$ we deduce that $\Sigma_2\ll T(\log T)^{k^2+1}$ which, when combined with our estimates for Σ_1 in the previous section, completes the proof of Theorem 1.1.

6. *Proof of Lemma 4.1*

In this section, we evaluate the sum

$$\mathcal{S}=\sum_{0<\gamma\le T}\zeta'(\rho)X(\rho)Y(1-\rho)$$

where $X(s) = \sum_{n\le N} x_n n^{-s}$, $Y(s) = \sum_{n\le N} y_n n^{-s}$, and x_n, y_n are defined by (4.2). The functional equation for the Riemann zeta-function is $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2^s \pi^{s-1}\sin(\pi s/2)\Gamma(1-s).$$

Differentiating the functional equation, we have

$$\zeta'(s) = -\chi(s)\Big(\zeta'(1-s) - \frac{\chi'}{\chi}(s)\zeta(1-s)\Big).$$

From this last equation it follows that

$$\mathcal{S} = -\sum_{0<\gamma\le T} \chi(\rho)\zeta'(1-\rho)X(\rho)Y(1-\rho) = -\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\zeta'}{\zeta}(s)\chi(s)\zeta'(1-s)X(s)Y(1-s)\,ds$$

where $\mathscr{C}$ denotes the positively oriented rectangle with vertices at $1-\kappa+i, \kappa+i, \kappa+iT$, and $1-\kappa+iT$ where $\kappa = 1+(\log T)^{-1}$. Here, without loss of generality, we can choose T so that the distance from T to the nearest ordinate γ of a zero $\rho = \beta+i\gamma$ of $\zeta(s)$ is uniformly $\gg (\log T)^{-1}$. Differentiating the functional equation, we have

$$\frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s) \tag{6.1}$$

and thus

$$\mathcal{S} = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\zeta'}{\zeta}(1-s)\chi(s)\zeta'(1-s)X(s)Y(1-s)\,ds$$

since the integral involving $\frac{\chi'}{\chi}(s)$ is zero by Cauchy's theorem. Using standard estimates for the integrand, the contribution to the above integral from horizontal edges of this contour is $\ll NT^{1/2+\varepsilon}$. Next, it follows from the functional equation and (6.1) that

$$\chi(s)\zeta'(1-s) = -\zeta'(s) + \frac{\chi'}{\chi}(s)\zeta(s) \tag{6.2}$$

By (6.1) and (6.2) it follows that the right-hand side of the previous integral is

$$\mathcal{S}_R = \frac{1}{2\pi i}\int_{\kappa+i}^{\kappa+iT} \Big(\frac{\chi'}{\chi}(s)^2\zeta(s) - 2\frac{\chi'}{\chi}(s)\zeta'(s) + \frac{\zeta'}{\zeta}(s)\zeta'(s)\Big)X(s)Y(1-s)\,ds\,; \tag{6.3}$$

the left-hand side is

$$\mathcal{S}_L = \frac{1}{2\pi i}\int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'}{\zeta}(1-s)\chi(s)\zeta'(1-s)X(s)Y(1-s)ds.$$

By the variable change $s \mapsto 1-s$ the left-hand side equals $-\overline{\mathcal{I}}_L$ where

$$\mathcal{I}_L = \frac{1}{2\pi i}\int_{\kappa+i}^{\kappa+iT} \chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)Y(s)X(1-s)ds.$$

It follows that

$$\mathcal{S} = \mathcal{S}_R - \overline{\mathcal{I}}_L + O(NT^{1/2+\varepsilon})$$

with $\mathcal{S}_R$ and $\mathcal{I}_L$ defined as above. To facilitate the evaluation of $\mathcal{I}_L$, we write

$$\frac{\zeta'}{\zeta}(s)\zeta'(s)Y(s) = \sum_{m=1}^{\infty} a(m)m^{-s}$$

where

$$a(m) = \sum_{uvw=m} \Lambda(u)\log(v)y_w.$$

Then

$$\mathcal{I}_L = \sum_{k\le N} \frac{x_k}{k} \sum_{m=1}^{\infty} a(m) \left\{ \frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \chi(1-s)\,(m/k)^{-s}\, ds \right\}. \tag{6.4}$$

To estimate this integral, we invoke the following lemma.

Lemma 6.1. *Let $r, \kappa_0 > 0$. Then*

$$\frac{1}{2\pi i}\int_{\kappa+i}^{\kappa+iT} \chi(1-s) r^{-s}\, ds = I_{[0,\frac{T}{2\pi}]}(r) e(-r) + O\left(r^{-\kappa}\left(T^{\kappa-1/2} + \frac{T^{\kappa+1/2}}{|T-2\pi r| + T^{1/2}} \right)\right)$$

uniformly for $\kappa_0 \le \kappa \le 2$ where $I_{[0,\frac{T}{2\pi}]}(r)$ is the indicator function of $[0, \frac{T}{2\pi}]$. Here, as usual, the function $e(x) = e^{2\pi i x}$.

This is a direct consequence of Lemma 2 of [**4**] and Stirling's formula for the gamma function. Using an argument similar to the proof of Lemma 2 of [**2**], it may be shown that the error term of the above lemma contributes an amount which is $O(NT^{1/2+\varepsilon})$ to (6.4). Thus

$$\mathcal{I}_L = \mathcal{M} + O(NT^{1/2+\varepsilon})$$

where

$$\mathcal{M} = \sum_{k\le N} \frac{x_k}{k} \sum_{m \le \frac{kT}{2\pi}} a(m) e\left(-\frac{m}{k}\right) \tag{6.5}$$

and so

$$\mathcal{S} = \mathcal{S}_R - \overline{\mathcal{M}} + O(NT^{1/2+\varepsilon}). \tag{6.6}$$

Next, we simplify $\mathcal{M}$ by expressing the additive character $e(-m/k)$ in terms of multiplicative characters. We write $m/k = m'/k'$ with $(m', k') = 1$ and invoke the identity

$$e\left(-\frac{m}{k}\right) = e\Big(-\frac{m'}{k'}\Big) = \frac{1}{\phi(k')} \sum_{\chi \bmod k'} \tau(\overline{\chi})\chi(-m')$$

where $\tau(\chi) = \sum_{a=1}^{k'} \chi(a) e(a/k')$ is the Gauss sum for χ modulo k'. Since $\tau(\chi_0) = \mu(k')$ for the principal character modulo χ_0 modulo k', we have

$$e\left(-\frac{m}{k}\right) = \frac{\mu(k')}{\phi(k')} + \frac{1}{\phi(k')} \sum_{\substack{\chi \bmod k' \\ \chi\ne\chi_0}} \tau(\overline{\chi})\chi(-m'). \tag{6.7}$$

The term $\frac{\mu(k')}{\phi(k')}$ provides the main contribution to $\mathcal{M}$ and the remaining terms contribute an amount which will be an error term. It follows that $\mathcal{M} = \mathcal{M}_0 + \mathcal{E}$ where

$$\mathcal{M}_0 = \sum_{k\le N} \frac{x_k}{k} \sum_{m\le \frac{kT}{2\pi}} a(m) \frac{\mu(k/(m,k))}{\phi(k/(m,k))} \tag{6.8}$$

and

$$\mathcal{E} = \sum_{k\le N} \frac{x_k}{k} \sum_{m\le\frac{kT}{2\pi}} a(m) \frac{1}{\phi(k')} \sum_{\substack{\chi \bmod k' \\ \chi\ne\chi_0}} \tau(\overline{\chi})\chi(-m'). \tag{6.9}$$

By (6.6) and the above decomposition of $\mathcal{M}$ it follows that

$$\mathcal{S} = \mathcal{S}_R - \overline{\mathcal{M}_0} - \overline{\mathcal{E}} + O(NT^{1/2+\varepsilon}). \tag{6.10}$$

The remainder of the article is devoted to computing asymptotic expressions for $\mathcal{S}_R$ and $\mathcal{M}_0$ and for providing an upper bound for $\mathcal{E}$. The evaluation of S_R is straighforward, whereas the evalution of $\mathcal{M}_0$ and $\mathcal{E}$ are more involved. The bound for $\mathcal{E}$, which requires GRH, is established using a method of Conrey, Ghosh & Gonek [**1**].

We will prove the following three results which together imply our theorem.

Proposition 6.2. *If x_n, y_n are real and satisfy $|x_n|, |y_n| \ll n^{\varepsilon}$ then*

$$\mathcal{S}_R = \frac{T}{2\pi} \sum_{mn \le N} \frac{x_m y_{mn}}{mn} \Big(\mathcal{Q}_2(\mathscr{L}) - 2\mathcal{Q}_1(\mathscr{L})(\log n) + (\Lambda * \log)(n) \Big) + O(NT^{\varepsilon})$$

where $\mathcal{Q}_2$ and $\mathcal{Q}_1$ are monic polynomials of degrees 2 and 1, respectively.

Proposition 6.3. *If x_n, y_n are real and satisfy $|x_n|, |y_n| \ll n^{\varepsilon}$ then*

$$\mathcal{M}_0 = \frac{T}{2\pi} \sum_{\substack{a,b \le N \\ (a,b)=1}} \frac{s(a;b)}{ab} \sum_{g \le \min(\frac{N}{a}, \frac{N}{b})} \frac{x_{ag} y_{bg}}{g} + \frac{T}{4\pi} \sum_{mn \le N} \frac{x_m y_{mn}}{mn} \log^2 \Big(\frac{T}{2\pi n} \Big) + \mathcal{E}_0 \tag{6.11}$$

where $s(a;b) \ll \Lambda_2(a) + (\log T)\Lambda(a)$, and $\mathcal{E}_0 \ll T \exp(-c\sqrt{\log T})$ for some $c > 0$.

Proposition 6.4. *Assume GRH. If x_n, y_n satisfy (4.2) and $0 < \vartheta < 1/2$, then*

$$\mathcal{E} \ll NT^{\frac{1}{2}+\varepsilon} \tag{6.12}$$

where $\mathcal{E}$ is the sum defined in (6.9).

Assuming these three results, we may now deduce Lemma 4.1.

Proof of Lemma 4.1. By (6.10) and Propositions 6.2, 6.3, and 6.4, we see that

$$\mathcal{S} = \frac{T}{2\pi} \sum_{mn \le N} \frac{x_m y_{mn}}{mn} \Big(\mathcal{Q}_2(\mathscr{L}) - 2\mathcal{Q}_1(\mathscr{L})(\log n) + (\Lambda * \log)(n) - \frac{1}{2}(\mathscr{L} - \log n)^2 \Big)$$
$$- \frac{T}{2\pi} \sum_{\substack{a,b \le N \\ (a,b)=1}} \frac{s(a;b)}{ab} \sum_{g \le \min(\frac{N}{a}, \frac{N}{b})} \frac{x_{ag} y_{bg}}{g} + O(T\exp(-c\sqrt{\log T}) + T^{\vartheta + \frac{1}{2} + \varepsilon})$$

for $0 < \vartheta < \frac{1}{2}$. Observing that

$$-\tfrac{1}{2}(\mathscr{L} - \log n)^2 = -\tfrac{1}{2}\mathscr{L}^2 + \mathscr{L}(\log n) - \tfrac{1}{2}(\log n)^2$$

and then setting $\frac{1}{2}\mathcal{P}_2(x) = \mathcal{Q}_2(x) - \frac{1}{2}x^2$, $\mathcal{P}_1(x) = 2\mathcal{Q}_1(x) - x$, and $r(a;b) = -s(a;b)$, the proof of Lemma 4.1 now follows. □

We now prove the Propositions.

Proof of Proposition 6.2. By (6.3) and the estimate

$$\frac{\chi'}{\chi}(s) = -\log\Big(\frac{t}{2\pi}\Big) + O(t^{-1})$$

which holds uniformly for $\frac{1}{2} \le \sigma \le 2$ and $t \ge 1$, it follows that

$$\mathcal{S}_R \sim \frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \Big(\log^2\Big(\frac{t}{2\pi}\Big)\zeta(s) + 2\log\Big(\frac{t}{2\pi}\Big)\zeta'(s) + \frac{\zeta'}{\zeta}(s)\zeta'(s) \Big) X(s)Y(1-s)\, ds \tag{6.13}$$

with an error term $O(NT^{\varepsilon})$. In order to evaluate $\mathcal{S}_R$, it suffices to evaluate integrals of the form

$$S_v(T;\{b_u\}) = \frac{1}{2\pi i}\int_{\kappa+i}^{\kappa+iT} B(s)X(s)Y(1-s)\log^v\left(\frac{t}{2\pi}\right)ds$$

where $B(s)=\sum_{u=1}^{\infty}|b_u|u^{-s}$ is a Dirichlet series satisfying $B(\sigma)\ll(\sigma-1)^{-\alpha}$ for $\sigma>1$ and some fixed $\alpha>0$. Interchanging the order of summation and integration, it follows that

$$S_v(T;\{b_u\}) = \sum_{u\ge 1}\frac{b_u}{u^{\kappa}}\sum_{m,n\le N}\frac{x_m y_n}{m^{\kappa}n^{1-\kappa}}\frac{1}{2\pi}\int_1^T\left(\frac{n}{um}\right)^{it}\log^v\left(\frac{t}{2\pi}\right)dt. \tag{6.14}$$

The integral may be evaluated as

$$\int_1^T x^{it}\log^v\left(\frac{t}{2\pi}\right)dt = \begin{cases} T\mathcal{P}_v(\mathscr{L})+O(1), & \text{if } x=1,\\ O(\mathscr{L}^v|\log x|^{-1}), & \text{otherwise,}\end{cases}$$

where $\mathcal{P}_v(x)$ is a degree v monic polynomial. It follows from this estimate that the terms for which $n\neq um$ in (6.14) contribute an amount which is $O(NT^{\varepsilon})$ to $\mathcal{S}_R$. The main term, arising from the terms with $n=um$, is thus

$$S_v(T;\{b_u\}) = \frac{T}{2\pi}\mathcal{P}_v(\mathscr{L})\sum_{um\le N}\frac{b_u x_m y_{mu}}{mu} + O(NT^{\varepsilon}).$$

Applying this estimate with $b_u=1, b_u=-2\log u$, and $b_u=(\Lambda*\log)(u)$ to each of the three terms in the integrand in (6.13), respectively, completes the proof of Proposition 6.2. □

The proofs of Propositions 6.3 and 6.4 will require the following result for decomposing Dirichlet series whose coefficients are convolutions.

LEMMA 6.5. *Let $J\in\mathbb{N}$, $J\ge 2$, and let $f_1,f_2,\ldots,f_J$ be arithmetic functions. Let $g\in\mathbb{N}$ and χ any completely multiplicative function. Then*

$$\sum_{\substack{m=1\\(m,k)=1}}^{\infty}\frac{(f_1*f_2*\cdots*f_J)(gm)\chi(m)}{m^s} = \sum_{g_1g_2\cdots g_J=g}\prod_{i=1}^{J}\sum_{\substack{m=1\\(m,kg_1\cdots g_{i-1})=1}}^{\infty}\frac{f_i(g_im)\chi(m)}{m^s}. \tag{6.15}$$

Note that if $i=1$ in the inner sum, then $g_1\cdots g_{i-1}=1$.

Proof. Note that for any two arithmetic functions f_1 and f_2, we have the identity

$$(f_1*f_2)(gm) = \sum_{g_1g_2=g}\sum_{\substack{u_1u_2=m\\(u_2,g_1)=1}} f_1(g_1u_1)f_2(g_2u_2). \tag{6.16}$$

To see this, we make the change of variables $\alpha=g_2u_2$ and notice that the right-hand side equals

$$\sum_{\alpha|gm} f_1(\tfrac{gm}{\alpha})f_2(\alpha)\sum_{\substack{g_2u_2=\alpha\\ g_2|g,\ u_2|m\\ (u_2,\frac{g}{g_2})=1}}1 = \sum_{\alpha|gm} f_1(\tfrac{gm}{\alpha})f_2(\alpha)\sum_{\substack{g_2u_2=\alpha\\ g_2|g,\ u_2|m\\ (\alpha,g)=g_2}}1.$$

Observe that the condition $u_2\mid m$ is extraneous. Thus the last expression equals $(f_1*f_2)(gm)$ since the inner sum possesses the single summand $g_2=(\alpha,g)$. Multiplying (6.16) by $\chi(m)m^{-s}$ and summing over $(m,k)=1$ establishes the Lemma for the case $J=2$. For $J>2$, the result follows by induction. □

Proof of Proposition 6.3. In (6.8) we make the variable change $\ell = (m,k)$, $m = \ell m'$, and $k = \ell k'$ where $(m',k') = 1$ to obtain

$$\mathcal{M}_0 = \sum_{\ell k' \le N} \frac{x_{\ell k'}}{\ell k'} \frac{\mu(k')}{\phi(k')} \sum_{\substack{m' \le \frac{k'T}{2\pi} \\ (m',k')=1}} a(\ell m').$$

Note that by (6.16) we have

$$a(\ell m') = (y * (\Lambda * \log))(\ell m') = \sum_{gh=\ell} \sum_{\substack{vu=m' \\ (v,h)=1}} y_{gv} (\Lambda * \log)(hu).$$

It follows that

$$\mathcal{M}_0 = \sum_{\ell k' \le N} \frac{x_{\ell k'}}{\ell k'} \frac{\mu(k')}{\phi(k')} \sum_{gh=\ell} \sum_{\substack{gv \le N \\ (v,k'h)=1}} y_{gv} \sum_{\substack{uv \le \frac{k'T}{2\pi} \\ (u,k')=1}} (\Lambda * \log)(hu). \tag{6.17}$$

Relabelling k' as k and regrouping terms, we find that

$$\mathcal{M}_0 = \sum_{ghk \le N} \frac{x_{ghk}}{ghk} \frac{\mu(k)}{\phi(k)} \sum_{\substack{v \le \frac{N}{g} \\ (v,kh)=1}} y_{gv} \sum_{\substack{u \le \frac{kT}{2\pi v} \\ (u,k)=1}} (\Lambda * \log)(hu). \tag{6.18}$$

In order to further simplify $\mathcal{M}_0$, it suffices to evaluate the sum

$$S_{h,k}(X) = \sum_{\substack{u \le X \\ (u,k)=1}} (\Lambda * \log)(hu)$$

where $T \ll X \ll TN$.

By Perron's formula, this sum may be written as

$$S_{h,k}(X) = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} \left(\sum_{\substack{u=1 \\ (u,k)=1}}^{\infty} \frac{(\Lambda * \log)(hu)}{u^s} \right) \frac{X^s}{s} ds + O\left(\frac{\tau(h) X (\log X)^6}{U} \right) \tag{6.19}$$

where $c = 1 + (\log X)^{-1}$. In order to estimate the integral, we will now find an alternate expression for the generating function

$$G(s) = \sum_{\substack{u=1 \\ (u,k)=1}}^{\infty} \frac{(\Lambda * \log)(hu)}{u^s}.$$

By Lemma 6.5, it follows that

$$G(s) = \sum_{ab=h} \sum_{(m,k)=1} \frac{\Lambda(am)}{m^s} \sum_{(m,ak)=1} \frac{\log(bm)}{m^s}.$$

If, for $s \in \mathbb{C}$ and $n \in \mathbb{N}$, we define the functions

$$\Phi(s;n) = \prod_{p|n} \left(1 - p^{-s}\right) \quad \text{and} \quad \eta(s;n) = \frac{\Phi'(s;n)}{\Phi(s;n)} = \sum_{p|n} \frac{\log p}{p^s - 1},$$

then a straightforward calculation shows that

$$\sum_{\substack{m=1 \\ (m,k)=1}}^{\infty} \frac{\Lambda(am)}{m^s} = \begin{cases} -\frac{\zeta'}{\zeta}(s) - \eta(s;k), & \text{if } a = 1, \\ \Lambda(a)(1 + \frac{\delta((a,k))}{p^s - 1}), & \text{if } a = p^\ell, \\ 0, & \text{otherwise,} \end{cases}$$

where $\delta(\cdot)$ is the arithmetic function defined by

$$\delta(1) = 1 \quad \text{and} \quad \delta(n) = 0 \ \text{ for } n \geq 2. \tag{6.20}$$

Moreover,

$$\sum_{\substack{m=1 \\ (m,ak)=1}}^{\infty} \frac{\log(bm)}{m^s} = \Phi(s;ak)\left((\log(b) - \eta(s;ak))\zeta(s) - \zeta'(s)\right).$$

Consequently, it follows that

$$\begin{aligned} G(s) = &\Big(-\frac{\zeta'}{\zeta}(s) - \eta(s;k)\Big)\Phi(s;k)\left((\log(h) - \eta(s;k))\zeta(s) - \zeta'(s)\right) \\ &+ \sum_{\substack{ab=h \\ a=p^\ell}} \Lambda(a)\Big(1 + \frac{\delta((a,k))}{p^s - 1}\Big)\Phi(s;ak)\big((\log(b) - \eta(s;ak))\zeta(s) - \zeta'(s)\big). \end{aligned}$$

We denote this last expression as $G(s) = G_1(s) + G_2(s)$. Expanding, we see that

$$G_1(s) = \Phi(s;k)\left(\frac{\zeta'(s)}{\zeta(s)}\zeta'(s) - (\log(h) - 2\eta(s;k))\zeta'(s) + \eta(s;k)(\eta(s;k) - \log(h))\zeta(s)\right).$$

We now simplify $G_2(s)$. If $a = p^\ell$ with p prime and $\ell \in \mathbb{N}$ then

$$\Phi(s;ak) = \begin{cases} \Phi(s;k), & \text{if } (a,k) > 1 \\ \Phi(s;k)(1 - p^{-s}), & \text{if } (a,k) = 1 \end{cases}.$$

and

$$\eta(s;ak) = \begin{cases} \eta(s;k), & \text{if } (a,k) > 1 \\ \eta(s;k) + \frac{\log p}{p^s - 1}, & \text{if } (a,k) = 1 \end{cases}.$$

Applying these observations, we can re-write $G_2(s)$ as

$$\begin{aligned} G_2(s) = &\Phi(s;k) \sum_{\substack{ab=h,\ a=p^\ell \\ (a,k)>1}} \Lambda(a)\left((\log(b) - \eta(s;k))\zeta(s) - \zeta'(s)\right) \\ &+ \Phi(s;k) \sum_{\substack{ab=h,\ a=p^\ell \\ (a,k)=1}} \Lambda(a)\Big(1 + \frac{1}{p^s - 1}\Big)(1 - p^{-s})\left(\Big(\log(b) - \eta(s;k) - \frac{\log p}{p^s - 1}\Big)\zeta(s) - \zeta'(s)\right) \\ = &\Phi(s;k)\left(\sum_{ab=h} \Lambda(a)\left((\log(b) - \eta(s;k))\zeta(s) - \zeta'(s)\right) - \Big(\sum_{\substack{ab=h,\ a=p^\ell \\ (a,k)=1}} \frac{\Lambda(a)\log p}{p^s - 1}\Big)\zeta(s)\right) \\ = &\Phi(s;k)\left(-\log(h)\zeta'(s) + \Big((\Lambda * \log)(h) - \eta(s;k)\log h - \sum_{\substack{ab=h,\ a=p^\ell \\ (a,k)=1}} \frac{\Lambda(a)\log p}{p^s - 1}\Big)\zeta(s)\right). \end{aligned}$$

Combining these new expressions for $G_1(s)$ and $G_2(s)$, we obtain

$$\begin{aligned} G(s) = \Phi(s;k)\Bigg(&\frac{\zeta'(s)}{\zeta(s)}\zeta'(s) - (2\log(h) - 2\eta(s;k))\zeta'(s) \\ &+ \Big((\Lambda * \log)(h) + \eta(s;k)^2 - 2\eta(s;k)\log(h) - \sum_{\substack{ab=h,\ a=p^\ell \\ (a,k)=1}} \frac{\Lambda(a)\log p}{p^s - 1}\Big)\zeta(s)\Bigg). \end{aligned} \tag{6.21}$$

With this formula for $G(s)$, we may now complete the evaluation of $S_{h,k}(X)$. By (6.19) and the calculus of residues, it follows that

$$S_{h,k}(X) = \operatorname*{Res}_{s=1}\left\{G(s)\frac{X^s}{s}\right\} + \frac{1}{2\pi i}\int_{\mathscr{C}} G(s)\frac{X^s}{s}ds + O\left(\frac{\tau(h)X(\log X)^6}{U}\right) \tag{6.22}$$

where $\mathscr{C}$ denotes the polygonal path of the three line segments connecting the vertices $c+iU, a+iU, a-iU$, and $c-iU$ and where $a = 1-1/\log U$. We note that when U is sufficiently large, $\mathscr{C}$ lies within the zero-free region of $\zeta(s)$. We choose $U = \exp(c'\sqrt{\log X})$ for an appropriate $c' > 0$, use (6.21), and apply well known bounds for $\frac{\zeta'}{\zeta}(s)$, $\zeta^{(j)}(s)$ for $j \geq 0$ along $\mathscr{C}$ to obtain

$$S_{h,k}(X) = \operatorname*{Res}_{s=1}\left\{G(s)\frac{X^s}{s}\right\} + O\Big(\tau(h)X\exp\big(-c''\sqrt{\log X}\big)\Big) \tag{6.23}$$

for some $c'' > 0$ and $k \leq N$.

To complete the proof of the proposition, it remains to compute the residue on the right-hand side of (6.23). It should be observed that the principal terms in the residue will arise from the residue at $s = 1$ of

$$\Phi(s;k)\left(\frac{\zeta'(s)}{\zeta(s)}\zeta'(s) - 2\log(h)\zeta'(s) + (\Lambda * \log)(h)\zeta(s)\right)\frac{X^s}{s}.$$

Since $\Phi(1;k) = \frac{\phi(k)}{k}$, it may be seen that the leading-order term of the residue of this expression is

$$\frac{X\phi(k)}{k}\Big(\tfrac{1}{2}(\log X)^2 + 2\log(X)\log h + (\Lambda * \log)(h)\Big).$$

However, by a standard residue calculation, the residue of $G(s)\frac{X^s}{s}$ may be computed exactly and we find that

$$\begin{aligned} S_{h,k}(X) = &\frac{X\phi(k)}{k}\Big(\tfrac{1}{2}(\log X)^2 + 2\log(X)\log h + (\Lambda * \log)(h) + \alpha(k)\log(X) + \beta(h,k)\Big) \\ &+ O(\tau(h)X\exp(-c'\sqrt{\log X})) \end{aligned}$$

for certain arithmetic functions $\alpha(k)$ and $\beta(h,k)$ which can be computed explicitly.† Inserting this expression for $S_{h,k}(X)$ into (6.18) with $X = \frac{kT}{2\pi v}$ yields

$$\begin{aligned} \mathcal{M}_0 \sim &\frac{T}{2\pi}\sum_{ghk \leq N}\frac{x_{ghk}\mu(k)}{ghk}\sum_{\substack{gv \leq N \\ (v,kh)=1}}\frac{y_{gv}}{v} \\ &\times\left(\frac{1}{2}\Big(\log\frac{kT}{2\pi v}\Big)^2 + 2\log\Big(\frac{kT}{2\pi v}\Big)\log h + (\Lambda * \log)(h) + \alpha(k)\log\Big(\frac{kT}{2\pi v}\Big) + \beta(h,k)\right) \end{aligned}$$

†In fact, it can be shown that $\alpha(k) = -\eta(1;k) - 1 - \gamma_0$ and

$$\beta(h,k) = \eta(1;k)\Big(\log(h) - \tfrac{1}{2}\eta(1;k) - \gamma_0\Big) - 2\log h - \tfrac{3}{2}\eta'(1;k) - \sum_{\substack{a|h,\, a=p^\ell \\ (a,k)=1}}\frac{\Lambda(a)\log p}{p-1} + C_0$$

where $C_0 = 3\gamma_1 + \gamma_0 + 1 + \gamma_0^2$. Here γ_0 is the Euler-Mascheroni constant and $-\gamma_1$ is the constant term in the Laurent series expansion of $\zeta'(s)$ about $s = 1$. The Maple code which computes $\alpha(k)$, $\beta(h,k)$, and the residue of $G(s)\frac{X^s}{s}$ is available on request from the authors.

with an error term $O(T\exp(-c\sqrt{\log T}))$ for some $c>0$. By the variable change $hk=a$ and by relabelling the variable v as b we see that

$$\mathcal{M}_0 = \frac{T}{2\pi}\sum_{ga\le N}\frac{x_{ga}}{ga}\sum_{\substack{gb\le N\\(a,b)=1}}\frac{y_{gb}}{b}s_0(a;b) + O(T\exp(-c\sqrt{\log T}))$$

where

$$s_0(a;b) = \sum_{hk=a}\mu(k)\left(\frac{1}{2}\Big(\log\frac{kT}{2\pi b}\Big)^2 + 2\log\Big(\frac{kT}{2\pi b}\Big)\log h + (\Lambda*\log)(h) + \alpha(k)\log\Big(\frac{kT}{2\pi b}\Big) + \beta(h,k)\right).$$

In order to simplify $s_0(a;b)$, we decompose $(\log\frac{kT}{2\pi b})^2 = \log^2 k + 2\log k\log\frac{T}{2\pi b} + (\log\frac{T}{2\pi b})^2$ and $\log(\frac{kT}{2\pi b}) = \log(k) + \log(\frac{T}{2\pi b})$ and then invoke the identities

$$\begin{aligned}
\Lambda_2(a) - 2\log(a)\Lambda(a) &= \sum_{hk=a}\mu(k)\log^2 k,\\
-\Lambda(a) &= \sum_{hk=a}\mu(k)\log k,\\
\delta(a) &= \sum_{hk=a}\mu(k),\\
(\log a)\Lambda(a) - \Lambda_2(a) &= \sum_{hk=a}\mu(k)\log k\log h,\\
\Lambda(a) &= \sum_{hk=a}\mu(k)\log h, \quad\text{and}\\
\Lambda_2(a) - \Lambda(a)\log(a) &= \sum_{hk=a}\mu(k)(\Lambda*\log)(h).
\end{aligned} \tag{6.24}$$

Furthermore, it may be shown that

$$\sum_{hk=a}\mu(k)\alpha(k)\log\Big(\frac{T}{2\pi b}\Big) \ll \Lambda_2(a) + (\log T)\Lambda(a)$$

and

$$\sum_{hk=a}\mu(k)\beta(h,k) \ll \Lambda_2(a) + (\log T)\Lambda(a).$$

These last two estimates and identities in (6.24) imply that

$$s_0(a;b) = \frac{1}{2}\log^2\Big(\frac{T}{2\pi b}\Big)\delta(a) + O\Big(\Lambda_2(a) + (\log T)\Lambda(a)\Big). \tag{6.25}$$

Thus

$$\begin{aligned}
\mathcal{M}_0 = \frac{T}{2\pi}&\sum_{\substack{a,b\le N\\(a,b)=1}}\frac{s(a;b)}{ab}\sum_{g\le\min(\frac{N}{a},\frac{N}{b})}\frac{x_{ga}y_{gb}}{g} + \frac{T}{4\pi}\sum_{gb\le N}\frac{x_g y_{gb}}{gb}\log^2\Big(\frac{T}{2\pi b}\Big)\\
&+ O(T\exp(-c\sqrt{\log T}))
\end{aligned}$$

where $s(a;b) = s_0(a;b) - \frac{1}{2}\log^2(\frac{T}{2\pi b})$. Moreover, by (6.25)

$$s(a;b) \ll \Lambda_2(a) + (\log T)\Lambda(a).$$

This establishes Proposition 6.3. □

Proof of Proposition 6.4. Recall that

$$\mathcal{E} = \sum_{k\le N}\frac{x_k}{k}\mathcal{E}_k$$

where

$$\mathcal{E}_k = \sum_{m \le \frac{kT}{2\pi}} a(m) \frac{1}{\phi(k')} \sum_{\substack{\chi \bmod k' \\ \chi \ne \chi_0}} \tau(\overline{\chi})\chi(-m'). \tag{6.26}$$

By Perron's formula, with $\alpha = 1 + \left(\log \frac{kT}{2\pi}\right)^{-1}$,

$$\mathcal{E}_k = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \mathcal{Q}^*(s, \alpha, k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} + O(kT^{\varepsilon})$$

where

$$\mathcal{Q}^*(s, \alpha, k) = \sum_{m=1}^{\infty} a(m) \left(\frac{1}{\phi(k')} \sum_{\substack{\chi \bmod k' \\ \chi \ne \chi_0}} \tau(\overline{\chi})\chi(-m') \right) m^{-s}, \tag{6.27}$$

$k' = k/(k, m)$, and $m' = m/(k, m)$. Shortly, we shall establish that $\mathcal{Q}^*(s, \alpha, k)$ is holomorphic and satisfies the bound

$$|\mathcal{Q}^*(s, \alpha, k)| \ll \sqrt{k} T^{\varepsilon} \tag{6.28}$$

in the region $\Re(s) \ge \frac{1}{2} + \varepsilon$. Assuming these facts, it follows from a contour shift and Cauchy's theorem that

$$\begin{aligned} \mathcal{E}_k &= \frac{1}{2\pi i} \int_{\frac{1}{2}+\varepsilon - iT}^{\frac{1}{2}+\varepsilon + iT} \mathcal{Q}^*(s, \alpha, k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} + O_j(k^{\frac{3}{2}} T^{\varepsilon}) \\ &\ll (\sqrt{k} T^{\varepsilon})(kT)^{\frac{1}{2}+\varepsilon} \int_{-T}^{T} \frac{dt}{|\frac{1}{2} + \varepsilon + it|} + k^{\frac{3}{2}} T^{\varepsilon} \\ &\ll k^{1+\varepsilon} T^{\frac{1}{2}+\varepsilon}. \end{aligned}$$

Inserting this expression in (6.9) yields

$$\mathcal{E} \ll T^{\frac{1}{2}+2\varepsilon} \sum_{k \le N} |x_k| \ll N T^{\frac{1}{2}+3\varepsilon}$$

since $|x_k| \ll T^{\varepsilon}$ and $k \ll T$. Since $\varepsilon > 0$ was abitrary, replacing 3ε with ε yields Proposition 6.4 based upon our assumptions for $\mathcal{Q}^*(s, \alpha, k)$.

To complete the argument it suffices to establish the holomorphy of $\mathcal{Q}^*(s, \alpha, k)$ in the region $\Re(s) \ge \frac{1}{2} + \varepsilon$ and to establish the bound (6.28). We begin by simplifying $\mathcal{Q}^*(s, \alpha, k)$. In the definition (6.27), we put $g = (k, m)$. Then, writing $m = gm'$, we obtain

$$\begin{aligned} \mathcal{Q}^*(s, \alpha, k) &= \sum_{g|k} \frac{1}{\phi(k/g)} \sum_{\substack{m=1 \\ (k,m)=g}} a(m) \Big(\sum_{\substack{\chi \bmod \frac{k}{g} \\ \chi \ne \chi_0}} \tau(\overline{\chi})\chi(-\frac{m}{g}) \Big) m^{-s} \\ &= \sum_{g|k} \frac{1}{\phi(k/g) g^s} \sum_{\substack{\chi \bmod \frac{k}{g} \\ \chi \ne \chi_0}} \tau(\overline{\chi})\chi(-1) \sum_{\substack{m'=1 \\ (m', \frac{k}{g})=1}}^{\infty} \frac{a(gm')\chi(m')}{(m')^s}. \end{aligned} \tag{6.29}$$

We now simplify the inner Dirichlet series using Lemma 6.5. Applying this lemma to $a = y * \log * \Lambda$, it follows that the innermost Dirichlet series over m' in the last line of (6.29) can be written as

$$\sum_{\substack{m'=1 \\ (m', \frac{k}{g})=1}}^{\infty} \frac{a(gm')\chi(m')}{(m')^s} = \sum_{g_1 g_2 g_3 = g} A_1(s, \tfrac{k}{g}, g_1) A_2(s, \tfrac{k}{g} g_1, g_2) A_3(s, \tfrac{k}{g} g_1 g_2, g_3) \tag{6.30}$$

where, for $u,v\in\mathbb{N}$,

$$A_1(s,u,v)=\sum_{\substack{m\le\frac{N}{v}\\(m,u)=1}}\frac{y_{mv}\chi(m)}{m^s},$$

$$A_2(s,u,v)=\sum_{\substack{m=1\\(m,u)=1}}^{\infty}\frac{\log(mv)\chi(m)}{m^s},$$

and

$$A_3(s,u,v)=\sum_{\substack{m=1\\(m,u)=1}}^{\infty}\frac{\Lambda(mv)\chi(m)}{m^s}.$$

At this point we simplify the functions A_i for $i=1,2,3$. Recall that $y_n=\tau_l(n;N^{\frac{1}{J}})$ where $J=\max(j,l)$. Let $I(n)$ denote the indicator function of the interval $[1,N^{\frac{1}{J}}]$ and observe that $y_n=(I*I*\cdots*I)(n)$ where I is convolved with itself $J-1$ times. By another application of Lemma 6.5, it follows that

$$A_1(s,u,v)=\sum_{v_1\cdots v_J=v}\prod_{i=1}^{J}\sum_{\substack{m=1\\(m,uv_1\cdots v_{i-1})=1}}^{\infty}\frac{I(mv_i)\chi(m)}{m^s}=\sum_{v_1\cdots v_J=v}\prod_{i=1}^{J}\sum_{\substack{m<\frac{N^{\frac{1}{J}}}{v_i}\\(m,uv_1\cdots v_{i-1})=1}}\frac{\chi(m)}{m^s}. \tag{6.31}$$

In addition, we have

$$\begin{aligned}A_2(s,u,v)&=(\log v)\sum_{\substack{m=1\\(m,u)=1}}^{\infty}\frac{\chi(m)}{m^s}+\sum_{\substack{m=1\\(m,u)=1}}^{\infty}\frac{\log(m)\chi(m)}{m^s}\\&=(\log v)\prod_{p\mid u}\Big(1-\frac{\chi(p)}{p^s}\Big)L(s,\chi)-\frac{d}{dz}\Big(\prod_{p\mid u}\Big(1-\frac{\chi(p)}{p^{s+z}}\Big)L(s+z,\chi)\Big)\Big|_{z=0}\\&=(\log v)F(s,u,\chi)L(s,\chi)-\frac{d}{dz}\Big(F(s+z,u,\chi)L(s+z,\chi)\Big)\Big|_{z=0}\end{aligned} \tag{6.32}$$

where

$$F(s,u,\chi)=\prod_{p\mid u}\Big(1-\frac{\chi(p)}{p^s}\Big).$$

Also,

$$A_3(s,u,v)=\begin{cases}-\frac{L'}{L}(s,\chi)-\frac{F'}{F}(s,u,\chi), & \text{if } v=1,\\ \frac{\log p}{1-\chi(p)p^{-s}}, & \text{if } v=p^\ell \text{ and } (u,p)=1,\\ \log p, & \text{if } v=p^\ell \text{ and } p\mid u,\\ 0, & \text{otherwise.}\end{cases} \tag{6.33}$$

We remark that $A_1(s,u,v)$ is clearly holomorphic and, also, that $A_2(s,u,v)$ is holomorphic since χ is a non-principal character. Moreover, assuming GRH, $A_3(s,u,v)$ is a holomorphic function in the region $\Re(s)\ge\frac12+\varepsilon$. It follows that $\mathcal{Q}^*(s,\alpha,k)$ is holomorphic in the region $\Re(s)\ge\frac12+\varepsilon$.

We now provide bounds for the A_i (for $i=1,2,3$) when $\Re(s)\ge\frac12+\varepsilon$. These bounds will depend on the standard convexity estimates which, on GRH, state that

$$|L(s,\chi)|,\Big|\frac{L'}{L}(s,\chi)\Big|\ll_\varepsilon(1+|t|)^\varepsilon\Big(\frac{k}{g}\Big)^\varepsilon \tag{6.34}$$

for $\Re(s) \geq \frac{1}{2} + \varepsilon$. Also we require the bounds

$$\begin{aligned} |F(s,u,\chi)| &\leq \prod_{p|u}\Big(1+\frac{1}{\sqrt{p}}\Big) \leq \tau(u) \ll u^{\varepsilon}, \\ \Big|\frac{F'}{F}(s,u,\chi)\Big| &= \Big|\sum_{p|u}\frac{\chi(p)\log p}{p^s-\chi(p)}\Big| \ll \sum_{p|u}\log p \ll u^{\varepsilon}, \\ |F'(s,u,\chi)| &\ll u^{\varepsilon} \end{aligned} \tag{6.35}$$

which are valid for $\Re(s) \geq \frac{1}{2}$. Using (6.31), we see that in order to bound $A_1(s,u,v)$, it suffices to establish a bound for the character sum

$$\sum_{\substack{m<X \\ (m,V)=1}} \chi(m)m^{-s}.$$

Assuming GRH, a bound for this sum without the condition $(m,V)=1$ is established on page 221 of [**2**]. The method is to apply Perron's formula, apply a contour shift, and invoke the Lindelöf bounds in (6.34) to bound the various contours. Following the argument in [**2**], assuming GRH, yields the similar estimate

$$\Bigg|\sum_{\substack{m<X \\ (m,V)=1}} \frac{\chi(m)}{m^s}\Bigg| \ll_{\varepsilon} \tau(V)T^{\varepsilon}\Big(\frac{k}{g}\Big)^{\varepsilon}X^{\varepsilon}\log(T)$$

for $\Re(s) \geq \frac{1}{2}+\varepsilon$ and $|\Im(s)| \ll T$. Inserting this into (6.31) yields

$$|A_1(s,u,v)| \ll \tau_J(v)T^{\varepsilon} \tag{6.36}$$

assuming GRH. It follows from (6.34) and (6.35) that

$$|A_2(s,u,v)|,\ |A_3(s,u,v)| \ll T^{\varepsilon} \tag{6.37}$$

for $\Re(s) \geq \frac{1}{2}+\varepsilon$, $|\Im(s)| \ll T$, and $u,v \leq T$. Thus by (6.30), (6.36), and (6.37) we deduce that the innermost Dirichlet series in the last line of (6.29) is

$$\Bigg|\sum_{\substack{m'=1 \\ (m',\frac{k}{g})=1}}^{\infty} \frac{a(gm')\chi(m')}{(m')^s}\Bigg| \ll T^{\varepsilon} \tag{6.38}$$

where $\varepsilon > 0$ may be different than the ε appearing in the bounds for $A_i(s,u,v)$ for $i=1,2,3$. Inserting this last expression back into (6.29), we conclude that

$$\begin{aligned} |\mathcal{Q}^*(s,\alpha,k)| &\ll \sum_{g|k}\frac{1}{\phi(k/g)g^{\frac{1}{2}}}\sum_{\substack{\chi \bmod \frac{k}{g} \\ \chi\neq\chi_0}}|\tau(\overline{\chi})|T^{\varepsilon} \\ &\ll T^{\varepsilon}\sum_{g|k}\frac{1}{\phi(k/g)g^{\frac{1}{2}}}\phi(k/g)\sqrt{k/g} \\ &\ll \sqrt{k}T^{\varepsilon}. \end{aligned}$$

This establishes the bound for $\mathcal{Q}^*(s,\alpha,k)$ in (6.28) which, in turn, completes the proof of Proposition 6.4. □

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