ON THE ORDER OF ADDITIVE BASES IN FINITE ABELIAN GROUPS

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ABSTRACT. Let G be a finite abelian group. For a subset $A \subset G$, we define $\mathrm{ord}^*_G(A)$ to be the smallest number h such that $G = hA$, and $\mathrm{ord}_G(A)$ to be the smallest number ℓ such that $G = \cup_{i=0}^{\ell} iA$, if these numbers exist. Here hA is the h-fold sumset of A. We address and obtain some partial results on the following questions:
(1) How large must $\mathrm{ord}^*_G(A)$ be, in terms of G?
(2) How small must $\mathrm{ord}^*_G(A)$ be, in terms of $\mathrm{ord}_G(A)$ and G?
These questions form part of a long line of research initiated by Erdős and Graham, who first studied the problems in $\mathbf{N}$.

1. Introduction

Let $(G, +)$ be an abelian (semi)group. If $A, B \subset G$, their sumset is $A + B := \{a + b : a \in A, b \in B\}$ and their difference set is $A - B := \{a - b : a \in A, b \in B\}$. For $x \in G$ we define $A + x = \{a + x : a \in A\}$. For a positive integer h, the h-fold sumset of A is $hA := A + \cdots + A$ (h times), i.e. the set of all elements that can be expressed as the sum of exactly h elements of A. Also, we define $0A = \{0\}$ and $[0, h]A = \bigcup_{i=0}^{h} iA$. In other words, $[0, h]A$ as the set of all elements that can be expressed as the sum of at most h elements of A. We write $A \sim B$ if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is finite.

In this paper, we are concerned with different notions of *additive bases*. A set $A \subset G$ is said to be:

(B1) a *strong basis* of order $\leq h$ of G if $hA = G$, and we write $\mathrm{ord}^*_G(A) = h$ if h is the smallest such number.
(B2) a *weak basis* of order $\leq h$ of G if $[0, h]A = G$, and we write $\mathrm{ord}_G(A) = h$ if h is the smallest such number.
(B3) a *strong asymptotic basis* of order $\leq h$ of G if $hA \sim G$, and we write $\mathrm{aord}^*_G(A) = h$ if h is the smallest such number.
(B4) a *weak asymptotic basis* of order $\leq h$ of G if $hA \sim G$, and we write $\mathrm{aord}^*_G(A) = h$ if h is the smallest such number.

In each case, we write $\mathrm{ord}^*_G(A)$ (resp. $\mathrm{ord}_G(A)$, $\mathrm{aord}^*_G(A)$, $\mathrm{aord}_G(A)$) $= \infty$ if no h exists, i.e. A is not a basis in the corresponding sense.[1]

Erdős and Graham [5, 6] considered bases in $\mathbf{N}$ (the nonnegative integers) and asked under what conditions a weak asymptotic basis is also a strong asymptotic basis. They proved [5,

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[1]We remark that in the literature, there are no standard nomenclature and notation for these notions. For example, in [3, 2], a set satisfying (B2) is called an h-basis, and a set satisfying (B1) is called an additive basis of order h. In [4, 9], sets satisfying (B1) and (B2) are called nice bases and nice weak bases, respectively. Also, as we will explain later, in [4, 9], the notations $\mathrm{ord}^*_G(A)$ and $\mathrm{ord}_G(A)$ are reserved for sets satisfying (3) and (4).

Theorem 1] that a weak asymptotic basis A in $\mathbf{N}$ is also a strong asymptotic basis if and only if

$$\gcd\{a-a' : a, a' \in A\} = 1. \quad (1)$$

Under this condition, Erdős and Graham then asked: what is the relation between $\mathrm{aord}^*_G(A)$ and $\mathrm{aord}_G(A)$? They proved [5, Theorem 2] that if $\mathrm{aord}_{\mathbf{N}}(A) = h$, then

$$\mathrm{aord}_{\mathbf{N}}(A) \leq \left(\frac{5}{4} + o(1)\right) h^2 \quad (2)$$

as $h \to \infty$. These questions and related ones have been extensively studied in $\mathbf{N}$; we refer the reader to [8] and [11] for comprehensive surveys of the subject. In particular, Plagne [12] improved (2) to $\mathrm{aord}_{\mathbf{N}}(A) \leq \left(\frac{1}{2} + o(1)\right) h^2$ and this remains the best bound to date. In [9], Lambert–Lê–Plagne studied these questions in general infinite abelian groups. In [4], Bienvenu–Girard–Lê extended the investigation further to a class of semigroups which includes all infinite abelian groups, as well as $\mathbf{N}$.

In contrast, in this paper our focus is on finite abelian groups. Note that the notion of strong/weak asymptotic bases is only interesting when G is infinite. On the other hand, the notion of strong/weak bases makes sense in all abelian groups. For this reason, we deal with strong/weak bases and reserve the notations $\mathrm{ord}^*_G(A)$ and $\mathrm{ord}_G(A)$ for their orders in this paper. We remark that when G is infinite, there is not much difference between $\mathrm{ord}^*_G(A)$ and $\mathrm{aord}^*_G(A)$ and between $\mathrm{ord}_G(A)$ and $\mathrm{aord}_G(A)$. Indeed, it is clear from the definitions that $\mathrm{aord}^*_G(A) \leq \mathrm{ord}^*_G(A)$ and $\mathrm{aord}_G(A) \leq \mathrm{ord}_G(A)$. On the other hand, by [9, Lemma 5], we have $\mathrm{ord}^*_G(A) \leq \mathrm{aord}^*_G(A) + 1$ and $\mathrm{ord}_G(A) \leq \mathrm{aord}_G(A) + 1$.

We first investigate the following quantity

$$f(G) := \max\{\mathrm{ord}^*_G(A) : A \subset G, \mathrm{ord}^*_G(A) < \infty\}, \quad (3)$$

which is clearly finite when G is finite. This situation is drastically different for infinite groups, since by [9, Theorem 1], $\mathrm{ord}^*_G(A)$ can take on any given positive integer value (see also [4, Proposition 33] for the same fact about semigroups). Also, it is easy to see that $f(G) = \max\{\mathrm{ord}_G(A) : A \subset G, \mathrm{ord}_G(A) < \infty\}$ (see Lemma 1).

Our first result gives an upper bound for $f(G)$ in terms of invariant factors of G. This bound might actually be an equality, though we have only confirmed it for some classes of groups. Recall that by the fundamental theorem of finite abelian groups, G has a decomposition

$$G \cong \mathbf{Z}_{d_1} \oplus \cdots \oplus \mathbf{Z}_{d_r},$$

where $d_1, \ldots, d_r > 1, d_1 \mid d_2 \cdots \mid d_r$ are called the invariant factors of G.

Theorem 1. *(a) We have* $f(G) \geq \sum_{i=1}^r (d_i - 1)$.
(b) We have $f(G) = \sum_{i=1}^r (d_i - 1)$ *when* G *has rank* ≤ 2 *(i.e.* $r \leq 2$*), or when* G *is a* p*-group (i.e. all* d_i *are powers of* p *for some prime* p*).*

Very recently and independently of us, András–Cziszter–Domokos–Szöllősi [1] have proved that $f(G) = \sum_{i=1}^r (d_i - 1)$ for all finite abelian groups G (see also the comment after Problem 1).

Next, in the spirit of Erdős-Graham, we seek to compare $\mathrm{ord}_G(A)$ and $\mathrm{ord}^*_G(A)$ when both of them are finite. In [4, Theorem 4], it was proved that if $\mathrm{ord}_G(A) = h$ and $\mathrm{ord}^*_G(A) < \infty$, then

$$\mathrm{ord}^*_G(A) \leq (1 + o(1))h^3 \quad (4)$$

when h goes to infinity. Also, if the exponent of G is a prime power ℓ, then [4, Theorem 6] states that

$$\mathrm{ord}_G^*(A) \leq \ell h + O(\ell^2). \tag{5}$$

While the proofs of (4) and (5) also work for finite groups, (5) is more relevant for finite groups since we know that $\mathrm{ord}_G^*(A)$ and $\mathrm{ord}_G(A)$ are bounded by $f(G)$. In the next theorems, we are able to prove more precise estimates than (4) and (5) in the following cases: $\ell = 2$ and $h = 2$. Recall that the exponent of a finite abelian group G is the smallest integer n such that $nx = 0$ for every $x \in G$. If $d_1 | d_2 \cdots | d_r$ are the invariant factors of G, then d_r is the exponent of G.

Theorem 2. *Let G be a finite abelian group of exponent 2, $G \neq \mathbb{Z}_2$. Let $A \subset G$ be such that $\mathrm{ord}_G(A) = h$ and $\mathrm{ord}_G^*(A) < \infty$. Then*

$$\mathrm{ord}_G^*(A) \leq 2h.$$

Furthermore, the constant 2 is the best possible.

Theorem 3. *Let G be a finite abelian group. If $\mathrm{ord}_G(A) \leq 2$ and $\mathrm{ord}_G^*(A) < \infty$, then $\mathrm{ord}_G^*(A) \leq 4$.*

In order to quantify the relation between $\mathrm{ord}_G(A)$ and $\mathrm{ord}_G^*(A)$, we introduce the function

$$g(G) =: \max\left\{\frac{\mathrm{ord}_G^*(A)}{\mathrm{ord}_G(A)} : A \subset G, \mathrm{ord}_G(A) < \infty, \mathrm{ord}_G^*(A) < \infty\right\} \tag{6}$$

We are able to determine $g(G)$ numerically for all cyclic groups $\mathbf{Z}_n$ where $n \leq 31$ (see Section 4.3).

The structure of the paper is as follows. In Section 2 we introduce some basic tools, including a generalization of the Erdős-Graham criterion (1). In Section 3, we prove Theorem 1. In Section 4, we prove results relating $\mathrm{ord}_G^*(A)$ and $\mathrm{ord}_G(A)$, i.e. Theorems 2 and 3. We also state some open problems (see Problems 1, 2 and 3.)

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2. Preliminaries

Throughout this paper, let G denote a finite abelian group. First we will state some immediate properties of $\mathrm{ord}_G(A)$ and $\mathrm{ord}_G^*(A)$. By definition, we have $\mathrm{ord}_G(A) = \mathrm{ord}_G^*(A \cup \{0\}) \leq \mathrm{ord}_G^*(A)$. Also, $\mathrm{ord}_G^*(A)$ is translation invariant, i.e. $\mathrm{ord}_G^*(A) = \mathrm{ord}_G^*(A + x)$ for any $x \in G$. From these properties we will show that the definition of $f(G)$ is not changed when $\mathrm{ord}_G^*(A)$ is replaced by $\mathrm{ord}_G(A)$.

Lemma 1. *We have $f(G) = \max\{\mathrm{ord}_G(A) : A \subset G, \mathrm{ord}_G(A) < \infty\}$.*

Proof. By definition $f(G) = \max\{\mathrm{ord}_G^*(A) : A \subset G, \mathrm{ord}_G(A) < \infty\}$, which is $\geq$ RHS. Suppose $f(G) = \mathrm{ord}_G^*(A)$ for some $A \subset G$. By translation invariant, we can assume $0 \in A$. Hence $f(G) = \mathrm{ord}_G(A) \leq$ RHS. □

Next we will state some observations about $A - A$.

Proposition 1. *Let $A \subset G$, then*

(1) *For every* $i \in \mathbf{N}$, *we have* $(iA - iA) \subset (i+1)A - (i+1)A$.
(2) $\langle A - A \rangle = \cup_{i=1}^{\infty}(iA - iA)$, *where* $\langle A - A \rangle$ *is the subgroup of* G *generated by* $A - A$.
(3) $\langle A - A \rangle = \langle A - a_0 \rangle$, *for any* $a_0 \in A$.
(4) $\langle A - A \rangle = G$ *if and only if* A *is not contained in a coset of a proper subgroup of* G.

Proof. (1) is clear. For (2), for each i, we have $iA - iA = i(A-A) \subset \langle A - A \rangle$, so $\cup_{i=1}^{\infty}(iA - iA) \subset \langle A - A \rangle$. On the other hand, if $x \in iA - iA, y \in jA - jA$, then $x - y \in (i+j)A - (i+j)A$. Hence $\cup_{i=1}^{\infty}(iA - iA)$ is a subgroup of G containing $A - A$, so $\cup_{i=1}^{\infty}(iA - iA) \supset \langle A - A \rangle$.

For (3), it is clear that $\langle A - A \rangle \supset \langle A - a_0 \rangle$. On the other hand, for any $a, a' \in A$, we have $a - a' = (a - a_0) - (a - a_0)$, so $A - A \subset \langle A - a_0 \rangle$ and $\langle A - A \rangle \subset \langle A - a_0 \rangle$.

We now prove (4). Suppose A is contained in a coset $x + H$ of a proper subgroup $H \lneq G$, then $A - A \subset H$ and $\langle A - A \rangle \subset H$. Suppose A is not contained in a coset of a proper subgroup of G. Let $a_0 \in A$ be arbitrary. Clearly $A \subset a_0 + \langle A - a_0 \rangle$, so $\langle A - A \rangle = \langle A - a_0 \rangle = G$. □

We recall the group analog of the Erdős-Graham criterion (1), which is essentially [9, Lemma 7]. Note that [9, Lemma 7] deals with $\mathrm{aord}_G(A)$ and $\mathrm{aord}_G^*(A)$ in infinite groups, but the adaptation to $\mathrm{ord}_G(A)$ and $\mathrm{ord}_G^*(A)$ is straightforward.

Lemma 2. *Suppose* $A \subset G, \mathrm{ord}_G(A) < \infty$. *Then the following conditions are equivalent:*

(1) $\mathrm{ord}_G^*(A) < \infty$,
(2) $\langle A - A \rangle = G$.

We also make use the following folklore result regarding sumsets.

Lemma 3. *Suppose* $A, B \subset G$ *satisfies* $|A| + |B| > G$. *Then* $A + B = G$.

Proof. Let $x \in G$ be arbitrary. Then $|A| + |x - B| = |A| + |B| > |G|$, so $A \cap (x - B) \neq \emptyset$ and therefore $x \in A + B$. □

For $S \subset G$, the *stabilizer* of S is

$$\mathrm{Stab}(S) = \{x \in G : x + S = S\}. \tag{7}$$

It is easy to see that $\mathrm{Stab}(S)$ is a subgroup of G. Also, it follows from the definition of the stabilizer that $S = S + \mathrm{Stab}(S)$.

3. On the maximum order of a basis

In this section we prove Theorem 1. Suppose $G = \mathbf{Z}_{d_1} \oplus \cdots \oplus \mathbf{Z}_{d_r}$ where $d_1 \mid d_2 \mid \cdots \mid d_r$. For each $1 \leq i \leq r$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i-th position. Let $A = \{0, e_1, \ldots, e_r\}$.

We first claim that $\mathrm{ord}_G(A) = \mathrm{ord}_G^*(A) = \sum_{i=1}^{r}(d_i - 1)$.

Each element $x \in G$ is of the form $(n_1, n_2, \ldots, n_r)$ with $0 \leq n_i \leq d_i - 1$ for all $i = 1, \ldots, r$. Clearly $x \in (\sum_{i=1}^{r} n_i)A$. Furthermore, $\sum_{i=1}^{r} n_i$ is the smallest number h such that $x \in hA$. Indeed, supposing $x \in hA$, there exist nonnegative integers $h_0, h_1, \ldots, h_r$ such that $h = h_0 + h_1 \cdots + h_r$ and

$$x = h_0 0 + h_1 e_1 + \cdots + h_r e_r.$$

Then we have $h_i \equiv n_i \pmod{d_i}$ and therefore $h_i \geq n_i$ for any $i = 1, \ldots, r$. Consequently, $h \geq \sum_{i=1}^r n_i$. Hence, $f(G) \geq \mathrm{ord}_G^*(A) = \sum_{i=1}^r (d_i - 1)$.

To prove the second part of Theorem 1 we use the notion of the Davenport constant. Given a finite abelian group G, the Davenport constant $D(G)$ is the smallest integer n such that, for any sequence $\{x_i\}_{i=1}^n$ of (not necessarily distinct) elements of G, there exists a nonempty subsequence $\{x_{i_j}\}_{j=1}^k$ such that $\sum_{j=1}^k x_{i_j} = 0$. We refer the reader to [7] for a coomprehensive survey of $D(G)$ and related quantities.

Claim: $f(G) \leq D(G) - 1$.

Suppose for the sake of contradiction that there exists $A \subset G$ with $h = \mathrm{ord}_G^*(A) \geq D(G)$. Without loss of generality, we may assume that $0 \in A$. Then, for any $x \in G$, there exists (not necessarily distinct) elements $a_1, \ldots, a_h \in A$ such that

$$x = a_1 + \cdots + a_h.$$

Since $h \geq D(G)$, there exists $1 \leq i_1 < \cdots < i_k \leq h$ such that $\sum_{j=1}^k a_{i_j} = 0$. Therefore, $x \in (h-k)A \subset (h-1)A$. Since this is true for all $x \in G$, we have $G = (h-1)A$. This contradicts the fact that $\mathrm{ord}_G^*(A) = h$.

To finish the proof, we note that $D(G) = 1 + \sum_{i=1}^r (d_i - 1)$ for the following groups (see [7, Theorem 3.1]):

- p-groups,
- groups of rank ≤ 2 (i.e. $r \leq 2$).

Consequently, for those groups, we have $f(G) = \sum_{i=1}^r (d_i - 1)$. This finishes the proof of Theorem 1.

We remark that it is not true in general that $D(G) = 1 + \sum_{i=1}^r (d_i - 1)$, and the gap between $D(G)$ and $\sum_{i=1}^r (d_i - 1)$ can be quite large (see [10]). Our current understanding of $D(G)$ is quite poor, even for groups of rank 3. The smallest group for which $D(G)$ is unknown is $\mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_6$. We close this section with the following question.

Problem 1. *Is it true that for all groups G, $f(G) = \sum_{i=1}^r (d_i - 1)$? In particular, what is $f(G)$ when $G = \mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_6$?*

Shortly after the submission of this paper, we learned that András–Cziszter–Domokos–Szöllősi [1] have very recently and independently studied Problem 1, even for groups which are not necessarily abelian. In particular, their [1, Theorem 3.2] confirms that the answer to Problem 1 is affirmative for abelian groups.

4. On the relation between $\mathrm{ord}_G(A)$ and $\mathrm{ord}_G^*(A)$

We begin with the following observation.

Lemma 4. *If $|G| > 2$, then $G = 2(G \setminus \{0\})$.*

Proof. If $|G| > 2$, then $|G \setminus \{0\}| = |G| - 1 > \frac{|G|}{2}$. By Lemma 3, we have $G = 2(G \setminus \{0\})$. □

4.1. **Proof of Theorem 2.** Suppose G has exponent 2 and $|G| > 2$. Let $\text{ord}_G A = h$, then $G = \{0\} \cup A \cup 2A \cup \cdots \cup hA$.

If $h = 1$, then $A \supset G \setminus \{0\}$ and $G = 2A$ by Lemma 4. We assume henceforth that $h \geq 2$. Since G has exponent 2, we have $iA \subset (i+2)A$ for any $i \in \mathbf{N}$. Therefore, $G = hA \cup (h-1)A$. We consider two cases.

Case 1: Suppose $iA \cap (i+1)A \neq \emptyset$ for some $i \leq h$. Let $c \in iA \cap (i+1)A$. Combining this with $G = hA \cup (h-1)A$, we get $G = G + c = (h+i)A$. Hence, $\text{ord}^*_G A \leq h+i \leq 2h$, as desired.

Case 2: Suppose for the sake of contradiction that $iA \cap (i+1)A = \emptyset$ for both $i = h$ and $i = h-1$. We have

$$G = hA \cup (h-1)A = hA \cup (h+1)A.$$

Furthermore, since these are disjoint unions, $(h+1)A = (h-1)A$. Next, since $\text{ord}^*_G A < \infty$, we have $G = \langle A - A \rangle = \cup_{i=1}^{\infty} (iA - iA)$ by Lemma 2. On the other hand, for any $i \geq 1$, choose k such that $h - 1 + 2k \geq i$. Then

$$iA - iA \subset (h-1+2k)A - (h-1+2k)A = (h-1)A - (h-1)A.$$

Therefore, $G = (h-1)A - (h-1)A$. Consequently, $A \cap ((h-1)A - (h-1)A) \neq \emptyset$, so $(h-1)A \cap hA \neq \emptyset$, contradicting our assumption.

To see that 2 is the best constant possible, we observe that for $A = G \setminus \{0\}$, we have $\text{ord}_G(A) = 1$ and $\text{ord}^*_G(A) = 2$ by Lemma 4.

4.2. **Proof of Theorem 3.** We start with the following observation.

Lemma 5. *Suppose $A \subset G$ satisfies $\langle A - A \rangle = G$. Then for all $n > 0$, we have $|(n+1)A| > |nA|$ unless $nA = G$.*

Proof. Suppose that $|(n+1)A| = |nA|$. Since

$$(n+1)A = \bigcup_{a \in A} (nA + a),$$

for any $a, b \in A$, we must have $nA + a = nA + b$. Hence, $b - a \in \text{Stab}(nA)$ (where the stabilizer is defined as in (7)). Thus, $A - A \subset \text{Stab}(nA)$. Since $\text{Stab}(nA)$ is a subgroup of G, we have $G = \langle A - A \rangle = \text{Stab}(nA)$. By the definition of the stabilizer, we have

$$nA = \bigcup_{x \in \text{Stab}(nA)} (x + nA) = \bigcup_{x \in G} (x + nA). \tag{8}$$

This implies that $nA = G$, as desired. □

We now proceed to prove Theorem 3. Suppose $\text{ord}_G(A) \leq 2$, then $G = \{0\} \cup A \cup 2A$. Since $\text{ord}^*_G(A) < \infty$, we have $\langle A - A \rangle = G$ by Lemma 2. If $G = 2A$, then $\text{ord}^*_G(A) \leq 2$. We may assume that $G \neq 2A$. By Lemma 5, we have $|2A| \geq |A| + 1$. Since $1 + |A| + |2A| \geq |G|$, we have $|2A| \geq \frac{1}{2}|G|$. We distinguish two cases.

Case 1: $|G|$ is odd. Then $|2A| > \frac{1}{2}|G|$, so $G = 4A$ by Lemma 3.

Case 2: $|G|$ is even. By Cauchy's theorem, G must have an element of order 2. Hence, $0 \in 2A$, so $G = A \cup 2A$. Since $|2A| > |A|$, we must have $|2A| > \frac{1}{2}|G|$, yielding $G = 4A$ again.

In either case, we have $\text{ord}^*_G(A) \leq 4$. So, we are done.

4.3. **On the function $g(G)$.** Recall the function $g(G)$ as defined in (6). Theorem 2 says that if G has exponent 2 and $|G| > 2$, then $g(G) = 2$.

Determining $g(G)$ for all groups G may be difficult. We ask the following.

Problem 2. *Let p be a fixed prime. Find the asymptotic of $g(\mathbf{Z}_p^d)$ as $d \to \infty$.*

Problem 3. *Find the asymptotic of $g(\mathbf{Z}_n)$ as $n \to \infty$.*

We were able to calculate $g(\mathbf{Z}_n)$ for all $n \leq 31$. In the table below, $g(\mathbf{Z}_n) = a/b$ means the maximum of $\frac{\mathrm{ord}^*_{\mathbf{Z}_n}(A)}{\mathrm{ord}_{\mathbf{Z}_n}(A)}$ is achieved at a set A such that $\mathrm{ord}^*_{\mathbf{Z}_n}(A) = a$ and $\mathrm{ord}_{\mathbf{Z}_n}(A) = b$. The table also shows that the values of $g(\mathbf{Z}_n)$ fluctuates and is not always an increasing function in n.

n	$g(\mathbb{Z}_n)$	n	$g(\mathbb{Z}_n)$	n	$g(\mathbb{Z}_n)$
2	$1/1 = 1.00000$	12	$11/5 = 2.20000$	22	$21/7 = 3.00000$
3	$2/1 = 2.00000$	13	$12/5 = 2.40000$	23	$22/7 = 3.14286$
4	$2/1 = 2.00000$	14	$13/5 = 2.60000$	24	$23/7 = 3.28571$
5	$4/2 = 2.00000$	15	$14/5 = 2.80000$	25	$24/8 = 3.00000$
6	$2/1 = 2.00000$	16	$15/5 = 3.00000$	26	$25/7 = 3.57143$
7	$6/3 = 2.00000$	17	$16/6 = 2.66667$	27	$26/8 = 3.25000$
8	$7/3 = 2.33333$	18	$17/6 = 2.83333$	28	$27/8 = 3.37500$
9	$8/4 = 2.00000$	19	$18/6 = 3.00000$	29	$28/8 = 3.50000$
10	$9/4 = 2.25000$	20	$19/7 = 2.71429$	30	$29/8 = 3.62500$
11	$10/4 = 2.50000$	21	$20/6 = 3.33333$	31	$30/8 = 3.75000$

TABLE 1. Values of $g(\mathbf{Z}_n)$ for $n \leq 31$

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