

SUPPLEMENT TO “TWO NEW CONDITIONS
SUPPORTING THE FIRST-ORDER APPROACH
TO MULTI-SIGNAL PRINCIPAL-AGENT PROBLEMS”

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This note contains some supplementary results mentioned in Conlon (2008). First, Section 1 gives Jewitt’s (1988) original (unpublished) proof of his one signal result, based on his Condition (2.10a), and Section 2 shows why it is hard to generalize this proof to the multi-signal case. Section 3 shows how to check for concavity of the signal technology in the agent’s effort level. Section 4 gives an alternative derivation of the results in Conlon’s (2008) Section 8, which avoids the vector calculus machinery from that paper’s Section 7. Section 5 presents a candidate canonical vector flow representing a signal technology, but argues that this canonical flow is of limited usefulness. Section 6 shows how to derive the state-space representation in Conlon’s (2008) Proposition 3, using the vector flow in that paper’s Lemma 3. Finally, Section 7 gives the details for the generalization of Jewitt’s Theorem 2, mentioned in Conlon’s (2008) Section 9.

1. JEWITT'S FULL ONE-SIGNAL CONDITIONS

THIS AND THE FOLLOWING SECTION show why it is difficult to generalize Jewitt's (1988) key one-signal Condition (2.10a) to the multi-signal case. The basic framework, including Jewitt's key condition, and his original unpublished proof of his one-signal result, are presented in this section. Section 2 then shows why it is difficult to generalize this proof and condition to the multi-signal case. Thus, Conlon (2008) must use the stronger condition, that signals are concave in the agent's effort, in its multi-signal generalization of Jewitt's conditions.

Note that Jewitt also drops his Condition (2.10a) in his treatment of the multi-signal case. In his main multi-signal result, his Theorem 3, he also replaces his (2.10a) with concavity of the signal technology, as does Conlon (2008) (though Jewitt also assumes independent signals). In his other multi-signal result, his Theorem 2, he replaces (2.10a) with the much stronger Convexity of the Distribution Function Condition.

Finally, it should be noted that these sections do not show that it is *impossible* to generalize Jewitt's Condition (2.10a) to the multi-signal case, but only argues that the most obvious generalizations do not work. Thus, a definitive resolution of this issue, either way, remains an interesting topic for future research.

There is one principal and one agent, with the agent choosing an effort level $a \geq 0$. Suppose initially that there is a one-dimensional signal \tilde{x} , with density and cumulative distribution functions

$$(1) \quad f(x|a) \text{ and } F(x|a),$$

respectively. Assume as usual that the support of $f(x|a)$ in (1) is independent of a , and that the density is bounded between two positive constants on its support.

The principal is risk neutral, while the agent has a von Neumann-Morgenstern utility function $u(s) - a$, with s the agent's income. Also, the principal pays the agent

using payment schedule $s(x)$. Thus, the principal's expected payoff is

$$(2) \quad V(s(\cdot), a) = \int [x - s(x)]f(x|a)dx,$$

and the agent's expected payoff is

$$(3) \quad U(s(\cdot), a) = \int u(s(x))f(x|a)dx - a.$$

The principal's problem is to choose a payment schedule, $s^*(\cdot)$, and target action, a^* , by the agent, to maximize (2), given two constraints:

$$(4) \quad \text{For the agent, } a^* \text{ maximizes } U(s^*(\cdot), a),$$

and

$$(5) \quad \text{The resulting expected payoff to the agent, } U(s^*(\cdot), a^*) \geq U_0,$$

where U_0 is the agent's reservation utility. Here (4) and (5) are the usual incentive compatibility and participation constraints.

The first order approach assumes that one can replace the constraint (4) with a "relaxed" constraint,

$$(6) \quad U_a(s^*(\cdot), a^*) = 0,$$

where subscripts denote partial derivatives. To ensure that (6) implies (4), it is sufficient for the agent's utility, $U(s^*(\cdot), a)$, to be a concave function of her effort a , when $s^*(\cdot)$ solves the *relaxed* problem involving (6). To get this, we first ensure that $u(s^*(x))$ is a concave function of x . For this, the following two conditions are sufficient:

CONDITION (a): $f_a(x|a)/f(x|a)$ is nondecreasing concave in x for each a ,

CONDITION (b): *Jewitt's (1988) function, $\omega(\cdot)$, is increasing concave, where*

$$(7) \quad \omega(z) = u((u')^{-1}(1/z)).$$

These ensure that, if $s^*(\cdot)$ is a solution to the relaxed problem, then $u(s^*(x))$ is concave in x (see Conlon (2008)). Jewitt also makes one more assumption:

CONDITION (c) – Jewitt's Condition (2.10a):

$$(8) \quad \int_{-\infty}^b F(x|a)dx \text{ is nonincreasing convex in } a \text{ for every } b.$$

Using integration by parts and letting $h(x; b) = \min(0, x - b)$, (8) is equivalent to

$$(9) \quad h^T(a; b) = \int h(x; b)f(x|a)dx \text{ is nondecreasing concave in } a \text{ for every } b.$$

Here $h(\cdot, b)$ is a kind of “test function,” since (9) implies that:

$$(10) \quad \text{if } \varphi(\cdot) \text{ is nondecreasing concave, } \varphi^T(a) = \int \varphi(x)f(x|a)dx \text{ is also.}$$

Jewitt cites an unpublished working paper (Jewitt and Kanbur (1988)) which shows that (8) is equivalent to (10). The equivalence of (9) to (10) is shown in the following lemma:

LEMMA 1: *Conditions (9) and (10) are equivalent.*

PROOF: The following is essentially the unpublished proof in the Jewitt-Kanbur paper (personal communication from Ian Jewitt), and has a structure similar to the proof of Lemma 1 in Conlon (2008). The basic idea is that any nondecreasing concave function $\varphi(\cdot)$ can clearly be approximated, up to a constant, by a positive linear combination, over different values of b , of the functions $h(\cdot; b)$, say:

$$(11) \quad \varphi(x) \approx \alpha_0 + \sum_{i=1}^n \alpha_i h(x; b_i), \text{ where } \alpha_i > 0 \text{ for } i \geq 1.$$

Thus, applying the transformation in (10) gives

$$(12) \quad \varphi^T(a) \approx \alpha_0 + \sum_{i=1}^n \alpha_i h^T(a; b_i),$$

with each term nondecreasing concave by (9). Taking limits, $\varphi^T(x)$ is also. *Q.E.D.*

PROPOSITION 1: *Assume (a) $f_a(x|a)/f(x|a)$ is nondecreasing concave in x , (b) Jewitt's function $\omega(\cdot)$ is increasing concave, and (c) Jewitt's Condition (2.10a) holds. Then any solution to the relaxed problem, maximizing (2) subject to (5) and (6), is also a solution to the full problem of maximizing (2) subject to (4) and (5).*

PROOF: Let $u(s^*(x)) = \varphi(x)$. Then (a) and (b) show that $\varphi(x)$ is nondecreasing concave, as in Conlon (2008). Thus, since Jewitt's Condition (2.10a) implies (9), which implies (10), it follows that $U(s^*(\cdot), a) = \varphi^T(a) - a$ is concave in a . *Q.E.D.*

Conditions (a) and (b) generalize easily to the multi-signal case (see Jewitt (1988) or Conlon (2008)). However, Condition (2.10a) is difficult to generalize, as shown next.

2. THE DIFFICULTY IN GENERALIZING JEWITT'S CONDITION (2.10a)

TO THE MULTI-SIGNAL CASE

This section shows why it is difficult to generalize Jewitt's Condition (2.10a) to the multi-signal case. For the multi-signal version of the problem, replace the random signal \tilde{x} by the random vector $\tilde{\mathbf{x}}$, with density $f(\mathbf{x}|a)$ and cumulative distribution function $F(\mathbf{x}|a)$. Again assume that the support of $f(\mathbf{x}|a)$ is independent of a , and that the density is bounded between two positive constants on its support. As in Sinclair-Desgagné (1994), let the monetary payoff to the principal as a function of \mathbf{x} be $\pi(\mathbf{x})$. Also, assume the principal pays the agent using payment function $s(\mathbf{x})$. Then, analogous to Section 1, the principal's expected payoff is

$$(13) \quad V(s(\cdot), a) = \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a) d\mathbf{x},$$

the agent's expected payoff is

$$(14) \quad U(s(\cdot), a) = \int u(s(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - a,$$

and the principal's problem is to choose a payment schedule, $s^*(\cdot)$, and target action, a^* , to maximize (13), given the constraints (4) and (5), where now $U(s(\cdot), a)$ is defined by (14), not (3). Also, as before, the first order approach assumes that one can replace the constraint (4) with a relaxed constraint, (6) (i.e., $U_a(s^*(\cdot), a^*) = 0$).

To generalize Jewitt's Condition (2.10a), we must determine conditions such that, for any *multivariable* function $\varphi(\mathbf{x})$,

$$(15) \quad \text{if } \varphi(\cdot) \text{ is nondecreasing concave, then } \varphi^T(a) = \int \varphi(\mathbf{x})f(\mathbf{x}|a)d\mathbf{x} \text{ is also.}$$

To see how difficult it is to generalize Jewitt's Condition (2.10a), we show that the obvious multivariable analogues to the test functions $h(x; b) = \min(0, x - b)$, used in (9) above, are insufficient to approximate, or "span," as positive linear combinations, all multivariable nondecreasing concave functions, so the argument in (11) and (12) does not go through. For related results, see Johansen (1974) and Bronshtein (1978), who show that, in the multi-variable case, the set of such generalized spanning test functions h must, remarkably, be *dense* in the set of all concave functions.

Focus for definiteness on the two variable case. One natural generalization of the $h(x; b)$ function is then the following:

$$(16) \quad h(x_1, x_2; \alpha_1, \alpha_2, \beta) = \min(0, \alpha_1 x_1 + \alpha_2 x_2 - \beta)$$

with $\alpha_1, \alpha_2 \geq 0$, so h is nondecreasing in (x_1, x_2) , and $\alpha_1 + \alpha_2 = 1$, say, as a harmless normalization. One would hope that all nondecreasing concave functions could be approximated as constants plus positive linear combinations of these h functions. If

this were possible, one could easily extend Lemma 1. However, this approach does not work, as shown by the following lemma.

LEMMA 2: *The nondecreasing concave function $k(x_1, x_2) = \min(x_1, x_2)$ cannot be approximated by a constant plus a positive linear combination of functions of the form $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ from (16), with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.*

PROOF: See Appendix A.

Thus, if one wants to use the functions in (16) to extend Lemma 1 to the multi-signal case, then the method of proof from Lemma 1 does not work. Of course, it could be possible for Lemma 1 to extend in this way to the multi-variable case, even if the method of proof does not. That is, one might hope that, if the transformed functions, $h^T(a; \alpha_1, \alpha_2, \beta) = \int h(\mathbf{x}; \alpha_1, \alpha_2, \beta) f(\mathbf{x}|a) d\mathbf{x}$, are concave in a for all $\alpha_1, \alpha_2 \geq 0$ and all β , then (15) holds. The next lemma shows that this is not the case.

LEMMA 3: *Consider the nondecreasing concave function given by $\varphi(x_1, x_2) = \min(x_1 - 1, x_2)$. There is a distribution, $f(\mathbf{x}|a)$, such that $h^T(a; \alpha_1, \alpha_2, \beta)$ is nondecreasing concave in a for all α_1, α_2 and β , with $\alpha_1, \alpha_2 \geq 0$, but $\varphi^T(a)$ is not concave in a . Also, this distribution can be made to satisfy the Monotone Likelihood Ratio (MLR) property, $f_a(\mathbf{x}|a)/f(\mathbf{x}|a)$ nondecreasing in \mathbf{x} .*

PROOF: See Appendix B.

Thus, the most natural generalization of (9) does not work. Of course, concave functions can always be approximated by *some* type of piecewise linear concave function, since the graph of a concave function forms the lower envelope of its tangent planes. Thus, we could generalize Lemma 1 by replacing the $h(x; b)$ functions used in (9) by arbitrary piecewise linear concave functions. However, this class of functions is *too* flexible, since it would be just as hard to check (15) for all piecewise linear concave functions as to check it for *all* concave functions.

A third possibility might be to replace the $h(\cdot; b)$ function in (9) with some sort of generalization of the $h(\cdot; b)$ function, which is not too flexible, but which is more flexible than the $h(\cdot, \cdot; \alpha_1, \alpha_2, \beta)$ functions in (16).

The next most flexible generalization might be a nondecreasing flat-top pyramid function. Define such a function to have a graph with a flat top (just as the graph of $h(x; b)$ is flat for $x \geq b$). Then require the faces surrounding this flat top to be separated by edges which do not intersect. This is a reasonable restriction, since otherwise we would come close to allowing all piecewise linear concave functions, which would be too broad a class to check, as argued above. As an example of such a nondecreasing flat-top pyramid function, consider

$$(17) \quad \min(0, 2x_1, 2x_2, x_1 + x_2 - 8).$$

The flat top in (17) is bounded by the lines $x_1 = 0$, $x_1 + x_2 = 8$, and $x_2 = 0$. The surrounding faces are separated by the edges $x_2 = x_1 + 8$ and $x_2 = x_1 - 8$, which do not intersect.

Of course, checking (15) for all such pyramid functions would also be impractical, even if this gave sufficient conditions for (15) in general. However, it probably does not give such sufficient conditions. For example, positive linear combinations of such functions are not enough to approximate the following nondecreasing concave function:

$$(18) \quad G(x_1, x_2) = \min(0, x_1, x_2, x_1 + x_2 + 1).$$

LEMMA 4: *$G(x_1, x_2)$ in (18) is nondecreasing concave, but it cannot be approximated by a positive linear combination of nondecreasing flat-top pyramid functions.*

PROOF: See Appendix C.

Thus, the argument in (11) and (12) again does not go through. Also, the proof of Lemma 4 could be adapted to the case of component pyramids *without* flat tops.

It would be interesting to construct a counterexample, analogous to the one in Lemma 3, to show definitively that, even if (15) holds for nondecreasing flat-top pyramids, it does not necessarily hold for arbitrary nondecreasing concave functions. However, the example in Appendix B is already quite complicated, and this additional counterexample would presumably be even more complicated. In any case, it seems to be extremely difficult to generalize Lemma 1 to the multi-signal case.

Thus, we have shown that it is difficult to generalize Jewitt's condition (2.10a) to the multi-signal case. Conlon (2008) therefore simply assumes that $\mathbf{x}(a, \vartheta)$ is concave in a . Note that Jewitt (1988) also makes this simple concavity assumption for his main multi-signal result, i.e. his Theorem 3. It would be interesting to see whether there are any classes of functions, perhaps more flexible than those considered here, which could be used to obtain a multi-signal generalization of Jewitt's Condition (2.10a).

3. CHECKING CONCAVITY IN a OF THE COORDINATES OF THE STATE-SPACE REPRESENTATION $\mathbf{x}(a, \vartheta)$

The state-space representation, $\mathbf{x}(a, \vartheta)$, in Proposition 3 of Conlon (2008), can be used to impose conditions for concavity of the $\mathbf{x}(a, \vartheta)$ function. This is a simple mechanical application of the following rather technical result. In the following proposition, let $\mathbf{F}(\mathbf{x}, a) = (F^1(\mathbf{x}, a), F^2(\mathbf{x}, a))'$ be a sufficiently smooth 2×1 vector function, with primes denoting transposes, and with $\mathbf{x} = (x_1, x_2)'$. Define $x_1(a, \theta_1, \theta_2)$ and $x_2(a, \theta_1, \theta_2)$ implicitly by $\mathbf{F}(\mathbf{x}, a) = (\theta_1, \theta_2)'$. Let \mathbf{v} be the 3×1 vector $(x_1, x_2, a)'$, let $\partial\mathbf{F}/\partial\mathbf{v}$ be the 2×3 Jacobian matrix of partial derivatives of $\mathbf{F}(\mathbf{x}, a) = \mathbf{F}(\mathbf{v})$ with respect to \mathbf{v} , and let \mathbf{H}^i be the 3×3 Hessian of $F^i(\mathbf{x}, a) = F^i(\mathbf{v})$ with respect to \mathbf{v} .

PROPOSITION 2: *Suppose $F_{x_1}^1 F_{x_2}^2 - F_{x_1}^2 F_{x_2}^1 > 0$. Then for each $i, j = 1, 2, i \neq j$,*

the function $x_i(a, \theta_1, \theta_2)$ is concave in a if and only if

$$(19) \quad \mathbf{z}'[\mathbf{H}^j F_{x_j}^i - \mathbf{H}^i F_{x_j}^j] \mathbf{z} \leq 0 \text{ for all } 3 \times 1 \text{ vectors } \mathbf{z} \text{ satisfying } (\partial \mathbf{F} / \partial \mathbf{v}) \mathbf{z} = \mathbf{0}.$$

PROOF: Differentiating $\mathbf{F}(\mathbf{v}) = (\theta_1, \theta_2)'$ implicitly, for fixed (θ_1, θ_2) , shows that

$$(20) \quad d\mathbf{v}/da = (dx_1/da, dx_2/da, 1)'$$
 is determined by $(\partial \mathbf{F} / \partial \mathbf{v})(d\mathbf{v}/da) = \mathbf{0}$,

where $d\mathbf{v}/da$ is determined *uniquely* since $F_{x_1}^1 F_{x_2}^2 - F_{x_1}^2 F_{x_2}^1 \neq 0$. Implicitly differentiating both coordinates of $(\partial \mathbf{F} / \partial \mathbf{v})(d\mathbf{v}/da) = \mathbf{0}$ from (20) gives

$$(d\mathbf{v}/da)' \mathbf{H}^1 (d\mathbf{v}/da) + (\partial F^1 / \partial \mathbf{v})(d^2 \mathbf{v}/da^2) = 0,$$

$$(d\mathbf{v}/da)' \mathbf{H}^2 (d\mathbf{v}/da) + (\partial F^2 / \partial \mathbf{v})(d^2 \mathbf{v}/da^2) = 0.$$

Solving these two equations for $d^2 \mathbf{v}/da^2 = (d^2 x_1/da^2, d^2 x_2/da^2, 0)'$ gives

$$(21) \quad d^2 x_i/da^2 = (d\mathbf{v}/da)' [\mathbf{H}^j F_{x_j}^i - \mathbf{H}^i F_{x_j}^j] (d\mathbf{v}/da) / [F_{x_1}^1 F_{x_2}^2 - F_{x_1}^2 F_{x_2}^1],$$

where $i, j = 1, 2, i \neq j$. The result follows from (20) and (21). *Q.E.D.*

To apply Proposition 2 to our case, replace $F^1(x_1, x_2, a)$ and $F^2(x_1, x_2, a)$ with $F^1(x_1|a)$ and $F^2(x_2|x_1, a)$. Then $F_{x_1}^1 F_{x_2}^2 - F_{x_1}^2 F_{x_2}^1 > 0$ reduces to $f^1(x_1|a) f^2(x_2|x_1, a) > 0$, with the f^i 's the obvious densities. One can therefore use (19) to check concavity in a of $\mathbf{x}(a, \vartheta)$, from (11) in Conlon (2008), by using the analogue to bordered Hessians, for example (see Debreu (1952)).

This may, however, be computationally messy. It might therefore be easier to use a shortcut to check concavity of $\mathbf{x}(a, \vartheta)$. E.g., one might begin with a state-space representation, or solve (11) in Conlon (2008) explicitly.

Thus, to find conditions for the concavity of $\mathbf{x}(a, \vartheta)$ in a , consider for example $x_2(a, \theta_1, \theta_2)$ from Proposition 3 of Conlon (2008). For simplicity represent this as

$$x_2(a, \theta_1, \theta_2) = (F^2)^{-1}(\theta_2 | (F^1)^{-1}(\theta_1 | a), a) = g(x_1(a, \theta_1), a) = g(h(a), a)$$

where

$$g(x_1, a) = (F^2)^{-1}(\theta_2|x_1, a) \text{ and } h(a) = (F^1)^{-1}(\theta_1|a) = x_1(a, \theta_1).$$

Here the θ_i 's have been suppressed in g and h for brevity. Concavity now requires:

$$\begin{aligned} \partial^2 x_2 / \partial a^2 &= g_1(h(a), a)h''(a) + g_{11}(h(a), a)h'(a)^2 \\ &\quad + g_{22}(h(a), a) + 2g_{12}(h(a), a)h'(a) \leq 0. \end{aligned}$$

Suppose \tilde{x}_1 and \tilde{x}_2 are positively related, so an increase in \tilde{x}_1 increases \tilde{x}_2 in the sense of first order stochastic dominance. Then $g_1(h(a), a)$ will be nonnegative. Thus if h is concave (so x_1 depends concavely on a), then the first term will be nonpositive. Next, if g is concave in each of its arguments (so x_2 depends concavely on x_1 and a), then the next two terms will be nonpositive. This leaves the last term. If the marginal distribution of \tilde{x}_1 is increasing in a in the sense of first order stochastic dominance, then $h'(a)$ is nonnegative. The key issue therefore becomes whether $g_{12}(h(a), a)$ is nonpositive. This requires roughly that the correlation between \tilde{x}_1 and \tilde{x}_2 be nonincreasing in a .

A similar condition is important in the CISP approach (see Proposition 7, Equation (29), in Conlon (2008)). Note also that if g is concave *jointly* in x_1 and a , then its Hessian is negative semi-definite, so the *sum* of the last three terms is again nonpositive.

4. ALTERNATIVE DERIVATION OF THE LOCAL CONDITIONS IN SECTION 8

This section presents alternative derivations of the results in Section 8 of Conlon (2008), which avoid the vector calculus machinery in Section 7 of that paper. However, I believe that these alternative derivations are less illuminating than the derivations in the paper. Proposition 3 treats NISP and is identical to Proposition 6 in the paper, while Proposition 4 handles CISP and is identical to Proposition 7. Proposition 5 below then presents conditions which are more general, but also more complicated, than those

in Propositions 3 and 4, and is identical to Lemma 3 in the paper. Throughout we focus on two-dimensional signals, $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$, on the unit square $\mathbf{S} = [0, 1] \times [0, 1]$. The results generalize in a straightforward manner to n signals.

Let $g(x|a)$ and $G(x|a)$ be the marginal density and cumulative distribution functions of \tilde{x} , and let $h(y|x, a)$ and $H(y|x, a)$ be the conditional density and cumulative distribution functions of \tilde{y} given $\tilde{x} = x$. Thus the joint density of (\tilde{x}, \tilde{y}) is $f(x, y|a) = g(x|a)h(y|x, a)$.

Let \mathbf{E} be an increasing set, and let $y = b(x)$, for $x \in [0, 1]$, be the downward sloping curve describing the Southwest boundary of $\mathbf{E} \cap \mathbf{S}$, so $b'(x) \leq 0$. If this Southwest boundary hits the upper boundary of \mathbf{S} , then let $b(x) = 1$ on the relevant interval $[0, x_a]$, and if it hits the lower boundary, let $b(x) = 0$ on the relevant interval $[x_b, 1]$. Assume that $b(\cdot)$ is continuous and piecewise differentiable (it can always be approximated by such a function). We start with the following lemma.

LEMMA 5: *Let $P = \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E})$ be the probability that the signal $\tilde{\mathbf{x}}$ is in \mathbf{E} . Then*

$$(22) \quad \begin{aligned} dP/da = \int_0^1 [G_a(x|a)H_x(b(x)|x, a) - g(x|a)H_a(b(x)|x, a)] dx \\ + \int_0^1 G_a(x|a)h(b(x)|x, a)b'(x)dx = I_A + I_B. \end{aligned}$$

PROOF: First,

$$P = \int_{x=0}^1 \int_{y=b(x)}^1 g(x|a)h(y|x, a)dydx = \int_0^1 g(x|a)[1 - H(b(x)|x, a)]dx.$$

Thus, assuming one can differentiate under the integral sign,

$$dP/da = \int_0^1 g_a(x|a)[1 - H(b(x)|x, a)]dx - \int_0^1 g(x|a)H_a(b(x)|x, a)dx = I_1 - I_2,$$

where I_1 and I_2 are the obvious integrals. Next, integrate I_1 by parts, yielding

$$\begin{aligned}
I_1 &= G_a(x|a)[1 - H(b(x)|x, a)]\Big|_{x=0}^1 + \int_0^1 G_a(x|a)H_x(b(x)|x, a)dx \\
&\quad + \int_0^1 G_a(x|a)h(b(x)|x, a)b'(x)dx = 0 + I_{1a} + I_{1b},
\end{aligned}$$

where the first term is zero since $G_a(0|a) = G_a(1|a) = 0$. Here $I_A = I_{1a} - I_2$ and $I_B = I_{1b}$. Thus, $dP/da = I_A + I_B$, as in (22). *Q.E.D.*

Note that the integration by parts on I_1 is useful since it is more natural to impose conditions on $G_a(x|a)$ and $H_x(y|x, a)$ than on $g_a(x|a)$. This is illustrated in the following proposition.

PROPOSITION 3: *Suppose $G(x|a)$ is nonincreasing in a , and $H(y|x, a)$ is nonincreasing in x and a . Then NISP holds.*

PROOF: First, since $G_a(x|a) \leq 0$, $H_x(y|x, a) \leq 0$ and $H_a(y|x, a) \leq 0$, the first integral in (22), $I_A \geq 0$. Next, since $G_a(x|a) \leq 0$ and $b'(x) \leq 0$, the second integral $I_B \geq 0$ as well. Thus, $dP/da \geq 0$, so NISP holds. *Q.E.D.*

Note that Proposition 3 is Proposition 6 of Conlon (2008), where its meaning is discussed in greater detail.

PROPOSITION 4: *Suppose $G_a(x|a)$ and $H_a(y|x, a)$ are negative (this follows from strict versions of the corresponding MLR properties for \tilde{x} and \tilde{y}). Assume also that $g(x|a)$ and $h(y|x, a)$ are strictly positive for $x, y \in [0, 1]$. Finally assume $H_x(y|x, a) < 0$ (so \tilde{x} and \tilde{y} are positively related). Then the conditions*

$$(23) \quad h_a(y|x, a)/h(y|x, a) \leq -G_{aa}(x|a)/G_a(x|a)$$

$$(24) \quad g_a(x|a)/g(x|a) \leq -H_{aa}(y|x, a)/H_a(y|x, a)$$

$$(25) \quad H_{ax}(y|x, a)/H_x(y|x, a) \leq -G_{aa}(x|a)/G_a(x|a)$$

are sufficient to ensure CISP.

PROOF: First, differentiate the integrand of I_B in (22) with respect to a , yielding $[G_{aa}(x|a)h(y|x, a) + G_a(x|a)h_a(y|x, a)]b'(x)$. Now, (23) plus $G_a(x|a) < 0$ and $b'(x) \leq 0$ ensure that this is less than or equal to zero, so I_B is nonincreasing in a . Similarly, (25) plus $H_x(y|x, a) < 0$ and $G_a(x|a) < 0$ ensure that the first term in the integrand of I_A is nonincreasing in a . Finally, (24) and $H_a(y|x, a) < 0$ ensure that the second term in I_A , after the minus sign, is nondecreasing in a . Thus, I_A is nonincreasing in a , so dP/da is nonincreasing in a , and CISP holds. *Q.E.D.*

Again, Proposition 4 is Proposition 7 of Conlon (2008), where its meaning is discussed in greater detail.

Next, (23) can be weakened if we are willing to strengthen (24) and visa versa, as explained in Conlon (2008). We show how to do this in Proposition 5 below, which replicates Lemma 3 of Conlon (2008). This proposition will build on the following lemma.

LEMMA 6: *Let $\phi(x, y, a)$ and $\psi(x, y, a)$ satisfy*

$$(26) \quad \phi_x(x, y, a) + \psi_y(x, y, a) = 0$$

and

$$(27) \quad \phi(0, y, a) = \phi(1, y, a) = \psi(x, 0, a) = \psi(x, 1, a) = 0.$$

Then

$$(28) \quad \int_0^1 \psi(x, b(x), a) dx - \int_0^1 \phi(x, b(x), a) b'(x) dx = I_\psi - I_\phi = 0.$$

PROOF: First, since $\psi(x, 1, a) = 0$,

$$(29) \quad I_\psi = - \int_0^1 [\psi(x, 1, a) - \psi(x, b(x), a)] dx = - \int_{x=0}^1 \int_{y=b(x)}^1 \psi_y(x, y, a) dy dx,$$

where the second step uses the Fundamental Theorem of Calculus.

Next, suppose $b(x)$ is strictly decreasing, and so invertible, and represent the Southwest boundary of \mathbf{E} by $x = c(y)$. This inverts $b(x)$ on the downward sloping part of the curve. Also let $c(y)$ equal zero for $y \geq b(0)$ and one for $y \leq b(1)$, if the curve hits the left or right boundary of the square \mathbf{S} . Then, letting $b(x) = y$, so $x = c(y)$, gives

$$(30) \quad I_\phi = \int_{b(0)}^{b(1)} \phi(c(y), y, a) dy = - \int_{b(1)}^{b(0)} \phi(c(y), y, a) dy = - \int_0^1 \phi(c(y), y, a) dy.$$

Here the last step uses $b(0) > b(1)$; it also uses, for $y \leq b(1)$, that $\phi(c(y), y, a) = \phi(1, y, a) = 0$, and similarly for $y \geq b(0)$. Next, since $\phi(1, y, a) = 0$, (30) equals

$$(31) \quad I_\phi = \int_0^1 [\phi(1, y, a) - \phi(c(y), y, a)] dy = \int_{y=0}^1 \int_{x=c(y)}^1 \phi_x(x, y, a) dx dy.$$

Also, an approximation argument can be used if $b(x)$ is not *strictly* decreasing. Now, both (29) and (31) are simply integrals over $\mathbf{E} \cap \mathbf{S}$, so using Fubini's Theorem,

$$I_\psi - I_\phi = - \int \int_{\mathbf{E} \cap \mathbf{S}} [\phi_x(x, y, a) + \psi_y(x, y, a)] dx dy = 0,$$

by (26).

Q.E.D.

PROPOSITION 5: *Suppose there are functions $\phi(x, y, a)$ and $\psi(x, y, a)$ satisfying (26) and (27), and such that*

$$(32) \quad G_a(x|a)h(y|x, a) + \phi(x, y, a)$$

and

$$(33) \quad g(x|a)H_a(y|x, a) - G_a(x|a)H_x(y|x, a) + \psi(x, y, a)$$

are less than or equal to zero for all $(x, y) \in \mathbf{S}$. Then $f(x, y|a)$ satisfies NISP. If there are $\phi(x, y, a)$ and $\psi(x, y, a)$ satisfying (26) and (27), and such that (32) and (33) are nondecreasing in a , then $f(x, y|a)$ satisfies CISP.

PROOF: Subtract (28) from (22) and use $b'(x) \leq 0$.

Q.E.D.

Proposition 5, which is Lemma 3 in Conlon (2008), now allows us to choose the ϕ and ψ functions to weaken (23) if we are willing to strengthen (24) and visa versa. For example it can be used to show that the GCDFC implies CISP, as in Conlon (2008).

5. IS THERE A CANONICAL VECTOR FLOW FOR SIGNAL TECHNOLOGIES

$f(\mathbf{x}|a)$ IN PRINCIPAL-AGENT MODELS?

Sections 7 and 8 of Conlon (2008) show that it is possible to represent any principal-agent technology $f(\mathbf{x}|a)$ by a vector flow $\mathbf{v}(\mathbf{x}, a)$, such that $f_a(\mathbf{x}|a) = -\text{div } \mathbf{v}(\mathbf{x}, a)$. This lets us confirm the *global* NISP and CISP conditions in that paper by checking certain easy-to-verify *local* conditions on the vector field $\mathbf{v}(\mathbf{x}, a)$.

Thus assume, for specificity, that the signal $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$ is two dimensional, and that $\tilde{\mathbf{x}}$ is distributed in the unit square, $\mathbf{S} = [0, 1] \times [0, 1]$, as in Conlon (2008). Say that a set \mathbf{E} is an “increasing set” if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in \mathbf{E}$. Then NISP is the condition that $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is nondecreasing in a , and CISP is the condition that $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is concave in a , for all increasing sets \mathbf{E} .

Next assume that, as a increases, the density $f(\mathbf{x}|a)$ follows the density flux $\mathbf{v}(\mathbf{x}, a) = (u(x, y, a), v(x, y, a))$, with $u(0, y, a) = u(1, y, a) = v(x, 0, a) = v(x, 1, a) = 0$, so no density flows out of \mathbf{S} . Thus $f_a(\mathbf{x}|a) = -\text{div } \mathbf{v}(\mathbf{x}, a)$, as in Conlon (2008).

The Divergence Theorem then shows that

$$(34) \quad \frac{d}{da} \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a) = - \int_0^1 \mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t) dt,$$

where $\mathbf{x}(t) = (x(t), y(t))$ traces the boundary of $\mathbf{E} \cap \mathbf{S}$ counterclockwise, $\mathbf{n}(t) = (y'(t), -x'(t))$ is the outward-pointing normal (perpendicular) to the boundary of $\mathbf{E} \cap \mathbf{S}$, and $\mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t)$ is the “dot product” of $\mathbf{v}(\mathbf{x}(t), a)$ and $\mathbf{n}(t)$, i.e.,

$$\mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t) = u(\mathbf{x}(t), a)y'(t) - v(\mathbf{x}(t), a)x'(t);$$

see Conlon (2008) for details. Lemma 2 in that paper uses (34) to show that, if the coordinates of $\mathbf{v}(\mathbf{x}, a)$ are everywhere nonnegative, then NISP holds, while if the coordinates of $\mathbf{v}(\mathbf{x}, a)$ are everywhere nonincreasing in a , then CISP holds.

It is reasonable to conjecture that Lemma 2 in the paper has a converse. That is, if $f(\mathbf{x}|a)$ satisfies NISP, then there should exist a vector field $\mathbf{v}(\mathbf{x}, a)$, as above, with coordinates everywhere nonnegative, while if $f(\mathbf{x}|a)$ satisfies CISP, then there should be a vector field $\mathbf{v}(\mathbf{x}, a)$ as above with coordinates nonincreasing in a . However, while these conjectures seem intuitively obvious, I have not been able to prove them.

In addition, the vector field, $\mathbf{v}(\mathbf{x}, a)$, constructed in Conlon (2008), Section 8, is rather ad hoc. This section therefore considers a possible *canonical* vector flow corresponding to $f(\mathbf{x}|a)$, and argues that this canonical flow is not obviously superior to the ad hoc flow constructed in Conlon (2008). Thus, at this point there seems to be no obvious alternative superior to the ad hoc flow in Conlon (2008). On the other hand, if the conditions in Conlon (2008) turn out to be inadequate for some future applications, then these future applications may themselves suggest new vector flows.

In any case, the difficulty with representing $f(\mathbf{x}|a)$ by a vector flow is precisely that there are too many vector flows capable of representing any technology $f(\mathbf{x}|a)$. This suggests representing $f(\mathbf{x}|a)$ using a vector flow which is in some sense *minimal*. Specifically, we look for a vector flow $\mathbf{v}(\mathbf{x}, a) = (u(\mathbf{x}, a), v(\mathbf{x}, a))$ which solves

$$\min_{\mathbf{v}(\cdot, \cdot)} \int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 d\mathbf{x}, \text{ subject to } f_a(\mathbf{x}|a) = -\text{div } \mathbf{v}(\mathbf{x}, a),$$

and

$$(35) \quad u(0, y, a) = u(1, y, a) = v(x, 0, a) = v(x, 1, a) = 0,$$

where $\|\mathbf{v}\|^2 = u^2 + v^2$ for $\mathbf{v} = (u, v)$. We can write the Lagrangian for this problem as

$$L = \int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 d\mathbf{x} + \int_0^1 \int_0^1 \Phi(\mathbf{x}, a) [f_a(\mathbf{x}|a) + \text{div } \mathbf{v}(\mathbf{x}, a)] d\mathbf{x} + \left(\begin{array}{c} \text{Boundary} \\ \text{Terms} \end{array} \right),$$

where $\Phi(\mathbf{x}, a)$ is the Lagrange multiplier for the constraint $f_a(\mathbf{x}|a) = -\text{div } \mathbf{v}(\mathbf{x}, a)$, and the “Boundary Terms” are the terms corresponding to the boundary constraints in (35).

Performing an integration by parts on the $\text{div } \mathbf{v}(\mathbf{x}, a)$ term shows that L equals

$$\int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 d\mathbf{x} + \int_0^1 \int_0^1 [\Phi(\mathbf{x}, a)f_a(\mathbf{x}|a) - \mathbf{v}(\mathbf{x}, a) \cdot \text{grad } \Phi(\mathbf{x}, a)] d\mathbf{x} + \left(\begin{array}{c} \text{Bndry} \\ \text{Terms} \end{array} \right).$$

Thus, following the usual variational argument, we consider a small change, or variation, $\delta\mathbf{v}(\mathbf{x}, a)$, in the vector field $\mathbf{v}(\mathbf{x}, a)$, leaving the boundary conditions fixed. If we consider only first order terms in $\delta\mathbf{v}$ then, at a minimum point, this should leave the Lagrangian L unchanged, so

$$0 = \delta L = \int_0^1 \int_0^1 [2\mathbf{v}(\mathbf{x}, a) - \text{grad } \Phi(\mathbf{x}, a)] \cdot \delta\mathbf{v}(\mathbf{x}, a) d\mathbf{x}.$$

Now, this must hold for all variations, $\delta\mathbf{v}(\mathbf{x}, a)$, in the vector flow, satisfying the boundary conditions. In particular, it must hold for a “blip” around an arbitrary point \mathbf{x}_0 , which equals zero outside of a neighborhood around \mathbf{x}_0 . Choosing $\delta\mathbf{v}(\mathbf{x}, a)$ judiciously inside this small neighborhood, in the usual way, then shows that $2\mathbf{v}(\mathbf{x}, a) - \text{grad } \Phi(\mathbf{x}, a) = 0$ for all $\mathbf{x} \in \mathbf{S}$, or

$$\mathbf{v}(\mathbf{x}, a) = (1/2)\text{grad } \Phi(\mathbf{x}, a).$$

That is, the minimal flow is a “potential flow” (Chorin and Marsden (1990, pp. 47-68)), with the potential given by the Lagrange multiplier. Next let Δ be the “Laplacian” operator, so

$$\begin{aligned} \Delta\Phi(\mathbf{x}, a) &= \Phi_{xx}(\mathbf{x}, a) + \Phi_{yy}(\mathbf{x}, a) \\ &= \text{div grad } \Phi(\mathbf{x}, a) = 2 \text{div } \mathbf{v}(\mathbf{x}, a) = -2 f_a(\mathbf{x}|a). \end{aligned}$$

Then this Lagrange multiplier potential solves the following “Neumann Problem” (Berg and McGregor (1966), Folland (1976)):

$$(36) \quad \Delta\Phi(\mathbf{x}, a) = -2f_a(\mathbf{x}|a),$$

subject to $\Phi_x(0, y) = \Phi_x(1, y) = \Phi_y(x, 0) = \Phi_y(x, 1) = 0$.

The fact that a minimal flow solves (36) is essentially “Dirichlet’s principle” (Folland (1976, pp. 112-117)). Thus, if one can find such a potential function $\Phi(\mathbf{x}, a)$, it becomes easy to check NISP and CISP. For example, NISP simply follows from $\Phi(\mathbf{x}, a)$ nondecreasing in \mathbf{x} , while CISP follows from $\text{grad } \Phi(\mathbf{x}, a)$ nonincreasing in a .

However, while there are various methods for solving Neumann problems (see, e.g., Folland (1976)), none seem easy to implement in practice. For example, one method uses a “Green’s function,” $G(\mathbf{x}, \mathbf{z})$, which gives

$$\Phi(\mathbf{x}, a) = -2 \int_0^1 \int_0^1 G(\mathbf{x}, \mathbf{z}) f_a(\mathbf{z}|a) d\mathbf{z}.$$

The problem then becomes to find an appropriate Green’s function. For a Neumann problem on a square, it does not seem to be possible to write this Green’s function in closed form, though it can be represented as a double Fourier series,

$$G(x, y; z, w) = 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} \frac{\cos \pi mx \cos \pi mz \cos \pi ny \cos \pi nw}{\pi^2(m^2 + n^2)},$$

where $\gamma_{00} = 0$, $\gamma_{m0} = \gamma_{0n} = 1/2$, and $\gamma_{mn} = 1$ otherwise (Roach (1982, p. 268)). It is not clear how to use such solutions to derive useful conditions for NISP or CISP.

In addition, while the solution above seems to be canonical in the sense that it minimizes the total flow necessary to represent the signal technology, it is not canonical in the sense of providing *necessary* as well as sufficient conditions for NISP and CISP. Thus, while $\Phi(\mathbf{x}, a)$ nondecreasing in \mathbf{x} is sufficient to assure NISP, it is not necessary. For example, consider a function $\Phi(\mathbf{x}, a)$ which is increasing in \mathbf{x} outside of some

small neighborhood around the point $(0.5, 0.5)$, say, but decreasing slightly inside of that neighborhood. Thus the vector field $\mathbf{v}(\mathbf{x}, a) = (1/2)\text{grad } \Phi(\mathbf{x}, a)$ has positive coordinates except in that small neighborhood. Clearly this can be done so that the integral, (34), above, will always be positive (recall that the normal vector $\mathbf{n}(t)$ in (34) is pointing Southwest whenever $\mathbf{v}(\mathbf{x}(t), a) \neq \mathbf{0}$). Thus NISP will hold, even though $\Phi(\mathbf{x}, a)$ is *not* always nondecreasing in \mathbf{x} .

This underscores the difficulty of finding easy-to-check *local* conditions to verify hard-to-check *global* conditions like NISP or CISP. Thus, it seems the best we can reasonably do at this point is to use ad hoc vector flows like those in Section 8 of Conlon (2008). Also, those ad hoc vector flows yield quite useful conditions, as argued in that paper. Nevertheless, in the process of considering different applications, even more useful conditions may be discovered.

6. DERIVING STATE-SPACE REPRESENTATIONS FROM VECTOR FLOWS

If we begin with the basic vector flow in Equation (26) of Conlon (2008), the state-space representation it implies is the representation in Proposition 3 of Conlon (2008), as mentioned in footnote 10 of that paper. This basic vector flow is

$$\mathbf{v}^b(x, y, a) = (-G_a(x|a)h(y|x, a), -g(x|a)H_a(y|x, a) + G_a(x|a)H_x(y|x, a)).$$

We want to show that, for this vector flow, the solution, $\mathbf{x}(a, \mathbf{x}_0)$, to the differential system

$$\dot{\mathbf{x}}_a(a, \mathbf{x}_0) = \frac{1}{f(\mathbf{x}(a, \mathbf{x}_0)|a)} \mathbf{v}^b(\mathbf{x}(a, \mathbf{x}_0), a),$$

subject to the initial condition, $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, is essentially the state space representation in Conlon's (2008) Proposition 3. Consider one coordinate at a time. The first coordinate of the above equation becomes

$$(37) \quad x_a(a, \mathbf{x}_0) = -\frac{G_a(x(a, \mathbf{x}_0)|a)h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{g(x(a, \mathbf{x}_0)|a)h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)} = -\frac{G_a(x(a, \mathbf{x}_0)|a)}{g(x(a, \mathbf{x}_0)|a)},$$

or

$$g(x(a, \mathbf{x}_0)|a)x_a(a, \mathbf{x}_0) + G_a(x(a, \mathbf{x}_0)|a) = 0.$$

Integrating this with respect to a gives

$$G(x(a, \mathbf{x}_0)|a) = \text{Constant in } a = G(x_0|a = 0).$$

Letting $G(x_0|a = 0) = \theta_1$, and noting that the marginal cumulative distribution function G , here, is the marginal cumulative distribution function F^1 in Proposition 3 of the paper, yields the first half of (11) in that proposition.

Similarly, the second coordinate of the above vector-differential equation gives

$$\begin{aligned} y_a(a, \mathbf{x}_0) \\ = -\frac{g(x(a, \mathbf{x}_0)|a)H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) - G_a(x(a, \mathbf{x}_0)|a)H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{g(x(a, \mathbf{x}_0)|a)h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}. \end{aligned}$$

Simplifying and using (37) gives

$$y_a(a, \mathbf{x}_0) = -\frac{H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)} - \frac{H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}x_a(a, \mathbf{x}_0),$$

or

$$\begin{aligned} h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)y_a(a, \mathbf{x}_0) \\ + H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) + H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)x_a(a, \mathbf{x}_0) = 0. \end{aligned}$$

Again, integrating with respect to a gives

$$H(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) = \text{Constant in } a = H(y_0|x_0, a = 0).$$

Finally, again letting $H(y_0|x_0, a = 0) = \theta_2$ gives the second half of (11) in the paper, and we're done.

7. GENERALIZING JEWITT'S THEOREM 2

Conlon (2008) provides generalizations of two of the three major sets of conditions justifying the first order approach to multi-signal principal-agent problems: the Sinclair-Desgagné (1994) conditions, and Jewitt's (1988) Theorem 3. It also briefly describes a generalization of the third major set of conditions, that in Jewitt's Theorem 2 (see Conlon (2008), Section 9). This note presents that generalization in detail. I would like to thank Ian Jewitt for suggesting this generalization to me.

Notation is as in Conlon (2008). Suppose the principal is risk neutral and observes a two-signal random vector (\tilde{x}, \tilde{y}) (note that this result does *not* generalize easily beyond the two-signal case). Jewitt, Theorem 2, assumes that the signals are independent. Let their cumulative distribution functions be $F^1(x|a)$ and $F^2(y|a)$, with sufficiently smooth densities $f^1(x|a)$ and $f^2(y|a)$. Also, recall Jewitt's function $\omega(z) = u((u')^{-1}(1/z))$ from (7) above. In the two-signal case, the principal's first order condition for the cost minimizing schedule $s^*(\cdot)$ to induce action a , given the *agent's* first order condition $U_a(s(\cdot), a) = 0$, yields:

$$(38) \quad u(s^*(x, y)) = \omega\left(\lambda + \mu \left[\frac{f_a^1(x|a)}{f^1(x|a)} + \frac{f_a^2(x|a)}{f^2(x|a)} \right]\right),$$

with λ and μ Lagrange multipliers, and $\mu \geq 0$, as argued in Jewitt (1988). With this setup, Jewitt's Theorem 2 is:

THEOREM (JEWITT'S THEOREM 2): *The first order approach is valid if $\omega(z)$ is concave in z and $F^1(x|a)$ and $F^2(y|a)$ satisfy the MLR property and the CDFC.*

To extend this theorem to the non-independence case, consider the general density and cumulative distribution functions, $f(x, y|a)$ and $F(x, y|a)$, and impose the conditions in the following two definitions:

DEFINITION 1: The distribution $F(x, y|a)$ satisfies the *Lower Quadrant Convexity Condition* (LQCC) if for every fixed (x, y) , the probability, $\text{Prob}(\tilde{x} \geq x \text{ or } \tilde{y} \geq y|a)$ is

concave in a , so the probability of the corresponding lower quadrant, $\text{Prob}(\tilde{x} \leq x \text{ and } \tilde{y} \leq y|a)$, is convex.

DEFINITION 2: The distribution $F(x, y|a)$ satisfies the *Submodular Likelihood Ratio* (SLR) property if the likelihood ratio, $f_a(x, y|a)/f(x, y|a)$, is submodular, so $\partial^2[f_a(x, y|a)/f(x, y|a)]/\partial x\partial y \leq 0$ for all x, y and a .

With these conditions, we can now state the generalization of Jewitt's Theorem 2.

PROPOSITION 6: *The first order approach is valid if $\omega(z)$ is concave in z and $F(x, y|a)$ satisfies MLR, SLR and LQCC.*

Note that LQCC follows if \tilde{x} and \tilde{y} are independent and each satisfies the CDFC, since $\text{Prob}(\tilde{x} \leq x|a)\text{Prob}(\tilde{y} \leq y|a)$ is convex in a if \tilde{x} and \tilde{y} satisfy CDFC. Also, if \tilde{x} and \tilde{y} are independent, then the SLR property is met, since in this case,

$$(39) \quad \frac{f_a(x, y|a)}{f(x, y|a)} = \frac{f_a^1(x|a)}{f^1(x|a)} + \frac{f_a^2(y|a)}{f^2(y|a)},$$

so the cross partial of the likelihood ratio in (39) is zero. Thus, the above Proposition 6 does generalize Jewitt's Theorem 2. To prove Proposition 6, we need a lemma. This lemma is similar to the $n = 2$ case of Theorem 3.3.15 in Müller and Stoyan (2002).

LEMMA 7: *Define the function $h(x, y; x_0, y_0) = 1$ if $x \geq x_0$ or $y \geq y_0$, zero otherwise. Suppose that $g(x, y)$ is nondecreasing in x and y and that $g_{xy}(x, y) \leq 0$. Then $g(\cdot, \cdot)$ can be approximated as a constant plus a positive linear combination of the $h(\cdot, \cdot; x_0, y_0)$ functions,*

$$(40) \quad g(x, y) \approx \alpha_0 + \sum_{i=1}^n \alpha_i h(x, y; x_i, y_i), \text{ with } \alpha_i > 0 \text{ for } i \geq 1.$$

PROOF OF LEMMA 7: In point of fact we represent $g(\cdot, \cdot)$ as an integral of the $h(\cdot, \cdot; x_0, y_0)$ functions. For simplicity suppose that all functions are defined on the square $\mathbf{S} = [0, 1] \times [0, 1]$ (a limiting argument can extend the result to functions on

any domain). Then

$$(41) \quad \begin{aligned} g(x, y) &= g(0, 0) + \int_0^1 g_x(u, 1)h(x, y; u, 1)du \\ &+ \int_0^1 g_y(1, v)h(x, y; 1, v)dv - \int_0^1 \int_0^1 g_{xy}(u, v)h(x, y; u, v)dudv. \end{aligned}$$

To see this, note that, using the definition of $h(x, y; u, v)$, the integrals in (41) can be rewritten, giving:

$$\begin{aligned} \text{Right Hand Side of (41)} &= g(0, 0) + \int_0^x g_x(u, 1)du + \int_0^y g_y(1, v)dv \\ &- \int_{u=0}^x \int_{v=0}^1 g_{xy}(u, v)dvdu - \int_{u=x}^1 \int_{v=0}^y g_{xy}(u, v)dvdu. \end{aligned}$$

Using the Fundamental Theorem of Calculus, this equals

$$\begin{aligned} &g(0, 0) + g(x, 1) - g(0, 1) + g(1, y) - g(1, 0) \\ &- \int_0^x [g_x(u, 1) - g_x(u, 0)]du - \int_x^1 [g_x(u, y) - g_x(u, 0)]du \\ &= g(0, 0) + g(x, 1) - g(0, 1) + g(1, y) - g(1, 0) \\ &- [g(x, 1) - g(x, 0) - g(0, 1) + g(0, 0)] \\ &- [g(1, y) - g(1, 0) - g(x, y) + g(x, 0)] = g(x, y). \end{aligned}$$

Now the coefficients $g_x(u, 1)$ and $g_y(1, v)$ in the first two integrals in (41) are nonnegative since $g(x, y)$ is nondecreasing in x and y . Also, the coefficient $g_{xy}(u, v)$ in the last integral in (41) is nonpositive, since $g_{xy}(x, y) \leq 0$. Since this last integral is being subtracted, it is clear that (41) expresses $g(x, y)$ as a constant plus a sum of integrals of functions of the form $h(x, y; u, v)$, as u and v vary, with positive coefficients. Hence it can be approximated as a constant plus a linear combination of such functions with positive coefficients, as in (40). *Q.E.D.*

PROOF OF PROPOSITION 6: First, the generalization of (38) to the nonindependence case is $u(s^*(x, y)) = \omega(\lambda + \mu[f_a(x, y|a)/f(x, y|a)])$. Using MLR, SLR and

concavity of $\omega(z)$ in this proves that $u(s^*(x, y))$ is nondecreasing and submodular in x and y . The lemma then shows that $u(s^*(x, y))$ can be approximated as in (40). Thus $U(s^*(\cdot), a)$ can be approximated by a linear combination, with positive coefficients, of the functions $h^T(a; x_i, y_i)$, and these are concave in a by the LQCC. Any solution to the relaxed problem therefore yields $U(s^*(\cdot), a)$ concave in a , so this solution satisfies the unrelaxed constraints, and so is a solution to the unrelaxed problem. The first order approach is therefore valid. *Q.E.D.*

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APPENDIX A: PROOF OF LEMMA 2

Intuitively, if the positive linear combination yielding $k(x_1, x_2) = \min(x_1, x_2)$ includes the function $h(x_1, x_2; \alpha_1^0, \alpha_2^0, \beta^0)$, then $k(x_1, x_2)$ should have a kink along the negatively sloped line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$. However, the only points where the $k(x_1, x_2)$ function is not perfectly flat are contained in the *positively* sloped line

$$\mathbf{K} = \{(x_1, x_2) : x_1 = x_2\}.$$

This yields a contradiction.

To make this argument precise, one must show that, if $k(x_1, x_2)$ can be *approximated* by a constant plus a positive linear combination of the $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ functions, then it can be expressed *exactly* as a constant plus an *integral* of these functions, where the integral is over $(\alpha_1, \alpha_2, \beta)$. This requires a compactness argument.

To facilitate this argument, consider approximations of $k(x_1, x_2)$ on the square

$$(x_1, x_2) \in \mathbf{S} = [-1, 1] \times [-1, 1].$$

It is enough to prove that there are no such approximations on \mathbf{S} . However, for approximations on \mathbf{S} , it is enough to consider values of $\beta \in [-1, 1]$. This is because the expression $\alpha_1 x_1 + \alpha_2 x_2$ is in $[-1, 1]$ for $(x_1, x_2) \in \mathbf{S}$, since $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. Thus, for $\beta < -1$, $h(x_1, x_2; \alpha_1, \alpha_2, \beta) = 0$ on \mathbf{S} , and for $\beta > 1$, $h(x_1, x_2; \alpha_1, \alpha_2, \beta) = \alpha_1 x_1 + \alpha_2 x_2 - \beta = h(x_1, x_2; \alpha_1, \alpha_2, 1) + 1 - \beta$ on \mathbf{S} . It is therefore enough to use the $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ functions with $(\alpha_1, \alpha_2, \beta)$ in the compact set

$$\mathbf{T} = \{(\alpha_1, \alpha_2, \beta) : \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1, -1 \leq \beta \leq 1\}.$$

Thus, consider a sequence of approximations to $k(x_1, x_2)$ given by

$$(A1) \quad k_n^A(x_1, x_2) = A_n^0 + B_n^0 \int_{\mathbf{T}} h(x_1, x_2; \alpha_1, \alpha_2, \beta) d\mu_n^0(\alpha_1, \alpha_2, \beta),$$

where μ_n^0 is a measure in the compact set, $\Delta\mathbf{T}$, of probability measures on \mathbf{T} under weak convergence. Assume k_n^A converges uniformly to k on \mathbf{S} . Of course, if the approximations in (A1) are ordinary sums with finite numbers of terms, then each μ_n^0 has support which is a finite set of points.

Unfortunately, the sequence $\{B_n^0\}$ might be unbounded, if the support of μ_n^0 becomes increasingly concentrated near $\beta = -1$. The following somewhat complicated argument, through equation (A3) below, shows that this is not a problem.

First let $\mathbf{T}^* = \{(\alpha_1, \alpha_2, \beta) \in \mathbf{T} : -0.5 \leq \beta \leq 1\}$. Then $B_n^0 \mu_n^0(\mathbf{T}^*)$ is a bounded sequence. For

$$\begin{aligned} B_n^0 \mu_n^0(\mathbf{T}^*) &= 2B_n^0 \int_{\mathbf{T}^*} [(-0.5 - \beta) - (-1 - \beta)] d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &= 2B_n^0 \int_{\mathbf{T}^*} [h(-0.5, -0.5; \alpha_1, \alpha_2, \beta) - h(-1, -1; \alpha_1, \alpha_2, \beta)] d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &\leq 2B_n^0 \int_{\mathbf{T}} [h(-0.5, -0.5; \alpha_1, \alpha_2, \beta) - h(-1, -1; \alpha_1, \alpha_2, \beta)] d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &= 2[k_n^A(-0.5, -0.5) - k_n^A(-1, -1)]. \end{aligned}$$

Here the second step follows by $\alpha_1 + \alpha_2 = 1$, $\beta \geq -0.5$ and the definition of the h function in (16), while the third step follows since the integrand is nonnegative and $\mathbf{T}^* \subseteq \mathbf{T}$. Finally, the last sequence above is bounded, since $k_n^A(x_1, x_2)$ converges uniformly to $k(x_1, x_2)$ on \mathbf{S} . Thus $B_n^0 \mu_n^0(\mathbf{T}^*)$ is a bounded sequence, as desired.

Next, $k(x_1, x_2) = 2k(0.5x_1, 0.5x_2)$, so $k(x_1, x_2)$ can be approximated on \mathbf{S} by

$$(A2) \quad \begin{aligned} 2k_n^A(0.5x_1, 0.5x_2) &= 2A_n^0 + 2B_n^0 \int_{\mathbf{T}} h(0.5x_1, 0.5x_2; \alpha_1, \alpha_2, \beta) d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &= 2A_n^0 + B_n^0 \int_{\mathbf{T}} h(x_1, x_2; \alpha_1, \alpha_2, 2\beta) d\mu_n^0(\alpha_1, \alpha_2, \beta), \end{aligned}$$

where the last step follows since $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ is homogeneous of degree one in (x_1, x_2, β) . Now, for $(x_1, x_2) \in \mathbf{S}$, the range of integration in (A2) can be restricted to \mathbf{T}^* , since $h(x_1, x_2; \alpha_1, \alpha_2, 2\beta) = 0$ for $\beta \in [-1, -0.5)$. Let $\mu_n^1 \in \Delta \mathbf{T}^*$ be the measure μ_n^0 , restricted to \mathbf{T}^* , and divided by $\mu_n^0(\mathbf{T}^*)$ to obtain another probability measure. Also, let $A_n^1 = 2A_n^0$ and $B_n^1 = B_n^0 \mu_n^0(\mathbf{T}^*)$. Then $k(x_1, x_2)$ is uniformly approximated on \mathbf{S} by the sequence

$$(A3) \quad A_n^1 + B_n^1 \int_{\mathbf{T}^*} h(x_1, x_2; \alpha_1, \alpha_2, 2\beta) d\mu_n^1(\alpha_1, \alpha_2, \beta).$$

Also $B_n^1 = B_n^0 \mu_n^0(\mathbf{T}^*)$ is bounded, as shown above. In addition, since (A3) and B_n^1 are bounded, A_n^1 must be bounded as well.

Now, (A_n^1, B_n^1, μ_n^1) is a sequence in a compact set (since $\Delta \mathbf{T}^*$, like $\Delta \mathbf{T}$, is compact). Thus it has a convergent subsequence. Let the limit of this subsequence be $(A_\infty^1, B_\infty^1, \mu_\infty^1)$. Then

$$(A4) \quad k(x_1, x_2) = A_\infty^1 + B_\infty^1 \int_{\mathbf{T}^*} h(x_1, x_2; \alpha_1, \alpha_2, 2\beta) d\mu_\infty^1(\alpha_1, \alpha_2, \beta) \text{ on } \mathbf{S},$$

by continuity of the integral with respect to weak convergence of measures.

Now, the support of μ_∞^1 must contain a point $(\alpha_1^0, \alpha_2^0, \beta^0)$ such that the line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$ passes through the interior of \mathbf{S} , since otherwise the integral in (A4) would be linear (or more precisely, affine) in (x_1, x_2) on \mathbf{S} . There is then a small neighborhood around $(\alpha_1^0, \alpha_2^0, \beta^0)$ with positive measure under μ_∞^1 .

However, the negatively sloped line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$ is not contained in the set $\mathbf{K} = \{(x_1, x_2) : x_1 = x_2\}$ of edges of the graph of the function $k(x_1, x_2) = \min(x_1, x_2)$. Thus, there must be a point, $(x_1^0, x_2^0) \in \mathbf{S}$, on the line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$, but not in \mathbf{K} . But $k(x_1, x_2)$ should then have some curvature at (x_1^0, x_2^0) , which is impossible, since $(x_1^0, x_2^0) \notin \mathbf{K}$. *Q.E.D.*

APPENDIX B: PROOF OF LEMMA 3

Let the density $f(x_1, x_2|a)$, on the rectangle $[0, 3] \times [0, 1]$, be given by

$$f(x_1, x_2|a) = \begin{cases} (1/3) + (2x_1 - 3)a\epsilon_0 + (1 - x_1)(2x_2 - 1)(2a - a^2)\epsilon_1, & \text{for } 0 \leq x_1 \leq 1, \\ (1/3) + (2x_1 - 3)a\epsilon_0, & \text{for } 1 \leq x_1 \leq 2, \\ (1/3) + (2x_1 - 3)a\epsilon_0 + (x_1 - 2)(2x_2 - 1)a^2\epsilon_2, & \text{for } 2 \leq x_1 \leq 3. \end{cases}$$

Note that this breaks up the rectangle $[0, 3] \times [0, 1]$ into three regions: Region I: the square $[0, 1] \times [0, 1]$, Region II: the square $[1, 2] \times [0, 1]$, and Region III: the square $[2, 3] \times [0, 1]$. As a rises, the mass of the density tends to move in a concave manner in Region I, linearly in Region II, and in a *convex* manner in Region III.

Now, $\varphi(x_1, x_2) = \min(x_1 - 1, x_2)$. This equals $x_1 - 1$ in Region I and it equals x_2 in Region III. In Region II, it equals $x_1 - 1$ above the diagonal $x_2 = x_1 - 1$ (i.e., for $x_2 \geq x_1 - 1$), and it equals x_2 below this diagonal. Call the part of Region II above the diagonal Region IIa, and below the diagonal, Region IIb. Then

$$\varphi^T(a) = E[\varphi(\tilde{\mathbf{x}})|a] = I_I + I_{IIa} + I_{IIb} + I_{III},$$

where $I_I = \int_{\text{Region I}} \varphi(\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x}$, etc. Now, the first three of these integrals are linear in a , while the fourth is strictly convex in a . As an illustration,

$$I_I = \int_0^1 \int_0^1 (x_1 - 1) [(1/3) + (2x_1 - 3)a\epsilon_0 + (1 - x_1)(2x_2 - 1)(2a - a^2)\epsilon_1] dx_1 dx_2,$$

and it is easy to check that the coefficient of a^2 in this expression integrates out to zero (since $\int_0^1 (2x_2 - 1) dx_2 = 0$), so only terms linear in a survive. Similarly,

$$I_{IIIb} = \int_1^2 \int_0^{x_1-1} x_2 [(1/3) + (2x_1 - 3)a\epsilon_0] dx_2 dx_1,$$

which is also clearly linear in a . Since the first three integrals are linear and the fourth is strictly convex, it follows that $\varphi^T(a)$ is strictly convex, and so, is not concave.

Next, suppose ϵ_2 is smaller than ϵ_1 . Then we will show that

$$\int_0^3 \int_0^1 h(x_1, x_2; \alpha_1, \alpha_2, \beta) f(x_1, x_2|a) dx_2 dx_1$$

is concave in a for all α_1, α_2 and β with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. To see this, note that the line $\alpha_1 x_1 + \alpha_2 x_2 = \beta$, or $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$, divides Regions I, II and III into at most two subregions each, for a total of six subregions. Of these six subregions, only the three below the line $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$ matter, since the integrand is zero above the line. Also, the only terms that ultimately matter are those involving a^2 , since the other terms are either constant or linear in a .

The terms involving a^2 come from the parts of Regions I and III that are below the line $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$. Suppose $(\beta - \alpha_1 x_1)/\alpha_2 \in [0, 1]$ for all $x_1 \in [0, 3]$ (the other cases can be handled similarly). Then the resulting coefficients of a^2 are:

$$- \int_0^1 \int_0^{(\beta - \alpha_1 x_1)/\alpha_2} (\alpha_1 x_1 + \alpha_2 x_2 - \beta) (1 - x_1) (2x_2 - 1) \epsilon_1 dx_2 dx_1$$

from Region I, and

$$\int_2^3 \int_0^{(\beta - \alpha_1 x_1)/\alpha_2} (\alpha_1 x_1 + \alpha_2 x_2 - \beta) (x_1 - 2) (2x_2 - 1) \epsilon_2 dx_2 dx_1$$

from Region III. Now, it is easily checked that the first expression is negative and the second is positive. Also, if $\epsilon_1 > \epsilon_2$, then the first expression is bigger in absolute value than the second. Thus, the total coefficient on a^2 is negative, and $h^T(a; \alpha_1, \alpha_2, \beta)$ is indeed concave.

It is also easy to check that, if ϵ_0 is sufficiently large compared to ϵ_1 and ϵ_2 , then the distribution function satisfies the MLR property as well. *Q.E.D.*

APPENDIX C: PROOF OF LEMMA 4

As was the case for Lemma 2, if the function can be approximated as a positive linear combination of nondecreasing flat-top pyramids, it can be represented exactly as a positive integral of flat-top pyramids.

Now, the edges of the graph of $G(x_1, x_2) = \min(0, x_1, x_2, x_1 + x_2 + 1)$ are given by (a) the ray $x_1 = 0, x_2 \geq 0$, (b) the ray $x_2 = 0, x_1 \geq 0$, (c) the ray $x_1 = -1, x_2 \leq -1$, (d) the ray $x_2 = -1, x_1 \leq -1$, and (e) the *finite* line segment from the point $(-1, -1)$ to the point $(0, 0)$. Key among these is the positively sloped finite line segment (e). Since the graphs of nondecreasing flat-top pyramids have no edges which are positively sloped finite line segments, it is impossible to get a function with such an edge from a positive linear combination of such flat-top pyramids. *Q.E.D.*

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