Van der Corput sets with respect to compact groups

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Abstract. We study the notion of van der Corput sets with respect to general compact groups.

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1. Introduction

Given a sequence $(x_n)_{n=1}^{\infty} \subset \mathbf{T} = \mathbf{R}/\mathbf{Z}$. The classical van der Corput's difference theorem in uniform distribution theory states that if $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed in $\mathbf{T}$ for all $h \in \mathbf{Z}^+$, then $(x_n)_{n=1}^{\infty}$ itself is also uniformly distributed in $\mathbf{T}$.

In [5], Kamae and Mendès France made the important observation that in order for $(x_n)_{n=1}^{\infty}$ to be uniformly distributed in $\mathbf{T}$, it suffices to have the uniform distribution in $\mathbf{T}$ of $(x_{n+h} - x_n)_{n=1}^{\infty}$ for h in a certain subset H of $\mathbf{Z}^+$. Such a set H is called a *van der Corput set*. A prototype result of this kind had already been proven by Delange, where one can take H to be the set of all multiples of a positive integer a. Other examples of van der Corput sets are

$$H_1 = \{n^2 : n \in \mathbf{Z}^+\}$$
$$H_2 = \{p - 1 : p \text{ prime}\}$$
$$H_3 = \{i - j : i, j \in I, i > j\} \quad \text{where } I \text{ is any infinite set of integers.}$$

On the other hand, it is known that the set of all odd numbers is not van der Corput. Also, no lacunary set is van der Corput.

Thanks to works of Kamae-Mendès France [5], Ruzsa [8, 9], Montgomery [7], Bergelson-Lesigne [1], Ninčević-Rabar-Slijepčević [10], many criteria for van der Corput sets are known. Extensive accounts of van der Corput sets can be found in [1] and [7]. For a modern treatment of van der Corput's difference theorem, see [2].

Let G be a compact (not necessarily abelian) group and μ_G be the normalized Haar measure on G. We recall that a sequence $(x_n)_{n=1}^{\infty}$ is said to be uniformly distributed in G if

$$\lim_{N\to\infty} \mu_G\left(\{1 \leq n \leq N : x_n \in C\}\right) = \mu_G(C)$$

for any open set $C \subset G$ with boundary measure 0. Van der Corput's difference theorem has been generalized to any compact group by Hlawka [4] (see also [6, Chapter 4, Section 2]), namely that if the sequences $(x_{n+h}x_n^{-1})_{n=1}^{\infty}$ are uniformly distributed in a compact group G for all $h \in \mathbf{Z}^+$, then the sequence $(x_n)_{n=1}^{\infty}$ is also uniformly distributed in G. Naturally, the notion of van der Corput sets also makes sense in any compact group. We make the following:

Definition 1. Let G be a compact topological group. We say a set $H \subset \mathbf{Z}^+$ is G-van der Corput (G-vdC for short) if the following is true. For any sequence $(x_n)_{n=1}^{\infty} \subset G$, if the sequence $(x_{n+h}x_n^{-1})_{n=1}^{\infty}$ is uniformly distributed for each $h \in H$, then the sequence $(x_n)_{n=1}^{\infty}$ is also uniformly distributed in G.

Given this definition, from now on usual van der Corput sets are referred to as $\mathbf{T}$-vdC sets. Presumably, the property of G-van der Corput depends on G. We will, however, prove the following:

Theorem 1. *If a set $H \subset \mathbf{Z}^+$ is $\mathbf{T}$-vdC, then it is also G-vdC for any compact group G.*

Kamae and Mendès France [5] found a connection between $\mathbf{T}$-vdC sets and *intersective sets* which are much studied in combinatorial number theory and ergodic Ramsey theory. A set $H \subset \mathbf{Z}^+$ is called *intersective* if for any dense subset A of the integers (that is, $\overline{\lim}_{N\to\infty} \frac{|A\cap\{1,\dots,N\}|}{N} > 0$), there exist two elements of A whose difference is in H. Alternatively, H is intersective if and only if it is a *set of recurrence*, that is, for any probability measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$, for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there is $h \in H$ such that $\mu(A \cap T^{-h}A) > 0$. Kamae and Mendès France showed that any $\mathbf{T}$-vdC set is intersective. The converse is not true: Bourgain [3] constructed a set that is intersective but not $\mathbf{T}$-vdC. Furthermore, he showed that the generic density conditions for $\mathbf{T}$-vdC and intersective sets are the same. We will extend Kamae and Mendès France's argument to prove the following.

Theorem 2. *If G is a compact, second countable group (that is, its topology has a coutable base), then every G-vdC set is intersective.*

As a consequence of Theorems 1 and 2, if G is a compact, second countable group then the class of all G-vdC sets lies in between the class of all $\mathbf{T}$-vdC sets and the class of all intersective sets. It is an interesting problem to determine if these two inclusions are strict, even for a specific choice of G, e.g. $G = \mathbf{Z}/2\mathbf{Z}$. This could be a difficult problem since Bourgain's construction of a set that is intersective but not $\mathbf{T}$-vdC is difficult.

In Section 2 we will recall some preliminaries on uniform distribution. In Section 3 and Section 4 we will prove Theorems 1 and 2.

2. Prelimiaries

We first recall Weyl's criterion for general compact groups. Let G be a compact group with normalized Haar measure μ_G. A *representation* of G of degree k is a continuous homomorphism $\mathbf{D}$ from G to the multiplicative group $GL(k)$ of all nonsingular complex matrices of order k. A representation $\mathbf{D}$ is called *unitary* if $\mathbf{D}(x)$ is unitary for all $x \in G$. Two representations $\mathbf{D}_1, \mathbf{D}_2$ of the same degree k are said to be *equivalent* if there exists a nonsingular $k \times k$ matrix S such that

$$\mathbf{D}_2(x) = S\mathbf{D}_1(x)S^{-1}$$

for all $x \in G$.

A representation $\mathbf{D}$ of degree k is called *reducible* if there exists a subspace V of $\mathbf{C}^k$ of dimension $0 < \dim(V) < k$ such that $\mathbf{D}(x)V \subset V$ for all $x \in G$. $\mathbf{D}$ is called *irreducible* if it is not reducible.

Let $\{\mathbf{D}^{(\lambda)} : \lambda \in \Lambda\}$ be a system of representations of G that is obtained by choosing exactly one representation from each equivalence class of irreducible unitary representation. Let $\mathbf{D}^{(0)}$ be the trivial representation. We then have:

Proposition 1 (Weyl's criterion, [6, Theorem 4.1.3]**).** *The sequence $(x_n)_{n=1}^{\infty} \subset G$ is uniformly distributed in G if and only if for any $\lambda \in \Lambda, \lambda \neq 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{D}^{(\lambda)}(x_n) = \mathbf{0}.$$

We will also need the following simple fact, whose proof we will omit:

Lemma 1. *If $\mathbf{D}$ is a non-trivial irreducible representation of G, then*

$$\int_G \mathbf{D}(x) d\mu_G = \mathbf{0}.$$

Next, we recall some criteria for $\mathbf{T}$-vdC sets. Though we will only need (A), (C) and (F), we will list all of them for completeness. For a set $H \subset \mathbf{Z}^+$, the following conditions are all equivalent to H being $\mathbf{T}$-vdC.

(A) (Ruzsa) For any sequence $(u_n)_{n=1}^{\infty}$ of bounded complex numbers, if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_{n+h}\overline{u_n} = 0$$

for any $h \in H$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_n = 0.$$

(B) (Ruzsa) For any sequence $(u_n)_{n=1}^{\infty}$ of complex numbers satisfying $\overline{\lim}_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|u_n|^2<\infty$, if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}u_{n+h}\overline{u_n}=0$$

for any $h\in H$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}u_n=0.$$

(C) (Kamae-Mendès France, Ruzsa) For any nonnegative, finite measure μ on $\mathbf{T}$, if $\widehat{\mu}(h)=\int_{\mathbf{T}}e(-hx)d\mu(x)=0$ for any h in H, then μ is continuous at 0 (that is, $\mu(\{0\})=0$).

(D) (Bergelson-Lesigne) For any nonnegative, finite measure μ on $\mathbf{T}$, if $\sum_{h\in H}|\widehat{\mu}(h)|<\infty$, then μ is continuous at 0.

(E) (Kamae-Mendès France, Ruzsa) For any $\epsilon>0$, there exists a nonnegative real trigonometric polynomial

$$T(x)=\sum_{n\in\mathbf{Z}}a_n e(nx)$$

supported on $H\cup(-H)\cup\{0\}$ (that is, a_n=0 for $n\notin H\cup(-H)\cup\{0\}$) satisfying $T(0)=1$ and $a_0\le\epsilon$.

(F) (Bergelson-Lesigne) For any $\epsilon>0$, there exists a finite, positive-definite sequence (a_n) (that is, $\sum_{n,n'\in Z}a_{n-n'}z_n\overline{z_{n'}}\ge 0$ for any sequence $(z_n)\subset\mathbf{C}$) supported on $H\cup(-H)\cup\{0\}$ (that is, a_n=0 for $n\notin H\cup(-H)\cup\{0\}$) satisfying

$$\sum_{n\in\mathbf{Z}}a_n=1 \qquad \text{and} \qquad a_0\le\epsilon.$$

(G) (Ninčević-Rabar-Slijepčević) H is operator recurrent, that is, for any Hilbert space $\mathcal{H}$, for any unitary operator U on $\mathcal{H}$, for any $x\in\mathcal{H}$ whose orthogonal projection on $\mathrm{Ker}(U-I)$ is non-zero, there exists $h\in H$ such that

$$\langle U^h x, x\rangle\neq 0.$$

3. Proof of Theorem 1

In proving Theorem 1, we use the following result due to Bergelson-Lesigne, which is a Hilbert space generalization of criterion (A).

Proposition 2 ([1, Corollary 1.31]). *Let $H\subset\mathbf{Z}^+$ be a $\mathbf{T}$-vdC set. Let $(u_n)_{n=1}^{\infty}$ be any bounded sequence in a Hilbert space $\mathcal{H}$. If*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+h},u_n\rangle=0$$

for any $h \in H$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} u_n = 0.$$

Bergelson-Lesigne deduced this from a *generalized van der Corput inequality* ([1, Proposition 1.30]) and criterion (F). We remark that in the case where $\mathcal{H}$ is finite-dimensional (which is all we need), this can also be proved by the method of *correlation functions* ([6, Section 2, Chapter 4]) and criterion (C).

For each k, $GL(k)$ is naturally a Hilbert space under the inner product

$$\langle A, B\rangle = \mathrm{Tr}(\overline{B}^t A).$$

Let H be any $\mathbf{T}$-vdC set. Let $(x_n)_{n=1}^{\infty} \subset G$ be a sequence such that $(x_{n+h}x_n^{-1})_{n=1}^{\infty}$ is uniformly distributed in G for any $h \in H$. Let $\mathbf{D}$ be any unitary representation of degree k of G. By Weyl's criterion (Proposition 1),

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{D}(x_{n+h}x_n^{-1}) = \mathbf{0}.$$

We have

$$0 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle \mathbf{D}(x_{n+h}x_n^{-1}), I_k\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle \mathbf{D}(x_{n+h}), \mathbf{D}(x_n)\rangle.$$

Since $\mathbf{D}$ is unitary, the sequence $(\mathbf{D}(x_n))$ is bounded. Proposition 2 implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{D}(x_n) = \mathbf{0}.$$

By Weyl's criterion, this implies that $(x_n)_{n=1}^{\infty}$ is uniformly distributed in G and H is G-vdC.

4. Proof of Theorem 2

Since G is second-countable, it follows from the Peter-Weyl theorem that the system of representatives $\{\mathbf{D}^{(\lambda)} : \lambda \in \Lambda\}$ of unitary representations of G is countable (since, in this case, $L^2(G)$ is separable).

In proving Theorem 2, we will generalize Kamae-Mendès France's probabilistic argument in [5]. Suppose $H \subset \mathbf{Z}^+$ is G-vdC but not intersective. Then there is a set $A \subset \mathbf{Z}^+$ of positive upper density such that $A \cap (A - h) = \emptyset$ for all $h \in H$.

Let $\{u_n\}_{n\in\mathbf{Z}^+}$ be a sequence of random variables taking values in G as follows. Put $u_n = 1_G$ for all $n \in A$. For $n \in \mathbf{Z}^+ \setminus A$, we select $u_n \in G$

uniformly and independently (with respect to μ_G). Fix $h \in H$. We have

$$u_{n+h}u_n^{-1} = \begin{cases} 1, & \text{if } n \in A \text{ and } n+h \in A; \\ u_{n+h}, & \text{if } n \in A \text{ and } n+h \notin A; \\ u_n^{-1}, & \text{if } n \notin A \text{ and } n+h \in A; \\ u_{n+h}u_n^{-1}, & \text{if } n \notin A \text{ and } n+h \notin A. \end{cases}$$

Since $A \cap (A - h) = \emptyset$, the first case does not occur. From here it can be shown using standard probablistic arguments that the random variables $u_{n+h}u_n^{-1}$ are independent and uniform in G. Indeed, uniformity is immediate. As for independence, one only needs to verify independence of families of random variables $\{u_{n_i+h}u_{n_i}^{-1}\}_{i=1}^{I}$ where the pairs of indices $\{n_i, n_i + h\}$ are not pairwise disjoint. We leave the details to the reader.

By the law of large numbers and Lemma 1, for any non-trivial $\lambda \in \Lambda$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{D}^{(\lambda)}(u_{n+h}u_n^{-1}) = \int_G \mathbf{D}^{(\lambda)}(x)d\mu_G = \mathbf{0}$$

almost surely.

Since Λ is countable, the above equation in conjunction with Weyl's criterion (Proposition 1) almost surely implies that for any $h \in H$, the sequence $(u_{n+h}u_n^{-1})_{n=1}^{\infty}$ is uniformly distributed in G. Since H is G-vdC, we have almost surely, $(u_n)_{n=1}^{\infty}$ is uniformly distributed in G.

Let $\mathbf{D}$ be any non-trivial irreducible unitary representation of G. On the one hand, by Weyl's criterion, we have almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{D}(u_n) = \mathbf{0}. \tag{1}$$

On the other hand, by the law of large numbers, and Lemma 1

$$\lim_{N\to\infty} \frac{1}{N - |A_N|} \sum_{\substack{n=1,\\ n\notin A}}^{N} \mathbf{D}(u_n) = \int_G \mathbf{D}(x)d\mu_G = \mathbf{0} \tag{2}$$

almost surely, where $A_N = A \cap \{1, 2, \ldots, N\}$. From (1) and (2) and the fact that $u_n = 1_G$ for all $n \in A$, we see that $\lim_{N\to\infty} \frac{|A_N|}{N} = 0$, which is a contradiction since A has positive upper density.

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