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CENTRAL VALUES OF DERIVATIVES OF DIRICHLET L-FUNCTIONS

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Let $\mathscr{C}_q^+$ be the set of even, primitive Dirichlet characters (mod q). Using the mollifier method, we show that $L^{(k)}(\frac{1}{2},\chi) \neq 0$ for almost all the characters $\chi \in \mathscr{C}_q^+$ when k and q are large. Here $L(s,\chi)$ is the Dirichlet L-function associated to the character χ.

Keywords: Dirichlet L-functions; non-vanishing of L-functions; mollifier method.

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1. Introduction and Statement of the Main Result

An important topic in number theory is the behavior of families of L-functions and their derivatives inside the critical strip. In particular, questions concerning the order of vanishing of L-functions at special points on the critical line have received a great deal of attention. In the case of Dirichlet L-functions, it is widely believed that $L(\frac{1}{2},\chi) \neq 0$ for all primitive characters χ. For quadratic characters χ, this appears to have been first conjectured by Chowla (see [3, Chap. 8]).

Though a proof of the non-vanishing of Dirichlet L-functions at the central point, $s = 1/2$, has remained elusive, there has been considerable progress in showing that $L(\frac{1}{2},\chi)$ is very often non-zero within various families of characters χ. In [10], Iwaniec and Sarnak show that at least $1/3$ of Dirichlet L-functions in the family of even primitive characters, to a large modulus q, do not vanish at the central point. This improves upon earlier work of Balasubramanian and Murty [1]. Soundararajan [16] has shown that at least $7/8$ of the central values in the family of quadratic Dirichlet L-functions are non-zero. More recently, Baier and Young [2] consider the family of Dirichlet L-functions associated to cubic and sextic characters and show that infinitely many (though not a positive proportion) of these functions are not zero at the central point.

In [15], Michel and VanderKam consider the behavior of the derivatives of completed Dirichlet L-functions, $\Lambda(s,\chi)$, at the central point. (See Sec. 2, below, for a definition.) In particular, they show that for $\varepsilon > 0$ and q sufficiently large depending on ε, the inequality

$$\sum_{\substack{\chi\in\mathscr{C}_q^+ \\ \Lambda^{(k)}(\frac{1}{2},\chi)\neq 0}} 1 \;\geq\; (P_k-\varepsilon)\cdot \sum_{\chi\in\mathscr{C}_q^+} 1 \tag{1.1}$$

holds, where the proportion

$$P_k = \frac{2}{3} - \frac{1}{36k^2} - \frac{c}{k^4}$$

for some absolute constant $c > 0$. As k tends to infinity, the proportion P_k approaches two thirds. This is analogous to a result of Conrey [4], who shows that almost all of the zeros of the kth derivative of the Riemann ξ-function are on the critical line, and to a result of Kowalski *et al.* [13] who show that almost half of the set $\{\Lambda^{(k)}(\frac{1}{2},f)\}$ is non-zero, where f runs over the set of primitive Hecke eigenforms of weight 2 relative to $\Gamma_0(q)$. This last result is best possible because half of these forms are even and half are odd. However, unlike the results in [4, 13], the inequality in (1.1) is not best possible since it is expected that $P_k = 1$ for every positive integer k.

In contrast to [15], we study the behavior of the functions $L^{(k)}(s,\chi)$, the derivatives of Dirichlet L-functions, at $s=\frac{1}{2}$. When k and q are sufficiently large, we show that $L^{(k)}(\frac{1}{2},\chi)\neq 0$ for almost all of the even, primitive characters χ. As is the case in [4, 13], our result is asymptotically best possible as k tends to infinity.

Theorem 1.1. *Let $k\in\mathbb{N}$. Then, for $\varepsilon>0$ and q sufficiently large* (*depending on ε*), *we have*

$$\sum_{\substack{\chi\in\mathscr{C}_q^+ \\ L^{(k)}(\frac{1}{2},\chi)\neq 0}} 1 \;\geq\; (P_k^*-\varepsilon)\cdot \sum_{\chi\in\mathscr{C}_q^+} 1, \tag{1.2}$$

where the proportion

$$P_k^* = 1 - \frac{1}{16k^2} - \frac{c}{k^4} \tag{1.3}$$

for some absolute constant $c>0$. In particular, $P_1^*\geq 0.7544$, $P_2^*\geq 0.9083$, $P_3^*\geq 0.9642$, $P_4^*\geq 0.9853$, $P_5^*\geq 0.9935$, and $P_{25}^*\geq 0.9999$.

Theorem 1.1 confirms a prediction of Conrey and Snaith which arises from the L-functions Ratios Conjectures (see [8, §8.1]). Their heuristic is based upon studying the behavior of the mollified moments of the derivatives of the Riemann zeta-function in t-aspect which they conjecture should behave similarly to the mollified moments of the derivatives of Dirichlet L-functions at the central point in q-aspect. This is in agreement with the conjectures of Keating and Snaith [11, 12] that suggest that both of these families of L-functions, the Riemann zeta-function in t-aspect and Dirichlet L-functions in q-aspect, should have the same underlying "unitary" symmetry and so their (mollified) moments should behave similarly. See [5] for a detailed discussion of these ideas. In particular, our Proposition 2.2 is a q-analogue of a result of Conrey and Ghosh[a] who computed the mollified moments of the derivatives of the Riemann zeta-function on the critical line.

We remark that Theorem 1.1 does not improve upon the main result of [15]. In fact, for $k \in \mathbb{N}$, the zeros of the functions $L^{(k)}(s,\chi)$ and $\Lambda^{(k)}(s,\chi)$ are expected to behave quite differently. To illustrate this point, let χ be a primitive character and assume that the Riemann Hypothesis (RH_χ) holds for the function $L(s,\chi)$. Then all the non-trivial zeros of $L(s,\chi)$ and all the zeros of $\Lambda(s,\chi)$ lie on the critical line $\text{Re } s = \frac{1}{2}$. In addition, $L(s,\chi)$ has an infinite number of trivial zeros on the negative real axis. Under the RH_χ, one can prove that all the zeros of $\Lambda^{(k)}(s,\chi)$ lie on the line $\text{Re } s = \frac{1}{2}$. In contrast, it can be shown that all but possibly a finite number of the non-real zeros of $L^{(k)}(s,\chi)$ are forced to lie in the half-plane $\text{Re } s \geq \frac{1}{2}$ and it is very likely the case that none of these zeros lie on the critical line.[b] In particular, it is reasonable to conjecture that $L^{(k)}(\frac{1}{2},\chi) \neq 0$ for all primitive characters χ and all $k \in \mathbb{N}$. However, if χ is an even, real-valued, primitive (i.e. quadratic) character, then the functional equation for $L(s,\chi)$ states that $\Lambda(s,\chi) = \Lambda(1-s,\chi)$. It follows from this that $\Lambda^{(k)}(\frac{1}{2},\chi) = 0$ whenever k is odd. Thus, the analogous conjecture for $\Lambda^{(k)}(\frac{1}{2},\chi)$ fails for infinitely many values of k and infinitely many characters χ.

1.1. *Notation and conventions*

We say a Dirichlet character $\chi \pmod q$ is even if $\chi(-1) = 1$. We let $\mathscr{C}_q$ denote the set of primitive characters $\pmod q$ and let $\mathscr{C}_q^+$ denote the subset of characters in $\mathscr{C}_q$ which are even. We put $\varphi^+(q) = \frac{1}{2}\varphi^*(q)$ where

$$\varphi^*(q) = \sum_{k|q} \varphi(k)\mu\Big(\frac{q}{k}\Big) = \big|\mathscr{C}_q\big|;$$

the proof of this appears in Lemma 4.1, below. It is not difficult to show that $\big|\mathscr{C}_q^+\big| = \varphi^+(q) + O(1)$. In addition, we write $\sum^+_{\chi \,(\text{mod } q)}$ to indicate that the summation is restricted to $\chi \in \mathscr{C}_q^+$ and we write $\sum^\star_{a \,(\text{mod } q)}$ and $\sum^\star_n$ to indicate that the summation is restricted to the residues $a \pmod q$ which are coprime to q and to n which are relatively prime to q, respectively.

[a]See [6, Eq. (7)].

[b]We can show that if q is sufficiently large, then the only zeros of $L'(s,\chi)$ on the critical line are the multiple zeros of $L(s,\chi)$. However, it is believed that the zeros of $L(s,\chi)$ are simple.

2. The Mollified Moments of $L^{(k)}(\frac{1}{2}, \chi)$

As may be expected, we prove Theorem 1.1 by computing certain mollified first and second moments of $L^{(k)}(\frac{1}{2}, \chi)$ over the characters $\chi \in \mathscr{C}_q^+$ and then we use Cauchy's inequality.

For $\chi \in \mathscr{C}_q^+$, the Dirichlet L-function $L(s, \chi)$ satisfies the functional equation

$$\Lambda(s, \chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \varepsilon_\chi \Lambda(1 - s, \overline{\chi}), \tag{2.1}$$

where $\bar{\chi}$ is conjugate character of χ, $\varepsilon_\chi = \tau(\chi) q^{-1/2}$, and $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{a (\text{mod } q)} \chi(a) e\left(\frac{a}{q}\right); \quad e(x) = e^{2\pi i x}.$$

Note that $|\varepsilon_\chi| = 1$ and, since χ is even, $\overline{\tau(\chi)} = \tau(\bar{\chi})$. For each $\chi \in \mathscr{C}_q^+$, we let

$$M(\chi) = M(\chi, P, y) := \sum_{n \le y} \frac{\mu(n)\chi(n)}{\sqrt{n}} P\left(\frac{\log y/n}{\log y}\right), \tag{2.2}$$

where P is an arbitrary polynomial satisfying the conditions $P(0) = 0$ and $P(1) = 1$. The purpose of the function $M(\chi)$ is to smooth out or "mollify" the large values of $L^{(k)}(\frac{1}{2}, \chi)$ as we average over $\chi \in \mathscr{C}_q^+$. Since $|\varepsilon_\chi L^{(k)}(\frac{1}{2}, \bar{\chi})| = |L^{(k)}(\frac{1}{2}, \chi)|$, if we let

$$S_1(k, q) = \sum_{\chi \,(\text{mod } q)}^{+} \varepsilon_\chi L^{(k)}\left(\frac{1}{2}, \bar{\chi}\right) M(\chi) \tag{2.3}$$

and

$$S_2(k, q) = \sum_{\chi \,(\text{mod } q)}^{+} \left| L^{(k)}\left(\frac{1}{2}, \chi\right)\right|^2 |M(\chi)|^2, \tag{2.4}$$

then Cauchy's inequality implies that

$$\sum_{\substack{\chi \,(\text{mod } q) \\ L^{(k)}(\frac{1}{2}, \chi) \neq 0}}^{+} 1 \ge \frac{|S_1(k, q)|^2}{S_2(k, q)}. \tag{2.5}$$

Thus, we require a lower bound for $|S_1(k, q)|$ and an upper bound for $S_2(k, q)$. The following propositions provide such estimates.

Proposition 2.1. *Let $k \in \mathbb{N}$. Then, for $y = q^\vartheta$ and $0 < \vartheta < 1$, we have*

$$S_1(k, q) = (-1)^k \varphi^+(q) \log^k q (1 + O((\log q)^{-1})),$$

where the implied constant depends on ϑ and k.

Proposition 2.2. *Let k be a positive integer and $\varepsilon > 0$ be arbitrary. Then, for $y = q^\vartheta$ and $0 < \vartheta < \frac{1}{2}$, we have*

$$S_2(k, q) = \mathcal{C}_k(\vartheta)\ \varphi^+(q) \log^{2k} q (1 + O((\log q)^{-1+\varepsilon})),$$

where

$$\mathcal{C}_k(\vartheta) = \frac{\vartheta^{-1}}{2k+1} \int_0^1 P'(x)^2 \, dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1} \int_0^1 P(x)^2 \, dx,$$

and the implied constant depends on ϑ, ε and k.

It is clear from (2.5) and the propositions that in order to prove Theorem 1.1 we need to choose the polynomial P, for each $k \geq 1$, which minimizes the constant $\mathcal{C}_k(\vartheta)$. This is done in Sec. 6. It turns out that except for a term which is exponentially small (as a function of k), the optimal choice of P is independent of the choice of ϑ. This is not surprising, since similar phenomena have been observed when mollifying high derivatives of the Riemann zeta-function and the Riemann ξ-function on the critical line, and also when mollifying high derivatives of families of L-functions at the central point (see [4, 6, 13, 15]).

3. Proof of Proposition 2.1

In this section we establish Proposition 2.1. The result we require is implicit in [15, §3, p. 135] where it is shown that[c]

$$\sum_{\chi \,(\mathrm{mod}\ q)}^{+} \Lambda^{(k)}\left(\frac{1}{2}, \chi\right) M(\chi) = \varphi^{+}(q)\Gamma\left(\frac{1}{4}\right) \hat{q}^{1/2} \log^k \hat{q}(1 + O((\log q)^{-1})) \tag{3.1}$$

for $k \in \mathbb{N}$ and $0 < \vartheta < 1$. Here $\hat{q} = \sqrt{q/\pi}$ and the implied constant depends on ϑ. From (2.1), we see that

$$\varepsilon_\chi L(s, \bar{\chi}) = H_q(s)\Lambda(1-s, \chi), \quad \text{where } H_q(s) = \frac{\hat{q}^{-s}}{\Gamma\left(\frac{s}{2}\right)}. \tag{3.2}$$

Using well-known estimates for the gamma function, it follows that

$$H_q^{(k)}\left(\frac{1}{2}\right) = (-1)^k \frac{\hat{q}^{-1/2}}{\Gamma\left(\frac{1}{4}\right)} \log^k \hat{q}(1 + O_k((\log q)^{-1})) \tag{3.3}$$

for each $k \in \mathbb{N}$. Now, combining (3.1)–(3.3) and using the Leibniz formula for differentiation, we find that

$$\begin{aligned}
&\sum_{\chi (\mathrm{mod}\ q)}^{+} \varepsilon_\chi L^{(k)}\left(\frac{1}{2}, \bar{\chi}\right) M(\chi) \\
&\quad = \sum_{\chi (\mathrm{mod}\ q)}^{+} \sum_{\ell=0}^{k} \binom{k}{\ell} H_q^{(\ell)}\left(\frac{1}{2}\right) (-1)^{k-\ell} \Lambda^{(k-\ell)}\left(\frac{1}{2}, \chi\right) M(\chi)
\end{aligned}$$

[c]It follows from the functional equation for $\Lambda(s, \chi)$ that the quantity $\mathscr{L}(P_k)$ in [15, §3] is equal to $2\sum_{\chi (\mathrm{mod}\ q)}^{+} \Lambda^{(k)}\left(\frac{1}{2}, \chi\right) M(\chi)$.

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} H_q^{(\ell)}\left(\frac{1}{2}\right) \sideset{}{^+}\sum_{\chi (\mathrm{mod}\ q)} \Lambda^{(k-\ell)}\left(\frac{1}{2}, \chi\right) M(\chi)$$

$$= (-1)^k \sum_{\ell=0}^{k} \binom{k}{\ell} \varphi^+(q) \log^k \hat{q}(1 + O((\log q)^{-1}))$$

$$= (-1)^k 2^k \varphi^+(q) \log^k \hat{q}(1 + O((\log q)^{-1})),$$

where the implied constant depends on ϑ and k. Since $2\log\hat{q} = \log q + O(1)$, we can conclude that

$$\sideset{}{^+}\sum_{\chi (\mathrm{mod}\ q)} \varepsilon_\chi L^{(k)}\left(\frac{1}{2}, \bar{\chi}\right) M(\chi) = (-1)^k\ \varphi^+(q) \log^k q(1 + O((\log q)^{-1})).$$

This establishes Proposition 2.1.

4. Some Preliminary Results

In this section, we collect some preliminary results which we will use to establish Proposition 2.2. In what follows, q is a large positive integer and $\alpha, \beta \in \mathbb{C}$ are taken to be small shifts satisfying $|\alpha|, |\beta| \leq (\log q)^{-1}$.

Our first lemma concerns the orthogonality of primitive characters.

Lemma 4.1. *For $(mn, q) = 1$ we have*

$$\sideset{}{^+}\sum_{\chi (\mathrm{mod}\ q)} \chi(m)\overline{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|m\pm n}} \mu(d)\varphi(r),$$

where the sums for the different signs $\pm$ are to be taken separately.

Proof. Let

$$f(h) = \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ h)} \chi(m)\overline{\chi}(n)$$

where $\sum^*$ denotes summation over primitive characters χ. Then for $(mn, q) = 1$ we have

$$\sum_{h|q} f(h) = \sum_{\chi (\mathrm{mod}\ q)} \chi(m)\overline{\chi}(n) = \begin{cases} \varphi(q), & \text{if } m \equiv n \ (\mathrm{mod}\ q), \\ 0, & \text{otherwise.} \end{cases}$$

Using Möbius inversion we obtain

$$\sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \chi(m)\overline{\chi}(n) = f(q) = \sum_{\substack{h|q \\ h|m-n}} \varphi(h)\mu\Big(\frac{q}{h}\Big).$$

It follows from this identity that

$$\left|\mathscr{C}_q\right| = \sum_{\chi (\text{mod } q)}{}^{*} 1 = \sum_{k|q} \varphi(k)\mu\left(\frac{q}{k}\right),$$

which justifies an above remark. Our lemma now follows by noting that

$$\sum_{\substack{\chi (\text{mod } q) \\ \chi(-1)=1}}{}^{*} \chi(m)\overline{\chi}(n) = \sum_{\chi (\text{mod } q)}{}^{*} \left[\frac{1+\chi(-1)}{2}\right]\chi(m)\overline{\chi}(n).$$

□

Lemma 4.2. *Let $G(s)$ be an even, entire function with rapid decay as $|s| \to \infty$ in any fixed vertical strip $A \leq \sigma \leq B$ and with $G(0) = 1$. Let*

$$W^{\pm}_{\alpha,\beta}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(s)H(s)g^{\pm}_{\alpha,\beta}(s)x^{-s}\frac{ds}{s}, \tag{4.1}$$

where

$$g^{+}_{\alpha,\beta}(s) = \frac{\Gamma\left(\frac{1/2+\alpha+s}{2}\right)\Gamma\left(\frac{1/2+\beta+s}{2}\right)}{\Gamma\left(\frac{1/2+\alpha}{2}\right)\Gamma\left(\frac{1/2+\beta}{2}\right)},$$

$$g^{-}_{\alpha,\beta}(s) = \frac{\Gamma\left(\frac{1/2-\alpha+s}{2}\right)\Gamma\left(\frac{1/2-\beta+s}{2}\right)}{\Gamma\left(\frac{1/2+\alpha}{2}\right)\Gamma\left(\frac{1/2+\beta}{2}\right)},$$

and

$$H(s) = \frac{\left(\frac{\alpha+\beta}{2}\right)^2 - s^2}{\left(\frac{\alpha+\beta}{2}\right)^2} \quad (\textit{for } \alpha+\beta \neq 0).$$

Then for $\chi_1, \chi_2 \in \mathscr{C}^{+}_q$ and $\alpha \neq -\beta$ we have that

$$\begin{aligned} L\left(\frac{1}{2}+\alpha,\chi_1\right)L\left(\frac{1}{2}+\beta,\chi_2\right) &= \sum_{m,n} \frac{\chi_1(m)\chi_2(n)}{m^{1/2+\alpha}n^{1/2+\beta}} W^{+}_{\alpha,\beta}\left(\frac{\pi mn}{q}\right) \\ &\quad + \varepsilon_{\chi_1}\varepsilon_{\chi_2}\left(\frac{q}{\pi}\right)^{-\alpha-\beta} \sum_{m,n} \frac{\overline{\chi_1}(m)\overline{\chi_2}(n)}{m^{1/2-\alpha}n^{1/2-\beta}} W^{-}_{\alpha,\beta}\left(\frac{\pi mn}{q}\right). \end{aligned}$$

Remarks. (1) An admissible choice of G in the above lemma is $G(s) = \exp(s^2)$.

(2) The purpose of the function $H(s)$ in the above lemma is to cancel the poles of the functions $\zeta_q(1 \pm (\alpha+\beta) + 2s)$ at $s = \mp(\alpha+\beta)/2$ which appear in the proof of the next lemma. This substantially simplifies our later calculations. A similar effect has been observed by Conrey *et al.* (see [7, §3]).

Proof. Consider the integral

$$I_{\alpha,\beta} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(s)H(s)\frac{\Lambda(1/2+\alpha+s,\chi_1)\Lambda(1/2+\beta+s,\chi_2)}{\Gamma\left(\frac{1/2+\alpha}{2}\right)\Gamma\left(\frac{1/2+\beta}{2}\right)}\frac{ds}{s}.$$

Shifting the line of integration to Re $s = -1$ and using Cauchy's theorem, it follows that

$$I_{\alpha,\beta} = R_0 + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} G(s)H(s)\frac{\Lambda(1/2+\alpha+s,\chi_1)\Lambda(1/2+\beta+s,\chi_2)}{\Gamma\left(\frac{1/2+\alpha}{2}\right)\Gamma\left(\frac{1/2+\beta}{2}\right)}\frac{ds}{s},$$

where R_0 is the residue of the integrand at $s = 0$. Evidently,

$$R_0 = \left(\frac{q}{\pi}\right)^{(1+\alpha+\beta)/2} L\left(\frac{1}{2}+\alpha,\chi_1\right)L\left(\frac{1}{2}+\beta,\chi_2\right).$$

By making the change of variables s to $-s$ and using (2.1), we have that

$$R_0 = I_{\alpha,\beta} + \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(s)H(s)\frac{\Lambda(1/2-\alpha+s,\overline{\chi_1})\Lambda(1/2-\beta+s,\overline{\chi_2})}{\Gamma\left(\frac{1/2+\alpha}{2}\right)\Gamma\left(\frac{1/2+\beta}{2}\right)}\frac{ds}{s}.$$

The lemma now follows by using (2.1) to express the Λ-functions in terms of Dirichlet series and then integrating term-by-term. □

Lemma 4.3. *Let*

$$S^+_{\alpha,\beta}(x) = \sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{W^+_{\alpha,\beta}(n^2/x)}{n^{1+\alpha+\beta}} \quad \text{and} \quad S^-_{\alpha,\beta}(x) = \sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{W^-_{\alpha,\beta}(n^2/x)}{n^{1-\alpha-\beta}}.$$

Then, for any $\varepsilon > 0$ and $\alpha \neq -\beta$, we have that

$$S^+_{\alpha,\beta}(x) = \zeta_q(1+\alpha+\beta) + O(\tau(q)x^{-1/2+\varepsilon})$$

and

$$S^-_{\alpha,\beta}(x) = g^-_{\alpha,\beta}(0)\zeta_q(1-\alpha-\beta) + O(\tau(q)x^{-1/2+\varepsilon}),$$

where $\tau(q)$ is the number of divisors of q and the function $\zeta_q(s)$ is defined by

$$\zeta_q(s) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right).$$

Proof. From (4.1), we observe that

$$S^{+}_{\alpha,\beta}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G(s)H(s)g^{+}_{\alpha,\beta}(s)x^{s}\zeta_q(1+\alpha+\beta+2s)\frac{ds}{s}.$$

We now shift the line of integration left to Re $s = -1/2+\varepsilon$, encountering only a simple pole of the integrand at $s = 0$. We note that the simple pole of $\zeta_q(1+\alpha+\beta+2s)$ at $s = -(\alpha+\beta)/2$ is canceled by a zero $H(s)$. The residue of the integrand at $s = 0$ is $\zeta_q(1+\alpha+\beta)$. Also, the integral along the new contour is trivially $\ll \tau(q)x^{-1/2+\varepsilon}$. This implies the first claim of the lemma. The second claim can be proved in a similar manner. □

Lemma 4.4. *Assume* $\alpha \neq -\beta$ *and let*

$$\mathscr{B}(m_1,n_1;\alpha,\beta) = \sum_{\chi\,(\mathrm{mod}\ q)}^{+} L\left(\frac{1}{2}+\alpha,\chi\right)L\left(\frac{1}{2}+\beta,\overline{\chi}\right)\chi(m_1)\overline{\chi}(n_1).$$

Then for $(m_1,n_1)=1$ *and* $(m_1n_1,q)=1$ *we have*

$$\mathscr{B}(m_1,n_1;\alpha,\beta) = \frac{\varphi^{+}(q)}{\sqrt{m_1n_1}}\left(\frac{\zeta_q(1+\alpha+\beta)}{m_1^{\beta}n_1^{\alpha}} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g^{-}_{\alpha,\beta}(0)\frac{\zeta_q(1-\alpha-\beta)}{m_1^{-\alpha}n_1^{-\beta}}\right)$$
$$+O(\beta(m_1,n_1)+q^{1/2+\varepsilon}),$$

where $\beta(m_1,n_1)$ *satisfies*

$$\sum_{m_1,n_1\le y}\frac{\beta(m_1,n_1)}{\sqrt{m_1n_1}} \ll yq^{1/2+\varepsilon}.$$

Proof. With $\chi_1=\chi$, $\chi_2=\bar{\chi}$, Lemmas 4.1 and 4.2 imply that

$$\mathscr{B}(m_1,n_1;\alpha,\beta) = \frac{1}{2}\sum_{q=dr}\mu(d)\varphi(r)\sum_{r\mid mm_1\pm nn_1}^{\star}\frac{W^{+}_{\alpha,\beta}\left(\frac{\pi mn}{q}\right)}{m^{1/2+\alpha}n^{1/2+\beta}}$$
$$+\frac{1}{2}\left(\frac{q}{\pi}\right)^{-\alpha-\beta}\sum_{q=dr}\mu(d)\varphi(r)\sum_{r\mid mn_1\pm nm_1}^{\star}\frac{W^{-}_{\alpha,\beta}\left(\frac{\pi mn}{q}\right)}{m^{1/2-\alpha}n^{1/2-\beta}}, \quad (4.2)$$

where $\sum^{\star}$ denotes summation over all $(mn,q)=1$. The main contribution to $\mathscr{B}(m_1,n_1;\alpha,\beta)$ comes from the diagonal terms $mm_1=nn_1$ and $mn_1=nm_1$ in the first and second sums on the right-hand side of (4.2), respectively. For $(m_1,n_1)=1$, this contribution is

$$\varphi^+(q)\left(\sum_{mm_1=nn_1}^{\star}\frac{W^+_{\alpha,\beta}\left(\frac{\pi mn}{q}\right)}{m^{1/2+\alpha}n^{1/2+\beta}}+\left(\frac{q}{\pi}\right)^{-\alpha-\beta}\sum_{mn_1=nm_1}^{\star}\frac{W^-_{\alpha,\beta}\left(\frac{\pi mn}{q}\right)}{m^{1/2-\alpha}n^{1/2-\beta}}\right)$$

$$=\varphi^+(q)\left(\frac{S^+_{\alpha,\beta}\left(\frac{q}{\pi m_1n_1}\right)}{n_1^{1/2+\alpha}m_1^{1/2+\beta}}+\left(\frac{q}{\pi}\right)^{-\alpha-\beta}\frac{S^-_{\alpha,\beta}\left(\frac{q}{\pi m_1n_1}\right)}{m_1^{1/2-\alpha}n_1^{1/2-\beta}}\right),$$

where $S^{\pm}_{\alpha,\beta}(x)$ are defined in Lemma 4.3. By Lemma 4.3, the above expression is equal to

$$\frac{\varphi^+(q)}{\sqrt{m_1n_1}}\left(\frac{\zeta_q(1+\alpha+\beta)}{m_1^{\beta}n_1^{\alpha}}+\left(\frac{q}{\pi}\right)^{-\alpha-\beta}g^-_{\alpha,\beta}(0)\frac{\zeta_q(1-\alpha-\beta)}{m_1^{-\alpha}n_1^{-\beta}}\right)+O(q^{1/2+\varepsilon}).$$

All the other terms in (4.2) contribute at most

$$\beta(m_1,n_1)=\sum_{mm_1\neq nn_1}\frac{(mm_1\pm nn_1,q)}{\sqrt{mn}}\left|W^{\pm}_{\alpha,\beta}\left(\frac{\pi mn}{q}\right)\right|.$$

Using the estimate $|W^{\pm}_{\alpha,\beta}(x)|\ll(1+|x|)^{-1}$ one can show that (see [10, Sec. 4])

$$\sum_{m_1,n_1\le y}\frac{\beta(m_1,n_1)}{\sqrt{m_1n_1}}\ll yq^{1/2+\varepsilon}(\log yq)^4.$$

The lemma now follows from the above estimates. □

Lemma 4.5. *Let*

$$S_j(d)=\sum_{\substack{n\le y/d\\(n,dq)=1}}\frac{\mu(n)}{n}(\log n)^jP\left(\frac{\log y/dn}{\log y}\right).$$

Then $S_j(d)=M_j(d)+O(E_j(d))$ *uniformly for* $d\le y$, *where*

$$M_0(d)=\frac{dq}{\varphi(dq)\log y}P'\left(\frac{\log y/d}{\log y}\right),\quad M_1(d)=-\frac{dq}{\varphi(dq)}P\left(\frac{\log y/d}{\log y}\right),$$

$M_j(d)=0$ *(for* $j\ge2$*), and*

$$E_j(d)=(\log y)^{j-2}(\log\log y)^2(1+(d/y)^{\theta}\log y)\prod_{p|dq}\left(1+\frac{1}{p^{1-2\delta}}\right)$$

with $\theta\gg1/\log\log y$ *and* $\delta=1/\log\log y$.

Proof. Consider the Dirichlet polynomial

$$G(z)=\sum_{\substack{n\le y/d\\(n,dq)=1}}\frac{\mu(n)}{n^{1+z}}P\left(\frac{\log y/dn}{\log y}\right).$$

Since, for $n \le y/d$, we have

$$P\left(\frac{\log y/dn}{\log y}\right) = \sum_{\ell \ge 1} \frac{a_\ell}{(\log y)^\ell} (\log y/dn)^\ell = \sum_{\ell \ge 1} \frac{a_\ell \ell!}{(\log y)^\ell} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{y}{dn}\right)^s \frac{ds}{s^{\ell+1}},$$

we can express $G(z)$ as

$$G(z) = \sum_{\ell \ge 1} \frac{a_\ell \ell!}{(\log y)^\ell} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{y}{d}\right)^s A(s+z) \frac{ds}{\zeta(1+z+s)s^{\ell+1}}$$

where

$$A(s) = \prod_{p|dq} \left(1 - \frac{1}{p^{1+s}}\right)^{-1}.$$

We note that $G(z)$ is precisely $G_j(1+z)$ in Lemma 10 of Conrey [4] (see the first expression in the proof), with x being replaced by y/d and $1/F(j,s)$ being replaced by $A(s-1)$. Using this, we obtain that

$$G^{(j)}(z) = M_j(d;z) + O(E_j(d)) \tag{4.3}$$

uniformly for $0 < |z| \ll 1/\log y$, where

$$M_0(d;z) = A(z)\left[zP\left(\frac{\log y/d}{\log y}\right) + \frac{1}{\log y} P'\left(\frac{\log y/d}{\log y}\right)\right],$$

$$M_1(d;z) = A(z) P\left(\frac{\log y/d}{\log y}\right), \quad \text{and}$$

$$M_j(d;z) = 0 \quad \text{for } j \ge 2.$$

Since $G(z)$ and $M_j(d;z)$ are both holomorphic in z, (4.3) also holds for $z = 0$. Observing that $S_j(d) = (-1)^j G^{(j)}(0)$ and $A(0) = dq/\varphi(dq)$, the lemma follows. □

Lemma 4.6. *Suppose that* $f(d) = \prod_{p|d} f(p)$ *with* $f(p) = 1 + O(p^{-c})$ *for some* $c > 0$ *and that*

$$J_j(y) = \sum_{d \le y}^{\star} \frac{\mu(d)^2}{d} f(d) \left(\log \frac{y}{d}\right)^j.$$

Then we have

$$J_j(y) = \frac{1}{j+1} \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p)}{p}\right) \prod_{p|q} \left(1 + \frac{f(p)}{p}\right)^{-1} (\log y)^{j+1} + O((\log y)^j).$$

Proof. We consider only the case where $j \ge 1$. The case $j = 0$ can be handled by following [14, proof of Lemma 3.11]. We first express $J_j(y)$ as a complex integral, namely

$$J_j(y) = \frac{j!}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{(d,q)=1} \frac{\mu(d)^2 f(d)}{d^{1+s}} y^s \frac{ds}{s^{j+1}}.$$

The sum over d is

$$\prod_{p\nmid q}\left(1+\frac{f(p)}{p^{1+s}}\right)=B(s)\zeta(1+s),$$

where

$$B(s)=\prod_{p}\left[\left(1-\frac{1}{p^{1+s}}\right)\left(1+\frac{f(p)}{p^{1+s}}\right)\right]\prod_{p|q}\left(1+\frac{f(p)}{p^{1+s}}\right)^{-1}.$$

Since $f(p)=1+O(p^{-c})$ for some $c>0$, $B(s)$ is absolutely and uniformly convergent in some half-plane containing the origin. We now shift the line of integration left to Re $s=-\delta$, crossing a pole of order $j+2$ at $s=0$. Here $\delta>0$ is some small, fixed constant chosen so that the arithmetical factor $B(s)$ converges absolutely for Re $s\geq-\delta$. Using Cauchy's theorem and the bound $\zeta(s)\ll(1+|t|)^{1/2+\delta}$ on the new line of integration, we obtain the estimate

$$J_j(y)=\frac{1}{j+1}B(0)(\log y)^{j+1}+O((\log y)^j).$$

The lemma now follows. □

5. Proof of Proposition 2.2

In this section, we prove Proposition 2.2. Throughout the proof, we let $y=q^{\vartheta}$ and assume that $0<\vartheta<\frac{1}{2}$. We begin by considering the mollified "shifted" second moment

$$J_{\alpha,\beta}(q)=\sum_{\chi(\mathrm{mod}\ q)}{}^{+}\,L\left(\frac{1}{2}+\alpha,\chi\right)L\left(\frac{1}{2}+\beta,\overline{\chi}\right)|M(\chi)|^2, \tag{5.1}$$

where $\alpha,\beta\in\mathbb{C}$ are small shifts satisfying $|\alpha|,|\beta|\leq(\log q)^{-1}$ and $\alpha\neq-\beta$. Applying Lemma 4.4, we have that

$$\begin{aligned}J_{\alpha,\beta}(q)&=\sum_{m,n\leq y}\frac{\mu(m)\mu(n)}{\sqrt{mn}}P\left(\frac{\log y/m}{\log y}\right)P\left(\frac{\log y/n}{\log y}\right)\mathscr{B}(m,n;\alpha,\beta)\\&=\Sigma_1(\alpha,\beta)+\Sigma_2(\alpha,\beta)+O(yq^{1/2+\varepsilon}),\end{aligned} \tag{5.2}$$

where

$$\Sigma_1(\alpha,\beta)=\varphi^{+}(q)\ \zeta_q(1+\alpha+\beta)\sum_{d\leq y}{}^{\star}\sum_{\substack{m,n\leq y/d\\(m,n)=1}}{}^{\star}\frac{\mu(dm)\mu(dn)}{dm^{1+\beta}n^{1+\alpha}}$$
$$\times P\left(\frac{\log y/dm}{\log y}\right)P\left(\frac{\log y/dn}{\log y}\right)$$

and

$$\Sigma_2(\alpha,\beta) = \varphi^+(q)\left(\frac{q}{\pi}\right)^{-\alpha-\beta} g^-_{\alpha,\beta}(0)\zeta_q(1-\alpha-\beta) \\ \times \sideset{}{^\star}\sum_{d\le y}\ \sideset{}{^\star}\sum_{\substack{m,n\le y/d\\(m,n)=1}} \frac{\mu(dm)\mu(dn)}{dm^{1-\alpha}n^{1-\beta}}\, P\left(\frac{\log y/dm}{\log y}\right) P\left(\frac{\log y/dn}{\log y}\right).$$

We can remove the restriction $(m,n) = 1$ by writing $K_{\alpha,\beta}(q) := \Sigma_1(\alpha,\beta) + \Sigma_2(\alpha,\beta)$ as

$$\varphi^+(q) \sideset{}{^\star}\sum_{cd\le y} \frac{\mu(c)\mu(cd)^2}{c^2 d} \\ \times \sum_{\substack{m,n\le y/cd\\(mn,cdq)=1}} \frac{\mu(m)\mu(n)}{mn} P\left(\frac{\log y/cdm}{\log y}\right) P\left(\frac{\log y/cdn}{\log y}\right) Z_{q,\alpha,\beta}(m,n,c), \tag{5.3}$$

where

$$Z_{q,\alpha,\beta}(m,n,c) = \frac{\zeta_q(1+\alpha+\beta)}{c^{\alpha+\beta}m^\beta n^\alpha} + \left(\frac{q}{\pi}\right)^{-\alpha-\beta} g^-_{\alpha,\beta}(0)\frac{\zeta_q(1-\alpha-\beta)}{c^{-\alpha-\beta}m^{-\alpha}n^{-\beta}}. \tag{5.4}$$

Though the function $\zeta_q(s)$ has a simple pole at $s = 1$, we note that $Z_{q,\alpha,\beta}(m,n,c)$ is holomorphic in both α and β in a small neighborhood of $\alpha = \beta = 0$ (as can be seen, for instance, by computing the Laurent series expansion of each of the terms on the right-hand side of (5.4) about $\alpha = \beta = 0$). Therefore, the expressions in (5.1) and (5.3) provide an analytic continuation of the function $J_{\alpha,\beta}(q) - K_{\alpha,\beta}(q)$ to the region $|\alpha|, |\beta| \le (\log q)^{-1}$; the function $K_{0,0}(q)$ must be defined in terms of the limit

$$Z_{q,0,0}(m,n,c) = \lim_{\alpha\to 0}\left(\frac{\zeta_q(1+2\alpha)}{(c^2mn)^\alpha} + \left(\frac{q}{\pi}\right)^{-2\alpha}\frac{\zeta_q(1-2\alpha)}{(c^2mn)^{-\alpha}}\right).$$

Moreover, by the maximum modulus principle and (5.2), we see that

$$|J_{\alpha,\beta}(q) - K_{\alpha,\beta}(q)| \ll_\varepsilon yq^{1/2+\varepsilon}$$

uniformly for $|\alpha|, |\beta| \le (\log q)^{-1}$. Hence, by Cauchy's Integral Theorem,

$$\frac{d^{2k}}{d\alpha^k d\beta^k}[J_{\alpha,\beta}(q) - K_{\alpha,\beta}(q)]\Big|_{\alpha=\beta=0} \\ = \frac{(k!)^2}{(2\pi i)^2}\int_{\mathscr{C}_\alpha}\int_{\mathscr{C}_\beta} \frac{J_{w_\alpha,w_\beta}(q) - K_{w_\alpha,w_\beta}(q)}{(w_\alpha w_\beta)^{k+1}} dw_\alpha dw_\beta \\ \ll_{k,\varepsilon} yq^{1/2+2\varepsilon},$$

where $\mathscr{C}_\alpha$ (respectively, $\mathscr{C}_\beta$) denotes the positively oriented circle in the complex plane centered at $\alpha = 0$ (respectively, $\beta = 0$) with radius $(\log q)^{-1}$. Thus, we have shown that

$$S_2(k, q) = \frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q)\bigg|_{\alpha=\beta=0} + O_{k,\varepsilon}(yq^{1/2+2\varepsilon}). \tag{5.5}$$

Writing

$$\begin{aligned}\frac{d^{2k}}{d\alpha^k d\beta^k}\ Z_{q,\alpha,\beta}(m,n,c)\bigg|_{\alpha=\beta=0} \\ = \sum_{h+i+j\le 2k+1} (a_{h,i,j}(\log c)^h + b_{h,i,j}(\log q/c)^h)(\log m)^i(\log n)^j\end{aligned}$$

for certain constants $a_{h,i,j}$ and $b_{h,i,j}$, we see that

$$\begin{aligned}&\frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q)\bigg|_{\alpha=\beta=0} \\ &= \varphi^+(q) \sum_{h+i+j\le 2k+1} \sideset{}{^\star}\sum_{cd\le y} (a_{h,i,j}(\log c)^h + b_{h,i,j}(\log cq)^h) \frac{\mu(c)\mu(cd)^2}{c^2 d} S_i(cd) S_j(cd),\end{aligned} \tag{5.6}$$

where S_i and S_j are defined in Lemma 4.5. It follows from Lemma 4.5 that

$$S_i(cd) \ll_i \frac{cdq}{\varphi(cdq)} (\log y)^{i-1},$$

from which it can be seen that the contribution of the terms with $h+i+j \le 2k$ to the sum on the right-hand side of (5.6) is

$$\ll_k (\log q)^{2k-1} q\varphi^+(q)/\varphi(q) \ll_{k,\varepsilon} \varphi^+(q)(\log q)^{2k-1+\varepsilon}.$$

The last estimate holds since $q/\varphi(q) \ll \log\log q$. It remains to consider the contribution of the terms with $h+i+j = 2k+1$. In the notation of Lemma 4.5, it can be shown that

$$\sideset{}{^\star}\sum_{cd\le y} \frac{S_i(cd)E_j(cd)}{c^2 d} \ll_{i,j,\varepsilon} (\log y)^{i+j-2+\varepsilon}$$

and

$$\sideset{}{^\star}\sum_{cd\le y} \frac{E_i(cd)E_j(cd)}{c^2 d} \ll_{i,j,\varepsilon} (\log y)^{i+j-3+\varepsilon}.$$

Hence the contribution of the error terms E_i and E_j, arising from Lemma 4.5, to the terms in (5.6) with $h+i+j=2k+1$ is $\ll_{k,\varepsilon} \varphi^+(q)(\log q)^{2k-1+\varepsilon}$. Thus,

$$\frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q)\Big|_{\alpha=\beta=0} = \varphi^+(q) \sum_{h+i+j=2k+1} \sum_{cd\le y}{}^{\star} (a_{h,i,j}(\log c)^h + b_{h,i,j}(\log cq)^h) \frac{\mu(c)\mu(cd)^2}{c^2 d} M_i(cd)M_j(cd) + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}).$$

Since $M_i(cd) = 0$ for $i > 1$, we need only to consider the terms with $0 \le i, j \le 1$. Moreover, the terms involving powers of $\log c$ can be ignored, as they contribute (due to the presence of c^{-2} in the sum) an amount which is $\ll_{k,\varepsilon} (\log q)^{2k-1+\varepsilon}$. Therefore, the above expression simplifies to

$$\frac{d^{2k}}{d\alpha^k d\beta^k} K_{\alpha,\beta}(q)\Big|_{\alpha=\beta=0} = T_1 + 2T_2 + T_3 + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}), \tag{5.7}$$

where

$$T_1 = \varphi^+(q) \sum_{cd\le y}{}^{\star} b_{2k+1,0,0}(\log q)^{2k+1} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_0(cd)^2,$$

$$T_2 = \varphi^+(q) \sum_{cd\le y}{}^{\star} b_{2k,1,0}(\log q)^{2k} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_0(cd)M_1(cd)$$

and

$$T_3 = \varphi^+(q) \sum_{cd\le y}{}^{\star} b_{2k-1,1,1}(\log q)^{2k-1} \frac{\mu(c)\mu(cd)^2}{c^2 d} M_1(cd)^2.$$

We first evaluate T_1. Using Lemma 4.5, we have that

$$\begin{aligned} T_1 &= \varphi^+(q) \frac{b_{2k+1,0,0} q^2 (\log q)^{2k+1}}{\varphi(q)^2 (\log y)^2} \sum_{cd\le y}{}^{\star} \frac{\mu(c)\mu(cd)^2 d}{\varphi(cd)^2} P'\left(\frac{\log y/cd}{\log y}\right)^2 \\ &= \varphi^+(q) \frac{b_{2k+1,0,0} q^2 (\log q)^{2k+1}}{\varphi(q)^2 (\log y)^2} \sum_{n\le y}{}^{\star} \frac{\mu(n)^2}{\varphi(n)} P'\left(\frac{\log y/n}{\log y}\right)^2. \end{aligned}$$

Now Lemma 4.6 implies that

$$\sum_{n\le y}{}^{\star} \frac{\mu(n)^2}{\varphi(n)} P'\left(\frac{\log y/n}{\log y}\right)^2 = \frac{\varphi(q)}{q}(\log y + O(1)) \int_0^1 P'(x)^2 dx.$$

Hence

$$T_1 = \varphi^+(q) \frac{b_{2k+1,0,0} q (\log q)^{2k+1}}{\varphi(q) \log y} \int_0^1 P'(x)^2 dx + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}). \tag{5.8}$$

Similarly, it can be shown that

$$T_2 = -\varphi^+(q)\frac{b_{2k,1,0}q(\log q)^{2k}}{\varphi(q)}\int_0^1 P'(x)P(x)dx + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon})$$
$$= -\varphi^+(q)\frac{b_{2k,1,0}q(\log q)^{2k}}{2\varphi(q)} + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}) \tag{5.9}$$

and that

$$T_3 = \varphi^+(q)\frac{b_{2k-1,1,1}q(\log q)^{2k-1}\log y}{\varphi(q)}\int_0^1 P(x)^2dx + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}). \tag{5.10}$$

Thus, combining (5.5), (5.7)–(5.10), and noting that

$$b_{2k+1,0,0} = \frac{\varphi(q)}{q(2k+1)}, \quad b_{2k,0,1} = -\frac{\varphi(q)}{2q} \quad \text{and} \quad b_{2k-1,1,1} = \frac{\varphi(q)k^2}{q(2k-1)},$$

it follows that, for $y = q^\vartheta$ and $0 < \vartheta < \frac{1}{2}$,

$$S_2(k,q) = \left(\frac{\vartheta^{-1}}{2k+1}\int_0^1 P'(x)^2dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1}\int_0^1 P(x)^2dx\right)\varphi^+(q)(\log q)^{2k} + O_{k,\varepsilon}(\varphi^+(q)(\log q)^{2k-1+\varepsilon}).$$

This completes the proof of Proposition 2.2.

6. Completing the Proof of Theorem 1.1: Optimizing the Mollifier

We are now in a position to complete the proof of Theorem 1.1. By Propositions 2.1 and 2.2, for $0 < \vartheta < \frac{1}{2}$, we see that

$$P_k^* \geq \left[\frac{\vartheta^{-1}}{2k+1}\int_0^1 P'(x)^2dx + \frac{1}{2} + \frac{\vartheta k^2}{2k-1}\int_0^1 P(x)^2dx\right]^{-1}. \tag{6.1}$$

For each choice of $k \in \mathbb{N}$, we wish to find a polynomial P satisfying $P(0) = 0$ and $P(1) = 1$ that maximizes the expression on the right-hand side of the above inequality. Equivalently, we wish to minimize the expression

$$F_k(P) := \frac{\vartheta^{-1}}{2k+1}\int_0^1 P'(x)^2dx + \frac{\vartheta k^2}{2k-1}\int_0^1 P(x)^2dx. \tag{6.2}$$

This optimization problem is solved explicitly in [15, Sec. 7] and, independently, in [6, p. 97]. We recall the argument given by Michel and Vanderkam in [15].

Using a standard approximation argument, the polynomial P can be replaced by any infinitely differentiable function with a rapidly convergent Taylor series on $[0,1]$. In this case, using the calculus of variations, the optimization problem can be explicitly solved and, for $k > 0$, the optimal choice of P is

$$P(t) = \frac{\sinh(\Lambda t)}{\sinh(\Lambda)}, \quad \text{where } \Lambda = \vartheta k\sqrt{\frac{2k+1}{2k-1}}.$$

Table 1. In the table, lower bounds for the proportions P_k and P_k^*, defined in Eqs. (1.1) and (1.2), respectively. These calculations were performed by using the expression for $F_k(P)$ given in (6.3) with $\vartheta = \frac{1}{2} - 1 \times 10^{-8}$.

k	Lower bound for P_k	Lower bound for P_k^*
1	$\frac{2}{3} \times 0.8216\ldots$	$0.7544\ldots$
2	$\frac{2}{3} \times 0.9369\ldots$	$0.9083\ldots$
3	$\frac{2}{3} \times 0.9758\ldots$	$0.9642\ldots$
4	$\frac{2}{3} \times 0.9901\ldots$	$0.9853\ldots$
5	$\frac{2}{3} \times 0.9956\ldots$	$0.9935\ldots$
10	$\frac{2}{3} \times 0.9995\ldots$	$0.9993\ldots$
15	$\frac{2}{3} \times 0.9997\ldots$	$0.9997\ldots$
20	$\frac{2}{3} \times 0.9998\ldots$	$0.9998\ldots$
25	$\frac{2}{3} \times 0.9999\ldots$	$0.9999\ldots$

With this choice of P, it follows that

$$F_k(P) = \frac{\Lambda \coth \Lambda}{\vartheta(2k+1)} = \frac{k \coth \Lambda}{\sqrt{4k^2 - 1}}. \tag{6.3}$$

As k gets large, the function $\coth \Lambda \to 1$ and so asymptotically (as $k \to \infty$) we have

$$F_k(P) = \frac{1}{2} + \frac{1}{16k^2} + O\left(\frac{1}{k^4}\right).$$

When combined with (6.1) and (6.2), this asymptotic formula is enough to establish the estimate for P_k^* in (1.3) and, thus, completes the proof of Theorem 1.1.

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