SUMS OF PRODUCTS OF FRACTIONAL PARTS

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ABSTRACT. We prove upper and lower bounds for certain sums of products of fractional parts by using majoring and minorizing functions from Fourier analysis. In special cases the lower bounds are sharp if there exist counterexamples to the Littlewood conjecture in Diophantine approximation. We show that our more general lower bounds are sharp if there exist multiplicatively badly approximable matrices as considered in recent papers of Y. Bugeaud and O. N. German. We prove upper bounds for sums of products of fractional parts that hold on the complement of a set of small measure. We also give an alternative proof of German's transference principle for multiplicatively badly approximable matrices.

1. INTRODUCTION

In this paper we prove upper and lower bounds for certain sums of products of fractional parts. As usual we write

$$\|x\| = \min\{|x-n| : n \in \mathbb{Z}\} \tag{1.1}$$

for the distance from the real number x to the nearest integer. Then $x \mapsto \|x\|$ is well defined on the quotient group $\mathbb{R}/\mathbb{Z}$, and $(x, y) \mapsto \|x - y\|$ is a metric on $\mathbb{R}/\mathbb{Z}$ that induces its quotient topology.

We will work in the following general setting. Let M and N be positive integers, and then define compact abelian groups

$$G_1 = (\mathbb{R}/\mathbb{Z})^{MN}, \quad \text{and} \quad G_2 = (\mathbb{R}/\mathbb{Z})^{M}. \tag{1.2}$$

We write μ_1 and μ_2, respectively, for Haar measures on the Borel subsets of these groups normalized so that $\mu_1(G_1) = \mu_2(G_2) = 1$. We write the elements of the group G_1 as $M \times N$ matrices with entries in $\mathbb{R}/\mathbb{Z}$. That is, we write

$$A = (\alpha_{mn}), \quad \text{where} \quad \alpha_{mn} \in \mathbb{R}/\mathbb{Z},$$

for a generic element of G_1. Obviously addition in the group G_1 coincides with addition of matrices. We write the elements of the group G_2 as $M \times 1$ column matrices. If $\boldsymbol{\xi}$ is an $N \times 1$ column vector in $\mathbb{Z}^N$ then

$$A \mapsto A\boldsymbol{\xi} = \left(\sum_{n=1}^{N} \alpha_{mn}\xi_n\right) \tag{1.3}$$

defines a continuous homomorphism from G_1 into G_2. If $\boldsymbol{\xi} \neq \mathbf{0}$ then it follows easily that (1.3) is surjective and measure preserving.

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For positive integers $L_1, L_2, \dots, L_M$, we define $F_{\boldsymbol{L}} : G_2 \to [1, \infty)$ by

$$F_{\boldsymbol{L}}(\boldsymbol{x}) = \prod_{m=1}^{M} \min\left\{L_m, (2\|x_m\|)^{-1}\right\}.$$

And we define $F : G_2 \to [1, \infty]$ by

$$F(\boldsymbol{x}) = \prod_{m=1}^{M} (2\|x_m\|)^{-1}. \tag{1.4}$$

If A is an element of the group G_1, if $Y \subseteq \mathbb{Z}^N$ is a finite set of integer lattice points, and $X = Y - Y$ is its difference set, we prove lower bounds for the sum

$$\sum_{\substack{\boldsymbol{\xi} \in X \\ \boldsymbol{\xi} \neq \boldsymbol{0}}} F_{\boldsymbol{L}}(A\boldsymbol{\xi}).$$

Our lower bounds depend on the cardinality $|Y|$, and on the integers $L_1, L_2, \dots, L_M$, but *not* on the point A in G_1. Here is the precise inequality.

Theorem 1.1. *Let $L_1, L_2, \dots, L_M$, be positive integers, $Y \subseteq \mathbb{Z}^N$ a finite, nonempty subset of integer lattice points with difference set $X = Y - Y$. Then for every point A in the group G_1, we have*

$$|Y| \prod_{m=1}^{M} \log(L_m + 1) - \prod_{m=1}^{M} L_m \leq \sum_{\substack{\boldsymbol{\xi} \in X \\ \boldsymbol{\xi} \neq \boldsymbol{0}}} F_{\boldsymbol{L}}(A\boldsymbol{\xi}). \tag{1.5}$$

Obviously (1.5) is of interest when the left hand side is positive, and so when $|Y|$ is sufficiently large. In the very special case $M = N = 1$ and $Y = \{0, 1, 2, \dots, K\}$, the inequality (1.5) states that

$$(K+1)\log(L+1) - L \leq \sum_{\substack{k=-K \\ k \neq 0}}^{K} \min\left\{L, (2\|k\alpha\|)^{-1}\right\} \leq \sum_{\substack{k=-K \\ k \neq 0}}^{K} (2\|k\alpha\|)^{-1} \tag{1.6}$$

for all α in $\mathbb{R}/\mathbb{Z}$. By taking $L = K$ in (1.6) we obtain a more precise form of an inequality obtained by Hardy and Littlewood in [10] and [11]. Their method, and later refinements of Haber and Osgood [9, Theorem 2], made assumptions on the continued fraction expansion of α. The method we develop here uses a special trigonometric polynomial with nonnegative Fourier coefficients that minorizes the function $\boldsymbol{x} \mapsto F_{\boldsymbol{L}}(\boldsymbol{x})$. Once such a trigonometric polynomial is determined, we appeal to ideas formulated by H. L. Montgomery in [15, Chapter 5, Theorem 9].

More generally, we can set $L_1 = L_2 = \cdots = L_M$ in (1.5), and then select this common value in an optimal way. This leads to the following lower bound for sums of the simpler expression $F(A\boldsymbol{\xi})$.

Corollary 1.2. *Let $Y \subseteq \mathbb{Z}^N$ be a finite, nonempty subset of integer lattice points with difference set $X = Y - Y$. Then for every point A in the group G_1 we have*

$$|Y|\left(\frac{\log|Y|}{M}\right)^{M} - |Y| \leq \sum_{\substack{\boldsymbol{\xi} \in X \\ \boldsymbol{\xi} \neq \boldsymbol{0}}} F(A\boldsymbol{\xi}). \tag{1.7}$$

We also prove upper bounds for sums of the form

$$\sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\neq \mathbf{0}}} F(A\boldsymbol{\xi}), \tag{1.8}$$

where X is a finite, nonempty subset of lattice points in $\mathbb{Z}^N$, but now we no longer assume that X is a difference set. If X contains a nonzero point then it is clear that the sum (1.8) is not a bounded function of A, and therefore no uniform upper bound for all A in G_1 is possible. However, we are able to give an upper bound, comparable with the lower bound (1.7), that holds at all points A in G_1 outside a subset of small μ_1-measure.

Theorem 1.3. *Let $X \subseteq \mathbb{Z}^N$ be a finite, nonempty subset of lattice points with cardinality $|X|$. If $0 < \varepsilon < 1$, then the inequality*

$$\sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\neq \mathbf{0}}} F(A\boldsymbol{\xi}) \le 8^M \varepsilon^{-1}|X| \sum_{m=0}^{M} \frac{\bigl(\log \varepsilon^{-1}|X|\bigr)^m}{m!} \tag{1.9}$$

holds at all point A in G_1 outside a set of μ_1-measure at most ε.

The inequality (1.9) can be used to prove metric theorems of various sorts. Here is an example.

Corollary 1.4. *Let $0 < \eta$, and let $1 < C_1$ and $1 < C_2$ be constants. Let $Y_1, Y_2, \dots, Y_\ell, \dots$ be a sequence of finite subsets of $\mathbb{Z}^N$ such that*

$$C_1^\ell \le |Y_\ell| \quad \text{for each } \ell = 1, 2, \dots, \tag{1.10}$$

and write

$$\mathcal{X} = \bigcup_{\ell=1}^{\infty} \bigl\{X \subseteq Y_\ell : |Y_\ell| \le C_2|X|\bigr\}.$$

Then for μ_1-almost all points A in G_1, the inequality

$$\sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\neq \mathbf{0}}} F(A\boldsymbol{\xi}) \ll_{A,\eta,M} |X|\bigl(\log 3|X|\bigr)^{M+1}\bigl(\log\log 27|X|\bigr)^{1+\eta} \tag{1.11}$$

holds for all subsets X in $\mathcal{X}$.

It is instructive to examine two special cases in more detail. If $M = 2$, $N = 1$, and $Y = \{0, 1, 2, \dots, K\}$, then (1.7) asserts that

$$(K+1)\bigl(\tfrac{1}{2}\log(K+1)\bigr)^2 - (K+1) \le \sum_{\substack{k=-K\\ k\neq 0}}^{K} \bigl(4\|k\alpha_1\|\|k\alpha_2\|\bigr)^{-1} \tag{1.12}$$

for all α_1 and α_2 in $\mathbb{R}/\mathbb{Z}$. And in the special case $M = 1$, $N = 2$, and

$$Y = \left\{ \begin{pmatrix} k_1\\ k_2\end{pmatrix} : 0 \le k_1 \le K,\ 0 \le k_2 \le K\right\},$$

the lower bound (1.7) takes the form

$$(K+1)^2\log(K+1) - (K+1)^2 \le \sum_{k_1=-K}^{K} \sum_{\substack{k_2=-K\\ \boldsymbol{k}\neq\mathbf{0}}}^{K} \bigl(2\|k_1\alpha_1 + k_2\alpha_2\|\bigr)^{-1} \tag{1.13}$$

for all α_1 and α_2 in $\mathbb{R}/\mathbb{Z}$. Theorem 1.3 shows that there always exist points α_1 and α_2 in $\mathbb{R}/\mathbb{Z}$ for which the lower bounds on the left of (1.12) and (1.13) cannot be significantly improved. In both cases the points α_1 and α_2 evidently depend on K. This raises the question: do there exist points α_1 and α_2 in $\mathbb{R}/\mathbb{Z}$ such that

$$\sum_{\substack{k=-K \\ k\neq 0}}^{K} \big(\|k\alpha_1\|\|k\alpha_2\|\big)^{-1} \ll_{\alpha_1,\alpha_2} K(\log K)^2 \tag{1.14}$$

as $K \to \infty$? And similarly, do there exist points α_1 and α_2 in $\mathbb{R}/\mathbb{Z}$ such that

$$\sum_{\substack{k_1=-K \\ \boldsymbol{k}\neq \boldsymbol{0}}}^{K} \sum_{k_2=-K}^{K} \|k_1\alpha_1 + k_2\alpha_2\|^{-1} \ll_{\alpha_1,\alpha_2} K^2 \log K \tag{1.15}$$

as $K \to \infty$? We will show that the inequality (1.14) holds whenever the pair $\{\alpha_1, \alpha_2\}$ is a counterexample to the Littlewood conjecture, (see [21] for a survey of recent work on this conjecture.) That is, if α_1 and α_2 are points in $\mathbb{R}/\mathbb{Z}$ such that

$$0 < \liminf_{k\to\infty} k\|k\alpha_1\|\|k\alpha_2\|, \tag{1.16}$$

then we will show that (1.14) holds. A basic transference theorem proved by O. N. German [8] shows that (1.16) also implies the estimate (1.15). We will prove generalizations of these results to a special class of $M \times N$ matrices A in the group G_1.

2. Multiplicatively badly approximable matrices

Let $A = (\alpha_{mn})$ be an $M \times N$ matrix in the compact group $G_1 = (\mathbb{R}/\mathbb{Z})^{MN}$, let $0 < \varepsilon_m \leq \frac{1}{2}$ be real numbers for $m = 1, 2, \ldots, M$, and let $K_1, K_2, \ldots, K_N$, be nonnegative integers, not all of which are zero. A general form of Dirichlet's theorem on Diophantine approximation states that if

$$1 \leq \varepsilon_1\varepsilon_2 \cdots \varepsilon_M (K_1+1)(K_2+1)\cdots(K_N+1), \tag{2.1}$$

then there exists a vector $\boldsymbol{\xi} \neq \boldsymbol{0}$ in $\mathbb{Z}^N$ such that

$$\|\alpha_{m1}\xi_1 + \alpha_{m2}\xi_2 + \cdots + \alpha_{mN}\xi_N\| \leq \varepsilon_m \quad \text{for } m = 1, 2, \ldots, M, \tag{2.2}$$

and

$$|\xi_n| \leq K_n \quad \text{for } n = 1, 2, \ldots, N. \tag{2.3}$$

Dirichlet's theorem is usually stated for matrices $A = (\alpha_{mn})$ having real entries. However, it is obvious that (2.2) depends only on the image of each matrix entry α_{mn} in $\mathbb{R}/\mathbb{Z}$.

We recall (see Perron [17], or Schmidt [19]) that the matrix $A = (\alpha_{mn})$ in G_1 is *badly approximable* if there exists a positive constant $\beta = \beta(A)$ such that the inequality

$$0 < \beta(A) \leq \max_{1\leq m\leq M} \Big\{\Big\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \Big\|\Big\}^M \max_{1\leq n\leq N} \big\{|\xi_n|\big\}^N \tag{2.4}$$

holds for all points $\boldsymbol{\xi} \neq \boldsymbol{0}$ in $\mathbb{Z}^N$. In [3] Y. Bugeaud introduced the stronger notion of multiplicatively badly approximable matrices. Following Bugeaud we say that

$A = (\alpha_{mn})$ in G_1 is *multiplicatively badly approximable* if there exists a positive constant $\gamma = \gamma(A)$ such that the inequality

$$0 < \gamma(A) \leq \left(\prod_{m=1}^{M} \Big\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \Big\| \right) \left(\prod_{n=1}^{N} \big(|\xi_n| + 1\big) \right) \tag{2.5}$$

holds for all points $\boldsymbol{\xi} \neq \mathbf{0}$ in $\mathbb{Z}^N$. It follows from the general form of Dirichlet's theorem that if A is multiplicatively badly approximable then $0 < \gamma(A) \leq 1$. Again it is obvious that (2.5) depends only on the image of each matrix entry α_{mn} in $\mathbb{R}/\mathbb{Z}$. For our purposes it is important that the matrix $A = (\alpha_{mn})$ belongs to a compact group, and so we work with matrices A in G_1.

If $M = N = 1$, then it is clear that $A = (\alpha_{11})$ is badly approximable if and only if A is multiplicatively badly approximable. And it is well known (see [4, Chapter I, Corollary to Theorem IV]) that $A = (\alpha_{11})$ is badly approximable if and only if the partial quotients in the continued fraction expansion of α_{11} are bounded. In the remaining cases, that is, in case $M + N \geq 3$, it was shown by O. Perron [17] (see also [19, Theorem 4B]) that badly approximable matrices $A = (\alpha_{mn})$ exist. However, in case $M+N \geq 3$, it is a difficult open problem to establish the existence of a multiplicatively badly approximable matrix in G_1. In the field of Laurent series with coefficients in an infinite field K, multiplicatively badly approximable matrices do exist (see [12, Theorem 2]).

Let $I \subseteq \{1, 2, \ldots, M\}$ and $J \subseteq \{1, 2, \ldots, N\}$ be nonempty subsets, and let $A(I, J)$ be the $|I| \times |J|$ submatrix of A with rows indexed by the elements of I and with columns indexed by the elements of J. If A is multiplicatively badly approximable, then it follows from (2.5) that

$$0 < \gamma(A) \leq \left(\prod_{m\in I} \Big\| \sum_{n\in J} \alpha_{mn}\xi_n \Big\| \right) \left(\prod_{n\in J} \big(|\xi_n| + 1\big) \right) \tag{2.6}$$

holds for all point $\boldsymbol{\xi} \neq \mathbf{0}$ in $\mathbb{Z}^N$ such that $n \mapsto \xi_n$ has support contained in J. Thus each submatrix $A(I, J)$ is also multiplicatively badly approximable. In particular, each matrix entry α_{mn} is a badly approximable point in $\mathbb{R}/\mathbb{Z}$.

The following result shows that if A is multiplicatively badly approximable, then an upper bound of the form (1.9) holds for all subsets $X \subseteq \mathbb{Z}^N$ that can be written as a product of intervals, and with an implied constant that depends only on A.

Theorem 2.1. *Let A be a multiplicatively badly approximable $M \times N$ matrix in the group G_1. Let $K_1, K_2, \ldots, K_N$, be nonnegative integers, not all of which are zero, and*

$$\mathcal{K} = \{\boldsymbol{\xi} \in \mathbb{Z}^N : 0 \leq \xi_n \leq K_n \text{ for } n = 1, 2, \ldots, N\}. \tag{2.7}$$

Then the inequality

$$\sum_{\substack{\boldsymbol{\xi}\in\mathcal{K}\\ \boldsymbol{\xi}\neq\mathbf{0}}} F(A\boldsymbol{\xi}) \ll_A |\mathcal{K}|\big(\log 2|\mathcal{K}|\big)^M \tag{2.8}$$

holds for all positive integer values of $|\mathcal{K}|$.

An important transference principle in Diophantine approximation (see [4, Chapter V, Corollary to Theorem II]) asserts that an $M \times N$ matrix A in G_1 is badly approximable if and only if the $N \times M$ transposed matrix A^T is badly approximable. This principle for multiplicatively badly approximable matrices was established by

O. N. German [8]. Here we give an alternative proof arguing by induction on M or N.

Theorem 2.2. *Let $A = (\alpha_{mn})$ be an $M \times N$ matrix in G_1. Then A is multiplicatively badly approximable if and only if the $N \times M$ transposed matrix A^T is multiplicatively badly approximable.*

In the special case $M = 1$ and $N = 2$, the statement of Theorem 2.2 was proved, in a different but equivalent form, by Cassels and Swinnerton-Dyer [6, section 7, Lemma 5]. The more general case in which A is a $1 \times N$ matrix is stated explicitly by de Mathan [14, Theorem 1.1]. If

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{pmatrix},$$

where $2 \le N$ and $1, \alpha_1, \alpha_2, \cdots, \alpha_N$, is a $\mathbb{Q}$-basis for a real algebraic number field of degree $N+1$, then it follows from work of Peck [16] that the image of A in G_1 is not multiplicatively badly approximable. A further refinement of this result was obtained by de Mathan [14, Theorem 1.4].

If α_1 and α_2 are points in $\mathbb{R}/\mathbb{Z}$ that satisfy (1.16), then it follows immediately that the 2×1 matrix

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

is multiplicatively badly approximable. Hence the 1×2 transposed matrix A^T is multiplicatively badly approximable. Then an application of Theorem 2.1 to the matrix A^T establishes the bound (1.15). This shows that both (1.14) and (1.15) hold if the pair $\{\alpha_1, \alpha_2\}$ is a counterexample to the Littlewood conjecture.

3. Majorizing and minorizing functions

In this section we establish variants of inequalities proved in [20]. And we obtain new inequalities for a special collection of trigonometric polynomials $\tau_{L-1}(x)$ defined in (3.24).

We define three entire functions $H(z)$, $J(z)$, and $K(z)$, by

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \operatorname{sgn}(m)(z-m)^{-2} + 2z^{-1} \right\},$$

$$J(z) = \tfrac{1}{2} H'(z), \quad \text{and} \quad K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2. \tag{3.1}$$

Each of these functions is real valued on the real axis and has exponential type 2π. The functions J and K are integrable on $\mathbb{R}$ and their Fourier transforms

$$\widehat{J}(t) = \int_{-\infty}^{\infty} J(x) e(-tx)\,\mathrm{d}x, \quad \text{and} \quad \widehat{K}(t) = \int_{-\infty}^{\infty} K(x) e(-tx)\,\mathrm{d}x,$$

are continuous functions supported on $[-1, 1]$. Here we write $e(x) = e^{2\pi i x}$. These Fourier transforms are given explicitly (see [20, Theorem 6]) by

$$\begin{aligned} \widehat{J}(t) &= \pi t(1-|t|)\cot \pi t + |t| \quad \text{if} \quad 0 < |t| < 1, \\ \widehat{K}(t) &= (1-|t|) \quad \text{if} \quad 0 \le |t| \le 1, \\ \widehat{J}(0) &= 1, \text{ and } \widehat{J}(t) = \widehat{K}(t) = 0 \text{ if } 1 \le |t|. \end{aligned} \tag{3.2}$$

It was shown in [20, Lemma 5] that the functions H and K satisfy the basic inequality

$$|\operatorname{sgn}(x) - H(x)| \le K(x) \tag{3.3}$$

for all real x. Let $\alpha < \beta$ be real numbers and define

$$\begin{aligned} \chi_{\alpha,\beta}(x) &= \tfrac{1}{2}\operatorname{sgn}(x-\alpha) + \tfrac{1}{2}\operatorname{sgn}(\beta - x) \\ &= \begin{cases} 1 & \text{if } \alpha < x < \beta, \\ \frac{1}{2} & \text{if } x = \alpha \text{ or } x = \beta, \\ 0 & \text{if } x < \alpha \text{ or } \beta < x. \end{cases} \end{aligned} \tag{3.4}$$

The function $\chi_{\alpha,\beta}(x)$ is the normalized characteristic function of the real interval with endpoints α and β. That is, it satisfies

$$\chi_{\alpha,\beta}(x) = \tfrac{1}{2}\chi_{\alpha,\beta}(x-) + \tfrac{1}{2}\chi_{\alpha,\beta}(x+)$$

at each real number x.

Lemma 3.1. *Let $\alpha < \beta$ be real numbers and let $0 < \delta$. Then there exist real entire functions $S(z)$ and $T(z)$ of exponential type at most $2\pi\delta$, such that*

$$S(x) \le \chi_{\alpha,\beta}(x) \le T(x) \tag{3.5}$$

for all real x, both S and T are integrable on $\mathbb{R}$, and their Fourier transforms

$$\widehat{S}(y) = \int_{\mathbb{R}} S(x)e(-xy)\ \mathrm{d}x, \quad \text{and} \quad \widehat{T}(y) = \int_{\mathbb{R}} T(x)e(-xy)\ \mathrm{d}x, \tag{3.6}$$

are both supported on the interval $[-\delta, \delta]$. Moreover, these functions satisfy

$$\widehat{S}(0) = \beta - \alpha - \delta^{-1}, \quad \text{and} \quad \widehat{T}(0) = \beta - \alpha + \delta^{-1}. \tag{3.7}$$

Proof. We define entire functions $S(z)$ and $T(z)$ by

$$S(z) = \tfrac{1}{2}H\big(\delta(z-\alpha)\big) - \tfrac{1}{2}H\big(\delta(z-\beta)\big) - \tfrac{1}{2}K\big(\delta(z-\alpha)\big) - \tfrac{1}{2}K\big(\delta(z-\beta)\big),$$

and

$$T(z) = \tfrac{1}{2}H\big(\delta(z-\alpha)\big) - \tfrac{1}{2}H\big(\delta(z-\beta)\big) + \tfrac{1}{2}K\big(\delta(z-\alpha)\big) + \tfrac{1}{2}K\big(\delta(z-\beta)\big).$$

Then the inequality (3.5) follows for all real x from (3.3) and (3.4). Because both $H(z)$ and $K(z)$ have exponential type 2π, it is clear that $S(z)$ and $T(z)$ have exponential type at most $2\pi\delta$.

The identity

$$\tfrac{1}{2}H\big(\delta(z-\alpha)\big) - \tfrac{1}{2}H\big(\delta(z-\beta)\big) = \delta \int_{\alpha}^{\beta} J\big(\delta(x-y)\big)\ \mathrm{d}y, \tag{3.8}$$

follows from (3.1). It shows that the left hand side of (3.8) is the convolution of the two integrable functions $x \mapsto \chi_{\alpha,\beta}(x)$ and $x \mapsto \delta J\big(\delta x\big)$. Hence (3.8) is integrable on $\mathbb{R}$. As $x \mapsto K\big(\delta x\big)$ is obviously integrable, we find that both $x \mapsto S(x)$ and $x \mapsto T(x)$ are integrable on $\mathbb{R}$. Then the identities in (3.7) follow from (3.2). □

Corollary 3.2. *Let $\alpha < \beta$ be real numbers and let $0 < \delta$. Then there exists an entire functions $U(z)$ of exponential type at most $\pi\delta$, such that*

$$\chi_{\alpha,\beta}(x) \le |U(x)|^2 \tag{3.9}$$

for all real x, and

$$\int_{\mathbb{R}} |U(x)|^2 \ \mathrm{d}x = \beta - \alpha + \delta^{-1}. \tag{3.10}$$

Proof. Let $T(z)$ be the real entire function in the statement of Lemma 3.1. As $x \mapsto T(x)$ takes nonnegative values on $\mathbb{R}$, it follows from a theorem of Fejér (see [2, pp. 124–126]) that there exists an entire function $U(z)$ of exponential type at most $\pi\delta$ such that

$$T(z) = U(z)U^*(z),$$

where $U^*(z) = \overline{U(\overline{z})}$. In particular, we have

$$T(x) = U(x)U^*(x) = |U(x)|^2$$

for all real x. Now (3.9) and (3.10) follow from (3.5) and the identity on the right hand side of (3.7), respectively. □

For each positive integer L we write $J_{L+1}(z) = (L+1)J\big((L+1)z\big)$, so that $J_{L+1}(z)$ has exponential type $2\pi(L+1)$. Then for all real t the Fourier transforms $\widehat{J}$ and $\widehat{J}_{L+1}$ are related by the identity

$$\widehat{J}\big((L+1)^{-1}t\big) = \widehat{J}_{L+1}(t).$$

Similar remarks apply to K and K_{L+1}. Using this notation we define trigonometric polynomials $j_L(x)$ and $k_L(x)$ by

$$j_L(x) = \sum_{m=-\infty}^{\infty} J_{L+1}(x+m) = \sum_{\ell=-L}^{L} \widehat{J}_{L+1}(\ell)e(\ell x), \tag{3.11}$$

and

$$k_L(x) = \sum_{m=-\infty}^{\infty} K_{L+1}(x+m) = \sum_{\ell=-L}^{L} \widehat{K}_{L+1}(\ell)e(\ell x) \ . \tag{3.12}$$

The identities on the right of (3.11) and (3.12) follow from the Poisson summation formula. We also define the periodic function $x \mapsto \psi(x)$ by

$$\psi(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

The trigonometric polynomials

$$\psi * j_L(x) = \int_{-1/2}^{1/2} \psi(x-y) j_L(y)\, dy = \sum_{\substack{\ell=-L \\ \ell \neq 0}}^{L} (-2\pi i \ell)^{-1} \widehat{J}_{L+1}(\ell)e(\ell x), \tag{3.13}$$

and $k_L(x)$, satisfy

$$\operatorname{sgn}\big(\psi * j_L(x)\big) = \operatorname{sgn}\big(\psi(x)\big), \tag{3.14}$$

$$\begin{aligned} 2\big|\psi(x) - \psi * j_L(x)\big| &\le (L+1)^{-1} k_L(x) \\ &= (L+1)^{-2}\left(\frac{\sin \pi(L+1)x}{\sin \pi x}\right)^2, \end{aligned} \tag{3.15}$$

and

$$\big|\psi * j_L(x)\big| \le \big|\psi(x)\big|, \tag{3.16}$$

for all x in $\mathbb{R}/\mathbb{Z}$. A proof of (3.14), (3.15), and (3.16), is given in [20, Theorem 18].

If $\alpha < \beta < \alpha + 1$ we define the normalized characteristic function of an interval in $\mathbb{R}/\mathbb{Z}$ with endpoints α and β by

$$\varphi_{\alpha,\beta}(x) = \begin{cases} 1 & \text{if } \alpha < x - n < \beta \text{ for some } n \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } \alpha - x \in \mathbb{Z} \text{ or if } \beta - x \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

The periodic functions $\varphi_{\alpha,\beta}(x)$ and $\psi(x)$ are related by the elementary identity

$$\varphi_{\alpha,\beta}(x) = (\beta - \alpha) + \psi(\alpha - x) + \psi(x - \beta), \tag{3.17}$$

which is a periodic analogue of (3.4). By combining (3.15) and (3.17) we obtain the inequality

$$\begin{aligned} &|\varphi_{\alpha,\beta}(x) - \varphi_{\alpha,\beta} * j_L(x)| \\ &\qquad \le |\psi(\alpha - x) - \psi * j_L(\alpha - x)| + |\psi(x - \beta) - \psi * j_L(x - \beta)| \\ &\qquad \le (2L + 2)^{-1}\{k_L(\alpha - x) + k_L(x - \beta)\} \end{aligned} \tag{3.18}$$

for all x in $\mathbb{R}/\mathbb{Z}$. Alternatively, (3.18) follows directly from [20, Theorem 19].

Lemma 3.3. *Let $\alpha < \beta < \alpha + 1$ and let $1 \le L$ be an integer. Then there exist real trigonometric polynomials $s(x)$ and $t(x)$ of degree at most L, such that*

$$s(x) \le \varphi_{\alpha,\beta}(x) \le t(x) \tag{3.19}$$

at each point x in $\mathbb{R}/\mathbb{Z}$. Moreover, the Fourier coefficients $\widehat{s}(0)$ and $\widehat{t}(0)$ are given by

$$\widehat{s}(0) = \beta - \alpha - (L + 1)^{-1} \quad \text{and} \quad \widehat{t}(0) = \beta - \alpha + (L + 1)^{-1}. \tag{3.20}$$

Proof. We define trigonometric polynomials of degree L by

$$s(x) = \varphi_{\alpha,\beta} * j_L(x) - (2L + 2)^{-1}\{k_L(\alpha - x) + k_L(x - \beta)\},$$

and

$$t(x) = \varphi_{\alpha,\beta} * j_L(x) + (2L + 2)^{-1}\{k_L(\alpha - x) + k_L(x - \beta)\}.$$

The inequality (3.19) follows immediately from (3.18). The Fourier coefficient $\widehat{s}(0)$ is then

$$\begin{aligned} \widehat{s}(0) &= \widehat{\varphi}_{\alpha,\beta}(0)\widehat{j}_L(0) - (2L + 2)^{-1}\{\widehat{k}(0) + \widehat{k}(0)\} \\ &= \beta - \alpha - (L + 1)^{-1}, \end{aligned}$$

and similarly for $\widehat{t}(0)$. This verifies the identity (3.20). □

A result closely related to Lemma 3.3 is obtained in [1, Lemma 5].

Corollary 3.4. *Let $\alpha < \beta < \alpha + 1$ and let $1 \le L$ be an integer. Then there exists a trigonometric polynomial*

$$u(x) = \sum_{\ell=0}^{L} \widehat{u}(\ell)e(\ell x), \tag{3.21}$$

such that

$$\varphi_{\alpha,\beta}(x) \le |u(x)|^2 \tag{3.22}$$

at each point x in $\mathbb{R}/\mathbb{Z}$, and

$$\int_{R/\mathbb{Z}} |u(x)|^2 \ \mathrm{d}x = \beta - \alpha + (L+1)^{-1}. \tag{3.23}$$

Proof. Let $t(x)$ be the trigonometric polynomial of degree at most L that occurs in the statement of Lemma 3.3. As $x \mapsto t(x)$ takes nonnegative values, the Fejér-Riesz theorem for trigonometric polynomials (see [7] and [18]) establishes the existence of a trigonometric polynomial $u(x)$ of the form (3.21), such that

$$t(x) = |u(x)|^2$$

at each point x in $\mathbb{R}/\mathbb{Z}$. The inequality (3.22) and the identity (3.23) follow immediately from the lemma. □

Because $x \mapsto \psi * j_L(x)$ is an odd trigonometric polynomial, we have $\psi * j_L(0) = 0$ and $\psi * j_L\big(\frac{1}{2}\big) = 0$. Therefore the function $x \mapsto \tau_{L-1}(x)$ defined by

$$\tau_{L-1}(x) = \left(\frac{-2\pi}{\sin 2\pi x}\right)\psi * j_L(x) = \frac{\psi * j_L(x)}{\psi * j_1(x)} \tag{3.24}$$

is a trigonometric polynomial of degree $L-1$. Then (3.14) implies that

$$\operatorname{sgn}\big(\psi * j_L(x)\big) = \operatorname{sgn}\big(\psi(x)\big) = \operatorname{sgn}\big(\psi * j_1(x)\big).$$

It follows that

$$0 < \tau_{L-1}(x) \tag{3.25}$$

for all x in $\mathbb{R}/\mathbb{Z}$ such that $x \neq 0$ and $x \neq \frac{1}{2}$. We also have

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi * j_L(x)\Big|_{x=0} = L,$$

and this leads easily to the identity

$$\tau_{L-1}(0) = L. \tag{3.26}$$

The Fourier coefficients $\widehat{\tau}_{L-1}(\ell)$ can be determined explicitly by writing

$$\psi * j_L(x) = -\sum_{\ell=1}^{L}\left(\frac{1}{\pi\ell}\right)\widehat{J}\left(\frac{\ell}{L+1}\right)\sin 2\pi\ell x.$$

Then we have

$$\begin{aligned}\tau_{L-1}(x) &= \sum_{\ell=1}^{L}\left(\frac{2}{\ell}\right)\widehat{J}\left(\frac{\ell}{L+1}\right)\left\{\frac{\sin 2\pi\ell x}{\sin 2\pi x}\right\}\\ &= \sum_{\ell=0}^{\infty}\left(\frac{2}{2\ell+1}\right)\widehat{J}\left(\frac{2\ell+1}{L+1}\right)\\ &\qquad + \sum_{\ell=1}^{\infty}\left(\frac{4}{2\ell+1}\right)\widehat{J}\left(\frac{2\ell+1}{L+1}\right)\sum_{m=1}^{\ell}\cos 2\pi(2m)x\\ &\qquad\quad + \sum_{\ell=1}^{\infty}\left(\frac{2}{\ell}\right)\widehat{J}\left(\frac{2\ell}{L+1}\right)\sum_{m=1}^{\ell}\cos 2\pi(2m-1)x,\end{aligned} \tag{3.27}$$

where the sums on the index ℓ contain only finitely many nonzero terms (because $t \mapsto \widehat{J}(t)$ is supported on $[-1,1]$.) After further reorganization we arrive at the identity

$$\begin{aligned}\tau_{L-1}(x) = \sum_{\ell=0}^{\infty}\Big(\frac{2}{2\ell+1}\Big)\widehat{J}\Big(\frac{2\ell+1}{L+1}\Big)\\ + \sum_{m=1}^{L-1}\bigg\{\sum_{\ell=0}^{\infty}\Big(\frac{4}{2\ell+m+1}\Big)\widehat{J}\Big(\frac{2\ell+m+1}{L+1}\Big)\bigg\}\cos 2\pi mx.\end{aligned} \tag{3.28}$$

As

$$\tau_{L-1}(x) = \widehat{\tau}_{L-1}(0) + 2\sum_{m=1}^{L-1}\widehat{\tau}_{L-1}(m)\cos 2\pi mx,$$

it is clear from (3.28) that

$$0 < \widehat{\tau}_{L-1}(m) \quad \text{for} \quad |m| \le L-1. \tag{3.29}$$

Therefore we conclude from the definition (3.24) that

$$\sup\big\{\tau_{L-1}(x) : x \in \mathbb{R}/\mathbb{Z}\big\} = \tau_{L-1}(0) = L. \tag{3.30}$$

We collect useful properties of $\tau_{L-1}(x)$ in the following lemma.

Lemma 3.5. *For each positive integer L the trigonometric polynomial*

$$\tau_{L-1}(x) = \Big(\frac{-2\pi}{\sin 2\pi x}\Big)\psi * j_L(x) = \frac{\psi * j_L(x)}{\psi * j_1(x)}$$

has degree $L-1$, and takes nonnegative values on $\mathbb{R}/\mathbb{Z}$. The inequality

$$\tau_{L-1}(x) \le \min\big\{L, \big(2\|x\|\big)^{-1}\big\}, \tag{3.31}$$

holds for all x in $\mathbb{R}/\mathbb{Z}$, and there is equality in the inequality (3.31) *at $x = 0$. The Fourier coefficients $\widehat{\tau}_{L-1}(\ell)$ are positive for $|\ell| \le L-1$, and*

$$\log(L+1) \le \widehat{\tau}_{L-1}(0) \le 1 + \log L. \tag{3.32}$$

Proof. The inequality (3.25) implies that $\tau_{L-1}(x)$ takes nonnegative values on $\mathbb{R}/\mathbb{Z}$, and (3.28) verifies that its Fourier coefficients are positive for $|\ell| \le L-1$. From the inequality (3.16) and the definition (3.24), we conclude that

$$2\|x\|\tau_{L-1}(x) \le \sup\Big\{\frac{4\pi y(\frac12 - y)}{\sin 2\pi y} : 0 < y < \tfrac12\Big\} = 1. \tag{3.33}$$

By combining (3.30) and (3.33) we obtain the basic inequality (3.31), and we also get the case of equality in (3.31) at $x = 0$.

To establish the inequality on the right of (3.32), we integrate both sides of (3.31) over $\mathbb{R}/\mathbb{Z}$.

The inequality on the left of (3.32) is trivial to check in case $L = 1$ and $L = 2$. Therefore in what follows we will assume that $3 \le L$. It will be useful to define the function $w : \mathbb{R}/\mathbb{Z} \to [0,\infty)$ by $w(0) = 0$, $w\big(\frac12\big) = 1$, and

$$w(x) = \frac{-2\pi\psi(x)}{\sin 2\pi x}, \quad \text{if } x \ne 0 \text{ and } x \ne \tfrac12. \tag{3.34}$$

It follows easily that $x \mapsto w(x)$ is continuous and positive on $\mathbb{R}/\mathbb{Z} \setminus \{0\}$. The inequality (3.15) leads to the identity

$$\psi\big(\tfrac{\ell}{L+1}\big) = \psi * j_L\big(\tfrac{\ell}{L+1}\big) \tag{3.35}$$

for $\ell = 1, 2, \dots, L$. Then (3.34) and (3.35) imply that

$$w\big(\tfrac{\ell}{L+1}\big) = \tau_{L-1}\big(\tfrac{\ell}{L+1}\big) \tag{3.36}$$

for $\ell = 1, 2, \dots, L$. Because $\tau_{L-1}(x)$ is a trigonometric polynomial of degree $L-1$, we have

$$\begin{aligned}(L+1)\widehat{\tau}_{L-1}(0) &= \sum_{m=1-L}^{L-1} \widehat{\tau}_{L-1}(m) \sum_{\ell=0}^{L} e\big(\tfrac{\ell m}{L+1}\big)\\ &= \sum_{\ell=0}^{L} \tau_{L-1}\big(\tfrac{\ell}{L+1}\big)\\ &= \tau_{L-1}(0) + \sum_{\ell=1}^{L} \tau_{L-1}\big(\tfrac{\ell}{L+1}\big).\end{aligned} \tag{3.37}$$

We now combine (3.26), (3.36), and (3.37), to obtain the identity

$$(L+1)\widehat{\tau}_{L-1}(0) = L + \sum_{\ell=1}^{L} w\big(\tfrac{\ell}{L+1}\big). \tag{3.38}$$

On the open interval $(0, 1)$ the continuous function $x \mapsto w(x)$ can be expressed as a power series in $(x - \frac{1}{2})^2$ with positive coefficients. Hence the function is convex and satisfies $w(x) = w(1 - x)$. Then it follows from Simpson's rule that

$$\begin{aligned}\sum_{\ell=1}^{L} w\big(\tfrac{\ell}{L+1}\big) &\ge w\big(\tfrac{1}{L+1}\big) + (L+1)\int_{\frac{1}{L+1}}^{\frac{L}{L+1}} w(x)\ \mathrm{d}x\\ &= w\big(\tfrac{1}{L+1}\big) + (L+1)\int_{\frac{1}{2}-\frac{1}{L+1}}^{\frac{1}{2}+\frac{1}{L+1}} w(x)\ \mathrm{d}x\\ &\qquad + (L+1)\int_{\frac{1}{L+1}}^{\frac{1}{2}-\frac{1}{L+1}} \big(w(x) + w\big(\tfrac{1}{2}+x\big)\big)\ \mathrm{d}x.\end{aligned} \tag{3.39}$$

From (3.34) we get

$$w\big(\tfrac{1}{L+1}\big) \ge (L+1)\big(\tfrac{1}{2} - \tfrac{1}{L+1}\big) = \tfrac{1}{2}(L+1) - 1. \tag{3.40}$$

The minimum of the even, convex function $x \mapsto w(x)$ on $(0, 1)$ occurs at $w(\frac{1}{2}) = 1$, and it follows that

$$(L+1)\int_{\frac{1}{2}-\frac{1}{L+1}}^{\frac{1}{2}+\frac{1}{L+1}} w(x)\ \mathrm{d}x \ge 2. \tag{3.41}$$

For the remaining integral on the right of (3.39) we find that

$$(3.42)\qquad \begin{aligned}(L+1)\int_{\frac{1}{L+1}}^{\frac12-\frac{1}{L+1}} \big(w(x)+w\big(\tfrac12+x\big)\big)\ \mathrm{d}x\\ &= \pi(L+1)\int_{\frac{1}{L+1}}^{\frac12-\frac{1}{L+1}} |\csc 2\pi x|\ \mathrm{d}x\\ &= \pi(L+1)\int_{\frac{2}{L+1}}^{\frac12} |\csc \pi x|\ \mathrm{d}x\\ &\ge (L+1)\int_{\frac{2}{L+1}}^{\frac12} x^{-1}\ \mathrm{d}x\\ &= (L+1)\log(L+1)-(L+1)(2\log 2).\end{aligned}$$

Finally, we combine the estimates (3.38), (3.39), (3.40), (3.41), and (3.42), to obtain the lower bound

$$(3.43)\qquad (L+1)\widehat{\tau}_{L-1}(0)\ge (L+1)\big(\tfrac32-2\log 2\big)+(L+1)\log(L+1).$$

Clearly (3.43) verifies the inequality on the left of (3.32). □

4. Large sieve inequalities

For each positive integer M, we define $P_M:(\mathbb{R}/\mathbb{Z})^M\to[0,1]$ by

$$(4.1)\qquad P_M(\boldsymbol{x})=\prod_{m=1}^{M}\|x_m\|.$$

And for each positive integer N, we define $Q_N:\mathbb{Z}^N\to\{1,2,\dots\}$ by

$$(4.2)\qquad Q_N(\boldsymbol{y})=\prod_{n=1}^{N}\big(|y_n|+1\big).$$

Let $K_1,K_2,\dots,K_N$, and $L_1,L_2,\dots,L_M$, be two sets of nonnegative integers. We define corresponding subsets $\mathcal{K}\subseteq\mathbb{Z}^N$ and $\mathcal{L}\subseteq\mathbb{Z}^M$ by

$$(4.3)\qquad \mathcal{K}=\{\boldsymbol{\xi}\in\mathbb{Z}^N : 0\le\xi_n\le K_n \text{ for } n=1,2,\dots,N\}$$

and

$$(4.4)\qquad \mathcal{L}=\{\boldsymbol{\eta}\in\mathbb{Z}^M : 0\le\eta_m\le L_m \text{ for } m=1,2,\dots,M\}.$$

It follows that the difference set $\mathcal{K}-\mathcal{K}$ is

$$\mathcal{K}-\mathcal{K}=\{\boldsymbol{\xi}\in\mathbb{Z}^N : |\xi_n|\le K_n \text{ for } n=1,2,\dots,N\},$$

and similarly for $\mathcal{L}-\mathcal{L}$. We write $|\mathcal{K}|$ and $|\mathcal{L}|$ for the cardinality of $\mathcal{K}$ and $\mathcal{L}$, respectively.

Using A in G_1, and the subsets $\mathcal{K}\subseteq\mathbb{Z}^N$ and $\mathcal{L}\subseteq\mathbb{Z}^M$, we define

$$(4.5)\qquad \gamma(A,\mathcal{K})=\min\big\{P_M(A\boldsymbol{\xi})Q_N(\boldsymbol{\xi}) : \boldsymbol{\xi}\in\mathcal{K}-\mathcal{K},\ \text{and } \boldsymbol{\xi}\ne\boldsymbol{0}\big\}.$$

It follows from the inequalities (2.1), (2.2), and (2.3), in the statement of Dirichlet's theorem that $0\le\gamma(A,\mathcal{K})\le 1$.

Lemma 4.1. *If A is a point in G_1 such that $0 < \gamma(A, \mathcal{K})$, then for all complex valued functions $\boldsymbol{\eta} \mapsto b(\boldsymbol{\eta})$ defined on $\mathcal{L}$, we have*

$$\sum_{\boldsymbol{\xi}\in\mathcal{K}} \Bigl|\sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) e\bigl(\boldsymbol{\eta}^T A \boldsymbol{\xi}\bigr)\Bigr|^2 \le \Bigl(\gamma(A,\mathcal{K})^{-\frac{1}{M}} |\mathcal{K}|^{\frac{1}{M}} + |\mathcal{L}|^{\frac{1}{M}}\Bigr)^M \sum_{\boldsymbol{\eta}\in\mathcal{L}} |b(\boldsymbol{\eta})|^2. \tag{4.6}$$

Proof. Let $\delta_1, \delta_2, \dots, \delta_M$, be real numbers such that $0 < \delta_m < 1$ and

$$\Bigl(\prod_{m=1}^{M} \delta_m\Bigr)\Bigl(\prod_{n=1}^{N} (K_n + 1)\Bigr) = \theta\gamma(A, \mathcal{K}), \tag{4.7}$$

where $0 < \theta < 1$. By Corollary 3.2, for each $m = 1, 2, \dots, M$, there exists an entire function $U_m(z)$ of exponential type at most $\pi\delta_m$ such that

$$1 \le |U_m(x)|^2 \tag{4.8}$$

for all real x satisfying $0 \le x \le L_m$, and

$$\int_{\mathbb{R}} |U_m(x)|^2 \, \mathrm{d}x = L_m + \delta_m^{-1}. \tag{4.9}$$

It follows that the function

$$U(\boldsymbol{x}) = \prod_{m=1}^{M} U_m(x_m)$$

belongs to $L^2\bigl(\mathbb{R}^M\bigr)$, and the Fourier transform $\widehat{U}(\boldsymbol{y})$ is supported on the compact subset

$$E = \bigl\{\boldsymbol{y} \in \mathbb{R}^M : |y_m| \le \tfrac{1}{2}\delta_m \text{ for each } m = 1, 2, \dots, M\bigr\}.$$

Moreover, (4.8) implies that the inequality

$$1 \le \bigl|U(\boldsymbol{\eta})\bigr|^2$$

holds at each point $\boldsymbol{\eta}$ in $\mathcal{L}$. It follows that the trigonometric polynomials

$$B(\boldsymbol{x}) = \sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) e\bigl(\boldsymbol{\eta}^T \boldsymbol{x}\bigr)$$

and

$$\widetilde{B}(\boldsymbol{x}) = \sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) U(\boldsymbol{\eta})^{-1} e\bigl(\boldsymbol{\eta}^T \boldsymbol{x}\bigr),$$

are related by the identity

$$B(\boldsymbol{x}) = \int_{\mathbb{R}^M} \widehat{U}(\boldsymbol{y}) \widetilde{B}(\boldsymbol{x} + \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

As the Fourier transform $\widehat{U}(\boldsymbol{y})$ is supported on the subset $E \subseteq \mathbb{R}^M$, using (4.9) and Cauchy's inequality we find that

$$\begin{aligned} \bigl|B(\boldsymbol{x})\bigr|^2 &\le \int_E \bigl|\widehat{U}(\boldsymbol{y})\bigr|^2 \, \mathrm{d}\boldsymbol{y} \int_E \bigl|\widetilde{B}(\boldsymbol{x} + \boldsymbol{w})\bigr|^2 \, \mathrm{d}\boldsymbol{w} \\ &= \prod_{m=1}^{M} \bigl(L_m + \delta_m^{-1}\bigr) \int_{\boldsymbol{x}+E} \bigl|\widetilde{B}(\boldsymbol{w})\bigr|^2 \, \mathrm{d}\boldsymbol{w}. \end{aligned} \tag{4.10}$$

Therefore the inequality (4.10) implies that

$$\sum_{\boldsymbol{\xi}\in\mathcal{K}} \bigl|B(A\boldsymbol{\xi})\bigr|^2 \le \prod_{m=1}^{M} \bigl(L_m + \delta_m^{-1}\bigr) \sum_{\boldsymbol{\xi}\in\mathcal{K}} \int_{A\boldsymbol{\xi}+E} \bigl|\widetilde{B}(\boldsymbol{w})\bigr|^2 \,\mathrm{d}\boldsymbol{w}. \tag{4.11}$$

We claim that the subsets in the collection

$$\bigl\{A\boldsymbol{\xi} + \boldsymbol{v} + E : \boldsymbol{\xi} \in \mathcal{K} \text{ and } \boldsymbol{v} \in \mathbb{Z}^M\bigr\} \tag{4.12}$$

are disjoint subsets of $\mathbb{R}^M$. Suppose that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are points in $\mathcal{K}$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are points in $\mathbb{Z}^M$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are points in E, and

$$A\boldsymbol{\xi}_1 + \boldsymbol{v}_1 + \boldsymbol{e}_1 = A\boldsymbol{\xi}_2 + \boldsymbol{v}_2 + \boldsymbol{e}_2. \tag{4.13}$$

We consider two cases. First we suppose that $\boldsymbol{\xi}_1 \ne \boldsymbol{\xi}_2$. Then (4.13) implies that that for each $m = 1, 2, \dots, M$, we have

$$\begin{aligned} \Bigl\|\sum_{n=1}^{N} \alpha_{mn}(\xi_{n1} - \xi_{n2})\Bigr\| &\le \Bigl|\sum_{n=1}^{N} \alpha_{mn}(\xi_{n1} - \xi_{n2}) + v_{1m} - v_{2m}\Bigr| \\ &= |e_{m1} - e_{m2}| \\ &\le \delta_m. \end{aligned} \tag{4.14}$$

As $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ is a nonzero point in $\mathcal{K} - \mathcal{K}$, (4.7) and (4.14) imply that

$$P_M\bigl(A(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\bigr)Q_N(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le \theta\gamma(A, \mathcal{K}) < \gamma(A, \mathcal{K}),$$

which is impossible. Next we suppose that $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$. In this case we get

$$|v_{1m} - v_{2m}| = |e_{1m} - e_{2m}| \le \delta_m < 1 \tag{4.15}$$

for each $m = 1, 2, \dots, M$. But (4.15) implies that $\boldsymbol{v}_1 = \boldsymbol{v}_2$, and therefore $\boldsymbol{e}_1 = \boldsymbol{e}_2$. We have shown that (4.13) implies that $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$, $\boldsymbol{e}_1 = \boldsymbol{e}_2$, and $\boldsymbol{v}_1 = \boldsymbol{v}_2$, and this verifies our claim.

Because the subsets in the collection (4.12) are disjoint in $\mathbb{R}^M$, the images of the subsets

$$\bigl\{A\boldsymbol{\xi} + E : \boldsymbol{\xi} \in \mathcal{K}\bigr\}$$

in the group $G_2 = (\mathbb{R}/\mathbb{Z})^M$ are also disjoint. In particular, the set

$$\bigcup_{\boldsymbol{\xi}\in\mathcal{K}} \bigl(A\boldsymbol{\xi} + E\bigr) \tag{4.16}$$

is contained in a fundamental domain for the quotient group $(\mathbb{R}/\mathbb{Z})^M$. Therefore we obtain the estimate

$$\begin{aligned} \sum_{\boldsymbol{\xi}\in\mathcal{K}} \int_{A\boldsymbol{\xi}+E} \bigl|\widetilde{B}(\boldsymbol{w})\bigr|^2 \,\mathrm{d}\boldsymbol{w} &\le \int_{(\mathbb{R}/\mathbb{Z})^M} \bigl|\widetilde{B}(\boldsymbol{w})\bigr|^2 \,\mathrm{d}\boldsymbol{w} \\ &= \sum_{\boldsymbol{\eta}\in\mathcal{L}} \bigl|b(\boldsymbol{\eta})\bigr|^2 \bigl|U(\boldsymbol{\eta})\bigr|^{-2} \\ &\le \sum_{\boldsymbol{\eta}\in\mathcal{L}} \bigl|b(\boldsymbol{\eta})\bigr|^2. \end{aligned} \tag{4.17}$$

By combining (4.11) and (4.17), we arrive at the inequality

$$\sum_{\boldsymbol{\xi}\in\mathcal{K}} \Bigl|\sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) e(\boldsymbol{\eta}^T A\boldsymbol{\xi})\Bigr|^2 \le \prod_{m=1}^{M} \bigl(L_m + \delta_m^{-1}\bigr) \sum_{\boldsymbol{\eta}\in\mathcal{L}} \bigl|b(\boldsymbol{\eta})\bigr|^2. \tag{4.18}$$

For each $m = 1, 2, \ldots, M$, we select δ_m so that

$$\delta_m^{-1} = X(L_m + 1) + 1, \tag{4.19}$$

where X is a positive real parameter at our disposal. This clearly verifies the requirement that $0 < \delta_m < 1$. We select X to be the unique positive real number such that

$$\prod_{m=1}^{M} \big(X + (L_m + 1)^{-1}\big) = \frac{|\mathcal{K}|}{\theta\gamma(A, \mathcal{K})|\mathcal{L}|}. \tag{4.20}$$

Then it is obvious that this choice of X must satisfy the inequality

$$X^M \le \frac{|\mathcal{K}|}{\theta\gamma(A, \mathcal{K})|\mathcal{L}|}. \tag{4.21}$$

And it follows easily that the identity (4.7) holds. Using (4.19) and (4.21) we find that

$$\begin{aligned} \prod_{m=1}^{M} \big(L_m + \delta_m^{-1}\big) &= |\mathcal{L}|(X + 1)^M \\ &\le |\mathcal{L}|\bigg(\Big(\frac{|\mathcal{K}|}{\theta\gamma(A, \mathcal{K})|\mathcal{L}|}\Big)^{\frac{1}{M}} + 1\bigg)^M \\ &= \Big((\theta\gamma(A, \mathcal{K}))^{-\frac{1}{M}}|\mathcal{K}|^{\frac{1}{M}} + |\mathcal{L}|^{\frac{1}{M}}\Big)^M. \end{aligned} \tag{4.22}$$

As $0 < \theta < 1$ is arbitrary, the inequality (4.6) follows from (4.18) and (4.22). □

Lemma 4.2. *Assume that A is a point in G_1 such that $0 < \gamma(A, \mathcal{K})$, and $\boldsymbol{y}$ is a point in $(\mathbb{R}/\mathbb{Z})^M$. Let $0 < \delta_m \le 1$ for each $m = 1, 2, \ldots, M$, and let $\boldsymbol{x} \mapsto \Phi\big(\boldsymbol{x}, \Delta(\boldsymbol{y})\big)$ denote the characteristic function of the subset*

$$\Delta(\boldsymbol{y}) = \{\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^M : \|x_m - y_m\| \le \tfrac{1}{2}\delta_m \text{ for each } m = 1, 2, \ldots, M\}. \tag{4.23}$$

Then for each subset $\mathcal{K}$ of the form (4.3), we have

$$\sum_{\boldsymbol{\xi}\in\mathcal{K}} \Phi\big(A\boldsymbol{\xi}, \Delta(\boldsymbol{y})\big) \le 4^M \gamma(A, \mathcal{K})^{-1}\mu_2\big(\Delta(\boldsymbol{y})\big)|\mathcal{K}| + 6^M, \tag{4.24}$$

where $\mu_2\big(\Delta(\boldsymbol{y})\big) = \delta_1\delta_2\cdots\delta_M$ is the normalized Haar measure of $\Delta(\boldsymbol{y}) \subseteq (\mathbb{R}/\mathbb{Z})^M$.

Proof. For each $m = 1, 2, \ldots, M$, let $x \mapsto \chi_m(x)$ be the characteristic function of the subset

$$\big\{x \in \mathbb{R}/\mathbb{Z} : \|x - y_m\| \le \tfrac{1}{2}\delta_m\big\}.$$

We apply Corollary 3.4 to the nonnegative valued function $\chi_m(x)$ with $L = L_m$. By that result there exists a trigonometric polynomial

$$u_m(x) = \sum_{\ell=0}^{L_m} \widehat{u}_m(\ell)e(\ell x)$$

such that, if $\|x - y_m\| \le \frac{1}{2}\delta_m$ then

$$1 \le |u_m(x)|^2,$$

and

$$\int_{\mathbb{R}/\mathbb{Z}} |u_m(x)|^2 \, \mathrm{d}x = \delta_m + (L_m + 1)^{-1}.$$

Now let $\mathcal{L} \subseteq \mathbb{Z}^M$ be defined by (4.4). It follows that there exist complex numbers $b(\boldsymbol{\eta})$, defined at each point $\boldsymbol{\eta}$ in $\mathcal{L}$, such that

$$\Phi\big(\boldsymbol{x}, \Delta(\boldsymbol{y})\big) \le \prod_{m=1}^{M} \big|u_m(x_m)\big|^2 = \Big|\sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) e(\boldsymbol{\eta}^T \boldsymbol{x})\Big|^2$$

at each point $\boldsymbol{x}$ in $(\mathbb{R}/\mathbb{Z})^M$. The inequality (4.6) implies that

$$\begin{aligned}\sum_{\boldsymbol{\xi}\in\mathcal{K}} \Phi\big(A\boldsymbol{\xi}, \Delta(\boldsymbol{y})\big) &\le \sum_{\boldsymbol{\xi}\in\mathcal{K}} \Big|\sum_{\boldsymbol{\eta}\in\mathcal{L}} b(\boldsymbol{\eta}) e\big(\boldsymbol{\eta}^T A\boldsymbol{\xi}\big)\Big|^2 \\ &\le \Big(\gamma(A,\mathcal{K})^{-\frac{1}{M}} |\mathcal{K}|^{\frac{1}{M}} + |\mathcal{L}|^{\frac{1}{M}}\Big)^M \sum_{\boldsymbol{\eta}\in\mathcal{L}} |b(\boldsymbol{\eta})|^2 \\ &\le 2^{M-1}\big(\gamma(A,\mathcal{K})^{-1}|\mathcal{K}| + |\mathcal{L}|\big) \prod_{m=1}^{M} \big(\delta_m + (L_m+1)^{-1}\big).\end{aligned} \tag{4.25}$$

The positive integers L_m are at our disposal. We select

$$L_m = \big[\delta_m^{-1}\big] \le \delta_m^{-1} < L_m + 1,$$

and the upper bound

$$\begin{aligned} 2^{M-1}\big(\gamma(A,\mathcal{K})^{-1}|\mathcal{K}| + |\mathcal{L}|\big) &\prod_{m=1}^{M} \big(\delta_m + (L_m+1)^{-1}\big) \\ &\le 4^M \gamma(A,\mathcal{K})^{-1} \mu_2\big(\Delta(\boldsymbol{y})\big)|\mathcal{K}| + 6^M \end{aligned} \tag{4.26}$$

follows easily. Then (4.24) follows from (4.25) and (4.26). □

5. Proof of Theorem 1.1 and Corollary 1.2

We define a positive integer valued function $\boldsymbol{\xi} \mapsto v(\boldsymbol{\xi})$ on elements $\boldsymbol{\xi}$ in the difference set $X = Y - Y$ by

$$\Big|\sum_{\boldsymbol{\eta}\in Y} e(\boldsymbol{x}^T \boldsymbol{\eta})\Big|^2 = \sum_{\boldsymbol{\xi}\in X} v(\boldsymbol{\xi}) e(\boldsymbol{x}^T \boldsymbol{\xi}). \tag{5.1}$$

It follows that $1 \le v(\boldsymbol{\xi}) \le |Y|$ at each point $\boldsymbol{\xi}$ in X, $v(\boldsymbol{0}) = |Y|$, and

$$\sum_{\boldsymbol{\xi}\in X} v(\boldsymbol{\xi}) = |Y|^2.$$

We also define a trigonometric polynomial $\sigma : (\mathbb{R}/\mathbb{Z})^M \to [0,\infty)$ by

$$\sigma(\boldsymbol{x}) = \prod_{m=1}^{M} \tau_{L_m-1}(x_m),$$

where $\tau_{L_m-1}(x)$ is the nonnegative trigonometric polynomial defined by (3.24). It follows from (3.29) that $\sigma(\boldsymbol{x})$ has positive Fourier coefficients supported on the subset

$$\mathcal{L} - \mathcal{L} = \big\{\boldsymbol{\ell} \in \mathbb{Z}^M : 1 - L_m \le \ell_m \le L_m - 1\big\}.$$

From the statement of Lemma 3.5, we conclude that

$$0 \le \sigma(\boldsymbol{x}) \le \prod_{M=1}^{M} \min\big\{L_m, (2\|x_m\|)^{-1}\big\} = F_{\boldsymbol{L}}(\boldsymbol{x}), \tag{5.2}$$

and

$$\prod_{m=1}^{M} \log(L_m + 1) \le \widehat{\sigma}(\mathbf{0}). \tag{5.3}$$

Using (5.3) we have

$$|Y|^2 \prod_{m=1}^{M} \log(L_m + 1) \le \widehat{\sigma}(\mathbf{0})|Y|^2 \le \sum_{\boldsymbol{\ell}\in\mathcal{L}-\mathcal{L}} \widehat{\sigma}(\boldsymbol{\ell}) \Big| \sum_{\boldsymbol{\eta}\in Y} e(\boldsymbol{\ell}^T A\boldsymbol{\eta}) \Big|^2. \tag{5.4}$$

Then using (5.1) and (5.2) we find that

$$\begin{aligned} \sum_{\boldsymbol{\ell}\in\mathcal{L}-\mathcal{L}} \widehat{\sigma}(\boldsymbol{\ell}) \Big| \sum_{\boldsymbol{\eta}\in Y} e(\boldsymbol{\ell}^T A\boldsymbol{\eta}) \Big|^2 &= \sum_{\boldsymbol{\xi}\in X} v(\boldsymbol{\xi}) \sum_{\boldsymbol{\ell}\in\mathcal{L}-\mathcal{L}} \widehat{\sigma}(\boldsymbol{\ell}) e(\boldsymbol{\ell}^T A\boldsymbol{\xi}) \\ &\le |Y| \sum_{\boldsymbol{\xi}\in X} \sigma(A\boldsymbol{\xi}) \\ &\le |Y| \sum_{\boldsymbol{\xi}\in X} F_{\boldsymbol{L}}(A\boldsymbol{\xi}) \\ &= |Y| \prod_{m=1}^{M} L_m + |Y| \sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\ne\mathbf{0}}} F_{\boldsymbol{L}}(A\boldsymbol{\xi}). \end{aligned} \tag{5.5}$$

The inequality (1.5) in the statement of Theorem 1.1 follows from (5.4) and (5.5).

To verify Corollary 1.2 we apply Theorem 1.1 with

$$L_m \le |Y|^{\frac{1}{M}} < L_m + 1, \quad \text{for each } m = 1, 2, \dots, M.$$

Then (1.5) implies that

$$|Y| \left(\frac{\log |Y|}{M} \right)^M - |Y| \le \sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\ne\mathbf{0}}} F_{\boldsymbol{L}}(A\boldsymbol{\xi}) \le \sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\ne\mathbf{0}}} F(A\boldsymbol{\xi}). \tag{5.6}$$

6. Proof of Theorem 1.3 and Corollary 1.4

Let G_1 be a compact abelian group, μ_1 a Haar measure on the Borel subsets of G_1 normalized so that $\mu_1(G_1) = 1$, and Γ_1 the dual group. That is, Γ_1 is the group of continuous homomorphisms $\gamma_1 : G_1 \to \mathbb{T}$, where

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

is the circle group. It follows from the Pontryagin duality theorem that Γ_1 is discrete. Let G_2, μ_2, and Γ_2, be another such triple.

Theorem 6.1. *Assume that $\varphi : G_1 \to G_2$ is a continuous, surjective homomorphism. Then for every function $f : G_2 \to \mathbb{C}$ in $L^1(G_2, \mu_2)$ we have*

$$\int_{G_1} f\big(\varphi(x)\big)\, \mathrm{d}\mu_1(x) = \int_{G_2} f(y)\, \mathrm{d}\mu_2(y). \tag{6.1}$$

Proof. Let γ_2 be a nonprincipal character in the dual group Γ_2. Then the composition

$$\gamma_2 \circ \varphi : G_1 \to \mathbb{T} \tag{6.2}$$

is clearly a continuous homomorphism, and is therefore a character in Γ_1. As γ_2 is not principal, there exists a point y_0 in G_2 such that $\gamma_2(y_0) \neq 1$. Because φ is surjective, there exists a point x_0 in G_1 such that $\varphi(x_0) = y_0$. Then we have

$$\gamma_2\big(\varphi(x_0)\big) = \gamma_2(y_0) \neq 1,$$

and it follows that the composition (6.2) is not the principle character in Γ_1. From the orthogonality relations for characters we find that

$$\int_{G_1} \gamma_2\big(\varphi(x)\big)\ \mathrm{d}\mu_1(x) = \int_{G_2} \gamma_2(y)\ \mathrm{d}\mu_2(y) = 0.$$

If γ_2 *is* the principal character in Γ_2, then it is obvious that the composition (6.2) is the principal character in Γ_1. Hence in this case we get

$$\int_{G_1} \gamma_2\big(\varphi(x)\big)\ \mathrm{d}\mu_1(x) = \int_{G_2} \gamma_2(y)\ \mathrm{d}\mu_2(y) = 1.$$

Thus we have

$$\int_{G_1} \gamma_2\big(\varphi(x)\big)\ \mathrm{d}\mu_1(x) = \int_{G_2} \gamma_2(y)\ \mathrm{d}\mu_2(y) \tag{6.3}$$

for all characters γ_2 in Γ_2. If $F_2 \subseteq \Gamma_2$ if a finite subset, and

$$T(y) = \sum_{\gamma_2 \in F_2} c(\gamma_2)\gamma_2(y)$$

is a finite linear combination of characters from Γ_2 with complex coefficients, then (6.3) implies that

$$\int_{G_1} T\big(\varphi(x)\big)\ \mathrm{d}\mu_1(x) = \int_{G_2} T(y)\ \mathrm{d}\mu_2(y). \tag{6.4}$$

Because the set of all finite linear combinations of characters from Γ_2 is dense in $L^1\big(G_2, \mu_2\big)$, it follows in a standard manner that (6.1) holds. ☐

Corollary 6.2. *Let $E \subseteq G_2$ be a Borel set. Then we have*

$$\mu_1\big(\varphi^{-1}(E)\big) = \mu_1\big\{x \in G_1 : \varphi(x) \in E\big\} = \mu_2(E). \tag{6.5}$$

Proof. This is (6.1) in the special case $f(y) = \chi_E(y)$, where χ_E is the characteristic function of the Borel set E. ☐

We now return to consideration of the groups

$$G_1 = (\mathbb{R}/\mathbb{Z})^{MN}, \quad \text{and} \quad G_2 = (\mathbb{R}/\mathbb{Z})^{M}, \tag{6.6}$$

specified in (1.2). We continue to write the elements of G_1 as $M \times N$ matrices with entries in the additive group $\mathbb{R}/\mathbb{Z}$. If $\boldsymbol{\xi}$ is a (column) vector in $\mathbb{Z}^N$ then

$$A \mapsto A\boldsymbol{\xi} = \left(\sum_{n=1}^{N} \alpha_{mn}\xi_n\right) \tag{6.7}$$

defines a continuous homomorphism from G_1 into G_2. If $\boldsymbol{\xi} \neq \mathbf{0}$ then it follows that (6.7) is surjective, and therefore the conclusions of Theorem 6.1 and Corollary 6.2 can be applied to this map.

Lemma 6.3. *Let* G_2 *be as in* (6.6). *If* $0 < \delta \le 1$ *then*

$$\mu_2\Big\{\boldsymbol{\beta} \in G_2 : \prod_{m=1}^{M} \big(2\|\beta_m\|\big) \le \delta\Big\} = \frac{1}{(M-1)!} \int_0^{\delta} \big(-\log x\big)^{M-1}\, \mathrm{d}x. \tag{6.8}$$

Proof. Because $G_2 = (\mathbb{R}/\mathbb{Z})^M$ is a product set, the set of coordinate functions

$$\boldsymbol{\beta} \mapsto \log\big(2\|\beta_m\|\big), \quad \text{for each } m = 1, 2, \dots, M,$$

is a collection of M independent, identically distributed, random variables on the probability space $\big(G_2, \mu_2\big)$. The density function of each of these random variables is

$$h(x) = \begin{cases} e^x & \text{if } x \le 0, \\ 0 & \text{if } 0 < x. \end{cases} \tag{6.9}$$

That is, for each index m, $1 \le m \le M$, and $-\infty < u < v < \infty$, we have

$$\mu_2\big\{\boldsymbol{\beta} \in G_2 : u < \log\big(2\|\beta_m\|\big) \le v\big\} = \int_u^v h(x)\, \mathrm{d}x.$$

Therefore the density function associated to the sum

$$\boldsymbol{\beta} \mapsto \sum_{m=1}^{M} \log\big(2\|\beta_m\|\big)$$

of M independent random variables, is the M-fold convolution

$$h * h * \cdots * h(x) = h^{(M)}(x).$$

In order to compute this density, observe that the Fourier transform of h is

$$\widehat{h}(y) = \int_{-\infty}^{\infty} h(x) e(-xy)\, \mathrm{d}x = (1 - 2\pi i y)^{-1}. \tag{6.10}$$

And therefore the Fourier transform of $h^{(M)}(x)$ is $(1 - 2\pi i y)^{-M}$. By differentiating both sides of (6.10) repeatedly with respect to y, we obtain the identity

$$\frac{1}{(M-1)!} \int_{-\infty}^{0} (-x)^{M-1} e^x e(-xy)\, \mathrm{d}x = (1 - 2\pi i y)^{-M}.$$

It follows that

$$h^{(M)}(x) = \frac{(-x)^{M-1} e^x}{(M-1)!} \quad \text{if } x \le 0,$$

and

$$h^{(M)}(x) = 0 \quad \text{if } 0 < x.$$

In particular, if $0 < \delta \le 1$ we get

$$\begin{aligned} \mu_2\Big\{\boldsymbol{\beta} \in G_2 : \sum_{m=1}^{M} \log\big(2\|\beta_m\|\big) \le \log \delta\Big\} \\ = \frac{1}{(M-1)!} \int_{-\infty}^{\log \delta} (-x)^{M-1} e^x\, \mathrm{d}x \\ = \frac{1}{(M-1)!} \int_0^{\delta} \big(-\log x\big)^{M-1}\, \mathrm{d}x. \end{aligned} \tag{6.11}$$

Then (6.11) is equivalent to (6.8). □

Corollary 6.4. *Let G_1 be as in* (6.6)*, and let $\boldsymbol{\xi}$ be a nonzero lattice point in $\mathbb{Z}^N$. If $1 \le \lambda < \infty$ then*

$$\mu_1\big\{A \in G_1 : \lambda \le F(A\boldsymbol{\xi})\big\} = \frac{1}{(M-1)!}\int_\lambda^\infty \big(\log x\big)^{M-1} x^{-2}\ \mathrm{d}x. \tag{6.12}$$

Proof. From Corollary 6.2 we have

$$\mu_1\big\{A \in G_1 : \lambda \le F(A\boldsymbol{\xi})\big\} = \mu_2\{\boldsymbol{\beta} \in G_2 : \lambda \le F(\boldsymbol{\beta})\}. \tag{6.13}$$

The measure of the set on the right of (6.13) follows from (6.8) by a simple change of variables. □

We are now in position to bound the μ_1-measure of the set

$$\Big\{A \in G_1 : \lambda \le \sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\ne\mathbf{0}}} F(A\boldsymbol{\xi})\Big\},$$

where $X \subseteq \mathbb{Z}^N$ is a finite, nonempty subset of lattice points.

Theorem 6.5. *Let G_1 be as in* (6.6)*, and let $X \subseteq \mathbb{Z}^N$ be a finite, nonempty subset of lattice points with cardinaltiy $|X|$. If $1 \le \lambda < \infty$ then*

$$\mu_1\Big\{A \in G_1 : \lambda \le \sum_{\substack{\boldsymbol{\xi}\in X\\ \boldsymbol{\xi}\ne\mathbf{0}}} F(A\boldsymbol{\xi})\Big\} \le \frac{|X|}{M!}\int_\lambda^\infty \big(\log x\big)^{M} x^{-2}\ \mathrm{d}x. \tag{6.14}$$

Proof. We may assume that X does not contain $\mathbf{0}$. Let $1 \le \eta < \infty$, and for each lattice point $\boldsymbol{\xi}$ in X, define

$$D(\eta, \boldsymbol{\xi}) = \big\{A \in G_1 : F(A\boldsymbol{\xi}) < \eta\big\}.$$

By Corollary 6.4 we have

$$\begin{aligned} \mu_1\big(G_1 \setminus D(\eta,\boldsymbol{\xi})\big) &= \mu_1\big\{A \in G_1 : \eta \le F(A\boldsymbol{\xi})\big\} \\ &= \frac{1}{(M-1)!}\int_\eta^\infty \big(\log x\big)^{M-1} x^{-2}\ \mathrm{d}x. \end{aligned} \tag{6.15}$$

Next we write

$$\mathcal{D}(\eta) = \bigcap_{\boldsymbol{\xi}\in X} D(\eta, \boldsymbol{\xi}),$$

so that using (6.15) we have

$$\begin{aligned} \mu_1\big(G_1 \setminus \mathcal{D}(\eta)\big) &= \mu_1\Big(\bigcup_{\boldsymbol{\xi}\in X}\big(G_1 \setminus D(\eta,\boldsymbol{\xi})\big)\Big) \\ &\le \sum_{\boldsymbol{\xi}\in X} \mu_1\big(G_1 \setminus D(\eta,\boldsymbol{\xi})\big) \\ &= \frac{|X|}{(M-1)!}\int_\eta^\infty \big(\log x\big)^{M-1} x^{-2}\ \mathrm{d}x. \end{aligned} \tag{6.16}$$

Using a standard argument we get the estimate

$$\begin{aligned}\mu_1\Big\{A\in\mathcal{D}(\eta):\lambda\le\sum_{\boldsymbol{\xi}\in X}F(A\boldsymbol{\xi})\Big\}&\\ &\le\lambda^{-1}\int_{\mathcal{D}(\eta)}\Big(\sum_{\boldsymbol{\xi}\in X}F(A\boldsymbol{\xi})\Big)\,\mathrm{d}\mu_1(A)\\ &\le\lambda^{-1}\sum_{\boldsymbol{\xi}\in X}\Big(\int_{D(\eta,\boldsymbol{\xi})}F(A\boldsymbol{\xi})\,\mathrm{d}\mu_1(A)\Big).\end{aligned}\tag{6.17}$$

From (6.1) and Corollary 6.4, the integral on the right of (6.17) is

$$\begin{aligned}\int_{D(\eta,\boldsymbol{\xi})}F(A\boldsymbol{\xi})\,\mathrm{d}\mu_1(A)&=\int_{G_1}\chi_{D(\eta,\boldsymbol{\xi})}\big(F(A\boldsymbol{\xi})\big)F(A\boldsymbol{\xi})\,\mathrm{d}\mu_1(A)\\ &=\int_{G_2}\chi_{D(\eta,\boldsymbol{\xi})}\big(F(\boldsymbol{\beta})\big)F(\boldsymbol{\beta})\,\mathrm{d}\mu_2(\boldsymbol{\beta})\\ &=\frac{1}{(M-1)!}\int_1^{\eta}\big(\log x\big)^{M-1}x^{-1}\,\mathrm{d}x.\end{aligned}\tag{6.18}$$

We combine (6.17) and (6.18) to obtain the inequality

$$\begin{aligned}\mu_1\Big\{A\in\mathcal{D}(\eta):\lambda\le\sum_{\boldsymbol{\xi}\in X}F(A\boldsymbol{\xi})\Big\}&\\ &\le\frac{|X|}{\lambda(M-1)!}\int_1^{\eta}\big(\log x\big)^{M-1}x^{-1}\,\mathrm{d}x.\end{aligned}\tag{6.19}$$

And then we combine (6.16) and (6.19) to get

$$\begin{aligned}&\mu_1\Big\{A\in G_1:\lambda\le\sum_{\boldsymbol{\xi}\in X}F(A\boldsymbol{\xi})\Big\}\\ &\quad\le\mu_1\Big\{A\in\mathcal{D}(\eta):\lambda\le\sum_{\boldsymbol{\xi}\in X}F(A\boldsymbol{\xi})\Big\}+\mu_1\big(G_1\setminus\mathcal{D}(\eta)\big)\\ &\quad\le\frac{|X|}{(M-1)!}\Big(\lambda^{-1}\int_1^{\eta}\big(\log x\big)^{M-1}x^{-1}\,\mathrm{d}x+\int_{\eta}^{\infty}\big(\log x\big)^{M-1}x^{-2}\,\mathrm{d}x\Big).\end{aligned}\tag{6.20}$$

The parameter η in the upper bound on the right of (6.20) is at our disposal. A simple calculation shows that the right hand side of (6.20) is minimized at $\eta=\lambda$. We find that

$$\begin{aligned}\lambda^{-1}\int_1^{\lambda}\big(\log x\big)^{M-1}x^{-1}\,\mathrm{d}x+\int_{\lambda}^{\infty}\big(\log x\big)^{M-1}x^{-2}\,\mathrm{d}x&\\ &=\frac{1}{M}\int_{\lambda}^{\infty}\big(\log x\big)^{M}x^{-2}\,\mathrm{d}x,\end{aligned}\tag{6.21}$$

and the theorem is proved. □

In applications of Theorem 6.5 the identity

$$\frac{1}{M!}\int_{\lambda}^{\infty}\big(\log x\big)^{M}x^{-2}\,\mathrm{d}x=\lambda^{-1}\sum_{m=0}^{M}\frac{(\log\lambda)^m}{m!}\tag{6.22}$$

is useful. We also note that if $1 \le \lambda_1 < \infty$ and $1 \le \lambda_2 < \infty$, then

$$\begin{aligned}\sum_{m=0}^{M} \frac{(\log\lambda_1 + \log\lambda_2)^m}{m!} &= \sum_{m=0}^{M}\sum_{l=0}^{m} \frac{(\log\lambda_1)^l}{l!}\frac{(\log\lambda_2)^{m-l}}{(m-l)!} \\ &= \sum_{l=0}^{M} \frac{(\log\lambda_1)^l}{l!} \sum_{m=l}^{M} \frac{(\log\lambda_2)^{m-l}}{(m-l)!} \\ &\le \left(\sum_{l=0}^{M} \frac{(\log\lambda_1)^l}{l!}\right)\left(\sum_{k=0}^{M} \frac{(\log\lambda_2)^k}{k!}\right).\end{aligned} \tag{6.23}$$

For example, if $0 < \delta < 1$ then there exists a unique real number λ such that $1 < \lambda < \infty$ and

$$\delta = \frac{1}{M!}\int_{\lambda}^{\infty} \big(\log x\big)^M x^{-2}\, \mathrm{d}x.$$

Using (6.22) we find that

$$\begin{aligned}\delta &< \lambda^{-1} \sum_{m=0}^{M} 2^{M-m} \frac{(\log\lambda)^m}{m!} \\ &= 2^M \lambda^{-1} \sum_{m=0}^{M} \frac{\left(\frac{1}{2}\log\lambda\right)^m}{m!} \\ &\le 2^M \lambda^{-\frac{1}{2}},\end{aligned}$$

and therefore

$$\lambda < 4^M \delta^{-2}. \tag{6.24}$$

Then using (6.23) and (6.24), we get

$$\begin{aligned}\lambda &= \delta^{-1} \sum_{m=0}^{M} \frac{(\log\lambda)^m}{m!} \\ &\le \delta^{-1} \sum_{m=0}^{M} \frac{\left(\log 4^M + \log\delta^{-2}\right)^m}{m!} \\ &\le 4^M \delta^{-1} \sum_{m=0}^{M} \frac{\left(2\log\delta^{-1}\right)^m}{m!} \\ &\le 8^M \delta^{-1} \sum_{m=0}^{M} \frac{\left(\log\delta^{-1}\right)^m}{m!}.\end{aligned} \tag{6.25}$$

We now prove Theorem 1.3. Let λ be selected so that

$$\varepsilon |X|^{-1} = \frac{1}{M!}\int_{\lambda}^{\infty} \big(\log x\big)^M x^{-2}\, \mathrm{d}x.$$

Then Theorem 6.5 implies that

$$\mu_1\left\{A \in G_1 : \lambda \le \sum_{\boldsymbol{\xi}\in X} F(A\boldsymbol{\xi})\right\} \le \varepsilon. \tag{6.26}$$

We apply the inequality (6.25) with $\delta = \varepsilon|X|^{-1}$. It follows from (6.26) that the inequality (1.9) holds at all point A in G_1 outside a set of μ_1-measure at most ε.

Next we prove Corollary 1.4. It follows from Corollary 6.2 that

$$\mu_1\bigl\{A \in G_1 : F(A\boldsymbol{\xi}) < \infty\bigr\} = 1$$

for each point $\boldsymbol{\xi} \neq \mathbf{0}$ in $\mathbb{Z}^N$. Therefore almost all points A in G_1 belong to the subset

$$\mathcal{Y} = \bigcap_{\substack{\boldsymbol{\xi} \in \mathbb{Z}^N \\ \boldsymbol{\xi} \neq \mathbf{0}}} \bigl\{A \in G_1 : F(A\boldsymbol{\xi}) < \infty\bigr\}. \tag{6.27}$$

For each $\ell = 1, 2, \dots$, we define

$$\varepsilon_\ell^{-1} = \log 3|Y_\ell|\bigl(\log\log 27|Y_\ell|\bigr)^{1+\eta}, \tag{6.28}$$

and

$$\mathcal{A}_\ell = \left\{A \in G_1 : 8^M \varepsilon_\ell^{-1}|Y_\ell| \sum_{m=0}^{M} \frac{\bigl(\log \varepsilon_\ell^{-1}|Y_\ell|\bigr)^m}{m!} < \sum_{\substack{\boldsymbol{\xi} \in Y_\ell \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi})\right\}.$$

Then using (1.9) and (1.10) we find that

$$\sum_{\ell=1}^{\infty} \mu_1\bigl(\mathcal{A}_\ell\bigr) \le \sum_{\ell=1}^{\infty} \varepsilon_\ell < \infty.$$

Thus by the Borel-Cantelli lemma almost all points A in G_1 belong to the subset

$$\mathcal{Z} = \bigcup_{L=1}^{\infty} \bigcap_{\ell=L}^{\infty} \bigl(G_1 \setminus \mathcal{A}_\ell\bigr).$$

Now suppose that A belongs to $\mathcal{Y} \cap \mathcal{Z}$. Then A belongs to only finitely many of the subsets $\mathcal{A}_\ell$. That is, there exists a positive integer $L = L(A)$ such that

$$\sum_{\substack{\boldsymbol{\xi} \in Y_\ell \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi}) \le 8^M \varepsilon_\ell^{-1}|Y_\ell| \sum_{m=0}^{M} \frac{\bigl(\log \varepsilon_\ell^{-1}|Y_\ell|\bigr)^m}{m!} \tag{6.29}$$

for $L \le \ell$. Because A also belongs to $\mathcal{Y}$, the numbers

$$\sum_{\substack{\boldsymbol{\xi} \in Y_\ell \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi}), \quad \text{where } \ell = 1, 2, \dots, L(A), \tag{6.30}$$

are finite. From (6.29) and (6.30) we conclude that

$$\sum_{\substack{\boldsymbol{\xi} \in Y_\ell \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi}) \ll_{A,M} \varepsilon_\ell^{-1}|Y_\ell|\bigl(\log \varepsilon_\ell^{-1}|Y_\ell|\bigr)^M$$

for *all* positive integers $\ell = 1, 2, \dots$. Then using (6.28) we find that

$$\sum_{\substack{\boldsymbol{\xi} \in Y_\ell \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi}) \ll_{A,\eta,M} |Y_\ell|\bigl(\log 3|Y_\ell|\bigr)^{M+1}\bigl(\log\log 27|Y_\ell|\bigr)^{1+\eta} \tag{6.31}$$

for all positive integers ℓ. Finally, if X is in the collection of subsets $\mathcal{X}$, then $X \subseteq Y_\ell$ for some positive integer ℓ, and therefore

$$\sum_{\substack{\boldsymbol{\xi} \in X \\ \boldsymbol{\xi} \neq \mathbf{0}}} F(A\boldsymbol{\xi}) \ll_{A,\eta,M} |Y_\ell|\bigl(\log 3|Y_\ell|\bigr)^{M+1}\bigl(\log\log 27|Y_\ell|\bigr)^{1+\eta}. \tag{6.32}$$

Because $|Y_\ell| \le C_2|X|$ for an absolute constant C_2, the bound (1.11) follows easily from (6.32).

7. Proof of Theorem 2.1

We assume throughout this section that A is an $M \times N$ matrix in $G_1 = (\mathbb{R}/\mathbb{Z})^{MN}$, and A is multiplicatively badly approximable. Using the notation introduced in (4.1) and (4.2), we conclude that

$$\gamma(A) = \inf\bigl\{P_M\bigl(A\boldsymbol{\xi}\bigr)Q_N(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathbb{Z}^N, \text{ and } \boldsymbol{\xi} \ne \mathbf{0}\bigr\} \tag{7.1}$$

is positive. In particular we have

$$0 < \gamma(A) \le \gamma(A, \mathcal{K}) \le 1 \tag{7.2}$$

for each subset $\mathcal{K} \subseteq \mathbb{Z}^N$ defined by (4.3) and $\gamma(A, \mathcal{K})$ defined by (4.5).

Let

$$\mathcal{D} = \{\boldsymbol{d} \in \mathbb{Z}^M : 1 \le d_m \text{ for each } m = 1, 2, \dots, M\}.$$

For each point $\boldsymbol{d}$ in $\mathcal{D}$ we write

$$|\boldsymbol{d}| = d_1 + d_2 + \cdots + d_M,$$

and we define

$$B(\boldsymbol{d}) = \bigl\{\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^M : 2^{-d_m-1} < \|x_m\| \le 2^{-d_m} \text{ for each } m = 1, 2, \dots, M\}.$$

As each subset $B(\boldsymbol{d})$ contains 2^M connected subsets of equal measure, it follows that

$$\mu_2\bigl(B(\boldsymbol{d})\bigr) = 2^M \prod_{m=1}^{M} (2^{-d_m} - 2^{-d_m-1}) = 2^{-|\boldsymbol{d}|}. \tag{7.3}$$

If $\boldsymbol{x}$ belongs to $B(\boldsymbol{d})$ we find that

$$2^{|\boldsymbol{d}|-M} \le F(\boldsymbol{x}) = \prod_{m=1}^{M} (2\|x_m\|)^{-1} < 2^{|\boldsymbol{d}|}. \tag{7.4}$$

If $\boldsymbol{d}$ and $\boldsymbol{e}$ are distinct elements of $\mathcal{D}$, then it is clear that $B(\boldsymbol{d})$ and $B(\boldsymbol{e})$ are disjoint subsets. Moreover, we have

$$\bigcup_{\boldsymbol{d}\in\mathcal{D}} B(\boldsymbol{d}) = \bigl\{\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^M : 0 < \|x_m\| \text{ for each } m = 1, 2, \dots, M\bigr\}.$$

As A is multiplicatively badly approximable, for each point $\boldsymbol{\xi} \ne \mathbf{0}$ in $\mathbb{Z}^N$ there is a unique point $\boldsymbol{d}$ in $\mathcal{D}$ such that $A\boldsymbol{\xi}$ belongs to $B(\boldsymbol{d})$. Because the co-ordinates of each point $\boldsymbol{d}$ in $\mathcal{D}$ are positive integers, it is clear that $M \le |\boldsymbol{d}|$. And if R is an integer such that $M \le R$, then

$$\sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 1 = \sum_{m=M}^{R} \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|= m}} 1 = \sum_{m=M}^{R} \binom{m-1}{m-M} = \binom{R}{M}. \tag{7.5}$$

Let $\boldsymbol{x} \mapsto \Phi\bigl(\boldsymbol{x}, B(\boldsymbol{d})\bigr)$ denote the characteristic function of the subset $B(\boldsymbol{d})$. The set $B(\boldsymbol{d})$ is contained in the union of 2^M subsets $\Delta(\boldsymbol{y})$ of the form (4.23). In particular, we have

$$B(\boldsymbol{d}) \subseteq \bigcup \bigl\{\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^M : \|x_m \pm (3)2^{-d_m-2}\| \le 2^{-d_m-2}\bigr\}, \tag{7.6}$$

where the union on the right of (7.6) is over the set of all 2^M choices of $\pm$ signs. We apply Lemma 4.2 to each subset on the right of (7.6). Then the inequality (4.24) in the statement of Lemma 4.2 and (7.2), imply that

$$\sum_{\boldsymbol{\xi}\in\mathcal{K}} \Phi\big(A\boldsymbol{\xi}, B(\boldsymbol{d})\big) \le 4^M\gamma(A)^{-1}\mu_2\big(B(\boldsymbol{d})\big)|\mathcal{K}| + 12^M \tag{7.7}$$

for each point $\boldsymbol{d}$ in $\mathcal{D}$.

From (7.1) we conclude that if $\boldsymbol{\xi} \ne \boldsymbol{0}$ belongs to $\mathcal{K}$, then

$$1 \le F(A\boldsymbol{\xi}) \le 2^{-M}\gamma(A)^{-1}Q_N(\boldsymbol{\xi}) \le 2^{-M}\gamma(A)^{-1}|\mathcal{K}|. \tag{7.8}$$

Let R be the unique positive integer such that

$$2^{R-1} < \gamma(A)^{-1}|\mathcal{K}| \le 2^R. \tag{7.9}$$

It follows that

$$R \le 1 + \frac{-\log\gamma(A) + \log|\mathcal{K}|}{\log 2} \ll_A \log 2|\mathcal{K}|, \tag{7.10}$$

and using (7.8) we get

$$F(A\boldsymbol{\xi}) \le 2^{R-M}. \tag{7.11}$$

For each point $\boldsymbol{\xi} \ne \boldsymbol{0}$ in $\mathcal{K}$ there exists a unique point $\boldsymbol{d}$ in $\mathcal{D}$ such that $A\boldsymbol{\xi}$ belongs to $B(\boldsymbol{d})$. From (7.4) and (7.11) we conclude that $|\boldsymbol{d}| \le R$. Then using (7.4) and (7.7), we obtain the inequality

$$\begin{aligned}\sum_{\substack{\boldsymbol{\xi}\in\mathcal{K}\\ \boldsymbol{\xi}\ne\boldsymbol{0}}} F(A\boldsymbol{\xi}) &= \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} \sum_{\substack{\boldsymbol{\xi}\in\mathcal{K}\\ \boldsymbol{\xi}\ne\boldsymbol{0}}} \Phi\big(A\boldsymbol{\xi}, B(\boldsymbol{d})\big)F(A\boldsymbol{\xi})\\ &\le \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 2^{|\boldsymbol{d}|} \sum_{\substack{\boldsymbol{\xi}\in\mathcal{K}\\ \boldsymbol{\xi}\ne\boldsymbol{0}}} \Phi\big(A\boldsymbol{\xi}, B(\boldsymbol{d})\big)\\ &\le 4^M\gamma(A)^{-1}|\mathcal{K}| \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 2^{|\boldsymbol{d}|}\mu_2\big(B(\boldsymbol{d})\big) + 12^M \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 2^{|\boldsymbol{d}|}\end{aligned} \tag{7.12}$$

From (7.3), (7.5), and (7.10), we find that

$$\sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 2^{|\boldsymbol{d}|}\mu_2\big(B(\boldsymbol{d})\big) = \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 1 = \binom{R}{M} \ll_A \big(\log 2|\mathcal{K}|\big)^M. \tag{7.13}$$

In a similar manner using (7.9), we get

$$\sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 2^{|\boldsymbol{d}|} \le 2^R \sum_{\substack{\boldsymbol{d}\in\mathcal{D}\\ |\boldsymbol{d}|\le R}} 1 \ll_A |\mathcal{K}|\big(\log 2|\mathcal{K}|\big)^M. \tag{7.14}$$

The inequality (2.8) in the statement of Theorem 2.1 follows from (7.12), (7.13), and (7.14).

8. Proof of Theorem 2.2

Let $A = (\alpha_{mn})$ be an $M \times N$ real matrix with $L = M + N$. We write B for the $L \times L$ real matrix partitioned into blocks as

$$B = \begin{pmatrix} \mathbf{1}_M & A \\ \mathbf{0} & \mathbf{1}_N \end{pmatrix},$$

where $\mathbf{1}_M$ and $\mathbf{1}_N$ are $M \times M$ and $N \times N$ identity matrices, respectively. Then we write $\Delta = [\delta_\ell]$ for an $L \times L$ diagonal matrix with positive diagonal entries δ_ℓ such that

$$\det \Delta = \prod_{\ell=1}^{L} \delta_\ell = 1. \tag{8.1}$$

Using B and Δ we define a lattice $\mathcal{M} \subseteq \mathbb{R}^L$ by

$$\mathcal{M} = \big\{\Delta B \boldsymbol{m} : \boldsymbol{m} \in \mathbb{Z}^L\big\}. \tag{8.2}$$

And we define the associated convex body $C_L \subseteq \mathbb{R}^L$ by

$$C_L = \big\{\boldsymbol{x} \in \mathbb{R}^L : |\boldsymbol{x}|_\infty \le 1\big\},$$

where

$$|\boldsymbol{x}|_\infty = \max\big\{|x_1|, |x_2|, \dots, |x_L|\big\}.$$

Let

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_L < \infty \tag{8.3}$$

be the successive minima of the lattice $\mathcal{M}$ with respect to the convex body C_L. We have

$$\det(\mathcal{M}) = \big|\det(\Delta B)\big| = 1, \quad \text{and} \quad \mathrm{Vol}_L\big(C_L\big) = 2^L. \tag{8.4}$$

Therefore it follows from (8.4) and Minkowski's inequality (see [5, Chapter VIII, Theorem V]) that

$$(L!)^{-1} \le \lambda_1 \lambda_2 \cdots \lambda_L \le 1. \tag{8.5}$$

The dual (or polar) lattice $\mathcal{M}^* \subseteq \mathbb{R}^L$ is given by

$$\mathcal{M}^* = \big\{\Delta^{-1} B^{-T} \boldsymbol{n} : \boldsymbol{n} \in \mathbb{Z}^L\big\},$$

where

$$B^{-T} = \begin{pmatrix} \mathbf{1}_M & \mathbf{0} \\ -A^T & \mathbf{1}_N \end{pmatrix}.$$

And the dual (or polar) convex body C_L^* is

$$C_L^* = \big\{\boldsymbol{x} \in \mathbb{R}^L : |\boldsymbol{x}|_1 \le 1\big\},$$

where

$$|\boldsymbol{x}|_1 = |x_1| + |x_2| + \cdots + |x_L|.$$

Let

$$0 < \lambda_1^* \le \lambda_2^* \le \cdots \le \lambda_L^* < \infty \tag{8.6}$$

be the successive minima associated to the lattice $\mathcal{M}^*$ and the convex body C_L^*. In this case we find that

$$\det(\mathcal{M}^*) = \big|\det(\Delta^{-1} B^{-T})\big| = 1, \quad \text{and} \quad \mathrm{Vol}_L\big(C_L^*\big) = \frac{2^L}{L!}.$$

Thus Minkowski's inequality for the dual successive minima is

$$1 \le \lambda_1^* \lambda_2^* \cdots \lambda_L^* \le L!. \tag{8.7}$$

The two sets of successive minima (8.3) and (8.6) are linked by an inequality of Mahler [13] (see also [5, Chapter VIII, Theorem VI]), which asserts that

$$1 \le \lambda_\ell \lambda_{L-\ell+1}^* \le L!$$

for each integer $\ell = 1, 2, \dots, L$. In particular, if $\ell = 1$ we have

$$1 \le \lambda_1 \lambda_L^*. \tag{8.8}$$

Then using (8.6), (8.7), and (8.8), we get

$$\frac{\bigl(\lambda_1^*\bigr)^{L-1}}{L!} \le \frac{\lambda_1^* \lambda_2^* \cdots \lambda_{L-1}^*}{L!} \le \frac{\lambda_1 \bigl(\lambda_1^* \lambda_2^* \cdots \lambda_L^*\bigr)}{L!} \le \lambda_1. \tag{8.9}$$

Now assume that A^T is multiplicatively badly approximable. We have already observed that each submatrix $A(I,J)^T$ is multiplicatively badly approximable. In particular, each column of A^T is multiplicatively badly approximable, and therefore each column of A^T is badly approximable. By the basic transference principle [4, Chapter V, Corollary to Theorem II], each row of the matrix A is badly approximable. Then (2.4) implies that

$$0 < \Bigl\| \sum_{n=1}^N \alpha_{mn} \xi_n \Bigr\| \tag{8.10}$$

for each point $\boldsymbol{\xi} \ne \mathbf{0}$ in $\mathbb{Z}^N$.

Because A^T is multiplicatively badly approximable, there exists a positive constant $\gamma\bigl(A^T\bigr)$ such that for each vector $\boldsymbol{u} \ne \mathbf{0}$ in $\mathbb{Z}^M$, we have

$$\gamma\bigl(A^T\bigr) \le \biggl(\prod_{m=1}^M \bigl(|u_m| + 1\bigr)\biggr)\biggl(\prod_{n=1}^N \Bigl\| \sum_{m=1}^M u_m \alpha_{mn} \Bigr\|\biggr). \tag{8.11}$$

We will show that there exists a positive constant $\gamma(A)$ such that for each vector $\boldsymbol{\xi} \ne \mathbf{0}$ in $\mathbb{Z}^N$, we have

$$\gamma(A) \le \biggl(\prod_{m=1}^M \Bigl\| \sum_{n=1}^N \alpha_{mn} \xi_n \Bigr\|\biggr)\biggl(\prod_{n=1}^N \bigl(|\xi_n| + 1\bigr)\biggr). \tag{8.12}$$

Our proof of (8.12) will be by induction on the positive integer M.

Let $\boldsymbol{\xi} \ne \mathbf{0}$ be a point in $\mathbb{Z}^N$ and let $\boldsymbol{\eta}$ be a point in $\mathbb{Z}^M$ such that

$$\Bigl| \eta_m + \sum_{n=1}^N \alpha_{mn} \xi_n \Bigr| = \Bigl\| \sum_{n=1}^N \alpha_{mn} \xi_n \Bigr\| \tag{8.13}$$

for each $m = 1, 2, \dots, M$. It will be convenient to define the vector $\boldsymbol{\psi}$ in $\mathbb{Z}^L$ by

$$\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\xi} \end{pmatrix},$$

where $\boldsymbol{\eta}$ belongs to $\mathbb{Z}^M$ and $\boldsymbol{\xi} \ne \mathbf{0}$ belongs to $\mathbb{Z}^N$. In view of (8.10), we define R to be the unique positive real number such that

$$R^L = \biggl(\prod_{m=1}^M \Bigl\| \sum_{n=1}^N \alpha_{mn} \xi_n \Bigr\|\biggr)\biggl(\prod_{n=1}^N \bigl(|\xi_n| + 1\bigr)\biggr). \tag{8.14}$$

Next we select the $L \times L$ diagonal matrix $\Delta = [\delta_\ell]$ so that

$$R = \delta_m \Big\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \Big\| \quad \text{if } 1 \le m \le M, \tag{8.15}$$

and

$$R = \delta_{M+n}\big(|\xi_n| + 1\big) \quad \text{if } 1 \le n \le N. \tag{8.16}$$

It follows from (8.14) that Δ satisfies the condition (8.1). As $\Delta B\boldsymbol{\psi}$ is a nonzero point in the lattice $\mathcal{M}$ defined by (8.2), we have

$$\lambda_1 \le |\Delta B\boldsymbol{\psi}|_\infty. \tag{8.17}$$

Then using (8.15) and (8.16), we find that (8.17) can be written as

$$\lambda_1 \le |\Delta B\boldsymbol{\psi}|_\infty = R. \tag{8.18}$$

It is well known (see [5, Chapter VIII, Lemma 1]) that there exists a point $\boldsymbol{w} \ne \boldsymbol{0}$ in $\mathbb{Z}^L$ such that

$$\big|\Delta^{-1}B^{-T}\boldsymbol{w}\big|_1 = \lambda_1^*. \tag{8.19}$$

Again it will be convenient to partition the column vector $\boldsymbol{w}$ as

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix},$$

where $\boldsymbol{u}$ is a point in $\mathbb{Z}^M$ and $\boldsymbol{v}$ is a point in $\mathbb{Z}^N$. Then (8.19) can be written as

$$\begin{aligned} R^{-1}\sum_{m=1}^{M} |u_m| \Big| \eta_m + \sum_{n=1}^{N} \alpha_{mn}\xi_n \Big| \\ + R^{-1}\sum_{n=1}^{N}\big(|\xi_n| + 1\big)\Big| v_n - \sum_{m=1}^{M} u_m\alpha_{mn}\Big| = \lambda_1^* \end{aligned} \tag{8.20}$$

If $\boldsymbol{u} = \boldsymbol{0}$ then $\boldsymbol{v} \ne \boldsymbol{0}$, and (8.20) leads to the inequality

$$R^{-1} \le R^{-1}\sum_{n=1}^{N}\big(|\xi_n| + 1\big)|v_n| = \lambda_1^*. \tag{8.21}$$

Then (8.9), (8.18), and (8.21), imply that

$$\frac{1}{L!} \le R^L. \tag{8.22}$$

Clearly (8.9), (8.18), and (8.22), show that R^L is bounded from below by a positive constant that depends on L, but not on the point $\boldsymbol{\xi} \ne \boldsymbol{0}$ in $\mathbb{Z}^N$.

For the remainder of the proof we assume that $\boldsymbol{u} \ne \boldsymbol{0}$. In this case it is clear from the definition of λ_1^* that we must have

$$\Big| v_n - \sum_{m=1}^{M} u_m\alpha_{mn}\Big| = \Big\| \sum_{m=1}^{M} u_m\alpha_{mn}\Big\| \tag{8.23}$$

for each integer $n = 1, 2, \ldots, N$. Therefore (8.13) and (8.23) imply that (8.20) can be written as

$$\sum_{m=1}^{M} \delta_m^{-1}|u_m| + \sum_{n=1}^{N} \delta_{M+n}^{-1}\Big\| \sum_{m=1}^{M} u_m\alpha_{mn}\Big\| = \lambda_1^*. \tag{8.24}$$

Then using (8.1) and (8.11) we have

$$
\begin{aligned}
\gamma\bigl(A^T\bigr) &\le \left(\prod_{m=1}^{M}\bigl(|u_m|+1\bigr)\right)\left(\prod_{n=1}^{N}\Bigl\|\sum_{m=1}^{M}u_m\alpha_{mn}\Bigr\|\right)\\
&= \left(\prod_{m=1}^{M}\delta_m^{-1}\bigl(|u_m|+1\bigr)\right)\left(\prod_{n=1}^{N}\delta_{M+n}^{-1}\Bigl\|\sum_{m=1}^{M}u_m\alpha_{mn}\Bigr\|\right).
\end{aligned}
\tag{8.25}
$$

We now argue by induction on M. If $M=1$ then $u_1\ne 0$. We use the identity (8.24), and we apply the arithmetic-geometric mean inequality to the right hand side of (8.25). In this way we obtain the inequality

$$
\begin{aligned}
L\gamma\bigl(A^T\bigr)^{\frac{1}{L}} &\le \delta_1^{-1}\bigl(|u_1|+1\bigr)+\sum_{n=1}^{N}\delta_{M+n}^{-1}\Bigl\|\sum_{m=1}^{M}u_m\alpha_{mn}\Bigr\|\\
&\le 2\delta_1^{-1}|u_1|+2\sum_{n=1}^{N}\delta_{M+n}^{-1}\Bigl\|\sum_{m=1}^{M}u_m\alpha_{mn}\Bigr\|\\
&= 2\lambda_1^*.
\end{aligned}
\tag{8.26}
$$

Then (8.9), (8.18), and (8.26), imply that R^L is bounded from below by a positive constant that does not depend on the point $\boldsymbol{\xi}\ne\mathbf{0}$ in $\mathbb{Z}^N$.

Finally, we assume that $2\le M$. We have already remarked that each submatrix $A(I,J)^T$ is multiplicatively badly approximable, and in particular the submatrix of A^T obtained by removing the column indexed by k, where $1\le k\le M$, is multiplicatively badly approximable. Hence by the inductive hypothesis the submatrix of A obtained by removing the row indexed by k is multiplicatively badly approximable. Thus there exists a positive constant $\gamma_k=\gamma_k(A)$ such that the inequality

$$
0<\gamma_k(A)\le\left(\prod_{\substack{m=1\\ m\ne k}}^{M}\Bigl\|\sum_{n=1}^{N}\alpha_{mn}\xi_n\Bigr\|\right)\left(\prod_{n=1}^{N}\bigl(|\xi_n|+1\bigr)\right)
\tag{8.27}
$$

holds for all $\boldsymbol{\xi}\ne\mathbf{0}$ in $\mathbb{Z}^N$. Using (8.27) we obtain the inequality

$$
\gamma_k(A)\Bigl\|\sum_{n=1}^{N}\alpha_{kn}\xi_n\Bigr\|\le\left(\prod_{m=1}^{M}\Bigl\|\sum_{n=1}^{N}\alpha_{mn}\xi_n\Bigr\|\right)\left(\prod_{n=1}^{N}\bigl(|\xi_n|+1\bigr)\right)=R^L
\tag{8.28}
$$

for each integer $k=1,2,\dots,M$. Again we use the identity (8.24), and we apply the arithmetic-geometric mean inequality to the right hand side of (8.25). This leads to the estimate

$$
\begin{aligned}
L\gamma\bigl(A^T\bigr)^{\frac{1}{L}} &\le \sum_{m=1}^{M}\delta_m^{-1}\bigl(|u_m|+1\bigr)+\sum_{n=1}^{N}\delta_{M+n}^{-1}\Bigl\|\sum_{m=1}^{M}u_m\alpha_{mn}\Bigr\|\\
&= \lambda_1^*+\sum_{m=1}^{M}\delta_m^{-1}\\
&= \lambda_1^*+R^{-1}\sum_{m=1}^{M}\Bigl\|\sum_{n=1}^{N}\alpha_{mn}\xi_n\Bigr\|.
\end{aligned}
\tag{8.29}
$$

We apply the inequalities (8.9), (8.18), and (8.28) to the right hand side of (8.29). In this way we arrive at the bound

$$
\begin{aligned}
L\gamma\big(A^T\big)^{\frac{1}{L}} &\le \lambda_1^* + R^{L-1}\sum_{m=1}^{M}\gamma_m(A)^{-1} \\
&\le \big(L!\big)^{\frac{1}{L-1}}R^{\frac{1}{L-1}} + R^{L-1}\sum_{m=1}^{M}\gamma_m(A)^{-1}.
\end{aligned}
\tag{8.30}
$$

The inequality (8.30) shows that R^L is bounded from below by a positive constant that does not depend on the point $\boldsymbol{\xi} \ne \mathbf{0}$ in $\mathbb{Z}^N$. This verifies (8.12), and completes the proof of Theorem 2.2.

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