Problems of the Millennium: The Riemann Hypothesis (2004)

by

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The Riemann Hypothesis (RH) is one of the seven millennium prize problems put forth by the Clay Mathematical Institute in 2000. Bombieri's statement [Bo1] written for that occasion is excellent. My plan here is to expand on some of his comments as well as to discuss some recent developments. RH has attracted further attention of late in view of the fact that three popular books were written about it in 2003 and a fourth one is forthcoming in 2005. Unlike the case of Fermat's Last Theorem, where the work of Frey, Serre and Ribet redefined the outlook on the problem by connecting it to a mainstream conjecture in the theory of modular forms (the "Shimura-Taniyama-Weil Conjecture"), in the case of RH we cannot point to any such dramatic advance. However, that doesn't mean that things have stood still.

For definiteness we recall the statement of RH. For $\Re(s) > 1$ the zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}, \tag{1}$$

the product being over the prime numbers. Riemann showed how to continue zeta analytically in s and he established the Functional Equation:

$$\Lambda(s) := \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s) = \Lambda(1-s)\,, \tag{2}$$

Γ being the Gamma function. RH is the assertion that all the zeros of $\Lambda(s)$ are on the line of symmetry for the functional equation, that is on $\Re(s) = \frac{1}{2}$. Elegant, crisp, falsifiable and far-reaching this conjecture is the epitome of what a good conjecture should be. Moreover, its generalizations to other zeta functions (see below) have many striking consequences making the conjecture even more important. Add to this the fact that the problem has resisted the efforts of many of the finest mathematicians and that it now carries a financial reward, it is not surprising that it has attracted so much attention.

We begin by describing the generalizations of RH which lead to the Grand Riemann Hypothesis. After that we discuss various consequences and developments. The first extension is to the zeta functions of Dirichlet, which he introduced in order to study primes in arithmetic progressions and which go by the name L-functions. Let χ be a primitive Dirichlet character of modulus q (i.e. $\chi(mn) = \chi(m)\,\chi(n)$, $\chi(1) = 1$, $\chi(m + bq) = \chi(m)$) then L is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}\,. \tag{3}$$

As with zeta, $L(s,\chi)$ extends to an entire function and satisfies the functional equation:

$$\Lambda(s,\chi) := \pi^{-(s+a_\chi)/2}\,\Gamma\left(\frac{s+a_\chi}{2}\right) L(s,\chi) \;=\; \epsilon_\chi q^{\frac{1}{2}-s}\,\Lambda(1-s,\bar{\chi})\,, \tag{4}$$

where $a_\chi = (1+\chi(-1))/2$ and ϵ_χ is the sign of a Gauss sum (see [Da]). These $L(s,\chi)$'s give all the degree one L-functions (i.e., in the product over primes each local factor at p is the inverse of a polynomial of degree one in p^{-s}). The general L-function of degree m comes from an automorphic form on the general linear group of m by m invertible matrices, GL_m. As an explicit example of a degree two L-function, consider the discriminant

$$\triangle(z) \;=\; e^{2\pi iz}\prod_{n=1}^{\infty}\left(1-e^{2\pi inz}\right)^{24} := \sum_{n=1}^{\infty}\tau(n)\,e^{2\pi inz}\,, \tag{5}$$

for z in the upper half plane. $\triangle(z)$ is a holomorphic cusp form of weight 12 for the modular group. That is

$$\triangle\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^{12}\,\triangle(z)$$

for $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$. Associated to $\triangle$ is its degree two L-function,

$$L(s,\triangle) \;=\; \sum_{n=1}^{\infty}\frac{\tau(n)}{n^{11/2}}\,n^{-s} \;=\; \prod_{p}\left(1-\frac{\tau(p)}{p^{11/2}}\,p^{-s} \,+\, p^{-2s}\right)^{-1}\,. \tag{6}$$

$L(s,\triangle)$ is entire and satisfies the functional equation

$$\Lambda(s,\triangle) := \Gamma_{\mathbb{R}}\left(s+\frac{11}{2}\right)\Gamma_{\mathbb{R}}\left(s+\frac{13}{2}\right) L(s,\triangle) \;=\; \Lambda(1-s,\triangle)\,,$$

where

$$\Gamma_{\mathbb{R}}(s) \;=\; \pi^{-s/2}\,\Gamma(s/2)\,. \tag{7}$$

The definition in the general case requires more sophisticated tools, which can be found in [We1] for example. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and let π be an automorphic cuspidal representation of $GL_m(\mathbb{A})$ with central character χ (see [J]). The representation π is equivalent to $\underset{v}{\otimes}\pi_v$ with $v=\infty(\mathbb{Q}_\infty=\mathbb{R})$ or $v=p$ and π_v an irreducible unitary representation of $GL_m(\mathbb{Q}_v)$. Corresponding to each prime p and local representation π_p one forms the local factor $L(s,\pi_p)$ which takes the form:

$$L(s,\pi_p) \;=\; \Pi_{j=1}^{m}\,(1-\alpha_{j,\pi}(p)p^{-s})^{-1} \tag{8}$$

for m complex parameters $\alpha_{j,\pi}(p)$ determined by π_p [T]. Similarly for $v=\infty$, π_∞ determines parameters $\mu_{j,\pi}(\infty)$ such that

$$L(s,\pi_\infty) \;=\; \Pi_{j=1}^{m}\,\Gamma_{\mathbb{R}}(s-\mu_{j,\pi}(\infty))\,. \tag{9}$$

The standard L-function associated to π is defined using this data by

$$L(s,\pi) \,=\, \prod_p L(s,\pi_p)\,. \tag{10}$$

The completed L-function is given as before by

$$\Lambda(s,\pi) \,=\, L(s,\pi_\infty)\, L(s,\pi)\,. \tag{11}$$

From the automorphy of π one can show that $\Lambda(s,\pi)$ is entire and satisfies a Functional Equation (see [J])

$$\Lambda(s,\pi) \,=\, \epsilon_\pi\, N_\pi^{\frac{1}{2}-s}\,\Lambda(1-s,\tilde{\pi}), \tag{12}$$

where $N_\pi \geq 1$ is an integer and is called the conductor of π, ϵ_π is of modulus 1 and is computable in terms of Gauss sums and $\tilde{\pi}$ is the contragredient representation $\tilde{\pi}(g) = \pi({}^t g^{-1})$.

General conjectures of Langlands assert these standard L-functions multiplicatively generate all L-functions (in particular Dedekind Zeta Functions, Artin L-functions, Hasse-Weil Zeta Functions, ...). So at least conjecturally one is reduced to the study of these. We can now state the general hypothesis (GRH).

GRAND RIEMANN HYPOTHESIS

Let π be as above then the zeros of $\Lambda(s,\pi)$ all lie on $\Re(s) = \frac{1}{2}$.

$L(s,\pi)$ is a generating function made out of the data π_p for each prime p and GRH naturally gives very sharp information about the variation of π_p with p. Many of its applications make direct use of this. However, there are also many applications where it is the size of $L(s,\pi)$ on $\Re(s) = \frac{1}{2}$ that is the critical issue (this coming from the ever growing number of formulae which relate values of L-functions at special points such as $s = \frac{1}{2}$ to arithmetic information and to periods, see [Wa] and [Wat] for example). GRH implies uniform and sharp bounds. Define the analytic conductor C_π of π to be $N_\pi \Pi_{j=1}^m (1 + |\mu_{j,\pi}(\infty)|)$. It measures the complexity of π. In what follows we fix m. For $X \geq 1$ the number of π with $C_\pi \leq X$ is finite. It is known [Mo] that for each $\epsilon > 0$ there is a constant B_ϵ such that $|L(\sigma + it, \pi)| \leq B_\epsilon C_\pi^\epsilon$ for $\sigma \geq 1 + \epsilon$ and $t \in \mathbb{R}$. The Grand Lindelöf Hypothesis (GLH) is the assertion that such bounds continue to hold for $\sigma \geq \frac{1}{2}$. Precisely, that for $\epsilon > 0$ there is B'_ϵ such that for $t \in \mathbb{R}$

$$|L\left(\frac{1}{2} + it, \pi\right)| \,\leq\, B'_\epsilon((1+|t|)C_\pi)^\epsilon\,. \tag{13}$$

GRH implies GLH with an effective B'_ϵ for $\epsilon > 0$.

It would be interesting to compile the long list of known consequences (many of which are quite indirect) of GRH and GLH. At the top of the list would be the prime number theorem with a sharp remainder term, the connection going back to Riemann's paper. He gives a formula for the number of primes less than X from which the prime number theorem would follow if one knew some deeper information about the zeros of zeta. In fact, the eventual proof of the prime number theorem went via this route with Hadamard and de la Vallée Poussin showing that $\zeta(s) \neq 0$ for $\Re(s) \geq 1$ (this is also known for all the $L(s,\pi)$'s). Other consequences of GRH for the family of Dirichlet L-functions followed, including the surprising result of Goldbach type due to Hardy and Littlewood, who showed that every sufficiently large odd integer is a sum of three primes. Later I. Vinogradov developed some fundamental novel methods which allowed him to prove this result unconditionally. A more recent application, again of GRH for Dirichlet L-functions, which was noted in Bombieri [Bo1] is the algorithm of Miller which determines whether a large integer n is prime in $O((\log n)^4)$ steps. Two years ago, Agrawal-Kayal and Saxena [A-K-S] put forth a primality testing algorithm (like Miller's it is based on Fermat's little Theorem) which they show without recourse to any unproven hypothesis, runs in $O((\log n)^{15/2})$ steps (the important point being that it is polynomial in $\log n$). Thus in practice GRH is used as a very reliable working hypothesis, which in many cases has been removed.

We give four examples of results in the theory of primes, Diophantine equations and mathematical physics which so far have been established only under GRH.

(A) (Serre), [Se, p. 632 and 715]: Given two non-isogeneous elliptic curves E and E' over $\mathbb{Q}$, there is a prime p which is $O((\log N_E N_{E'})^2)$ for which E and E' have good reduction at p and their number of points mod p are different. Here N_E is the conductor of E.

(B) (Artin's primitive root conjecture): If $b \neq \pm 1$ or a perfect square then b is a primitive root for infinitely many primes p [Ho].

(C) Ramanujan points out that congruence tests prevent $x^2 + y^2 + 10z^2$ representing any positive integer of the form $4^\lambda(16\mu + 6)$. He asks which numbers not of this form are not represented and lists 16 such. There are in fact exactly 18 such exceptions [O-S], 3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719. (The story for a general ternary quadratic form is similar but a little more complicated)

(D) The problem of the rate of equidistribution in the semi-classical limit of quantum eigenstates for a Hamiltonian which is classically chaotic, is a central one in "quantum chaos." The only Hamiltonian for which progress has been achieved is that of geodesic motion on an arithmetic hyperbolic surface where the equidistribution and its exact rate are known assuming GLH [Wat].

Towards problems (B), (C) and (D) slightly weaker but impressive unconditional results are known. In [HB] following [G-M] it is shown that there are at most three squarefree positive b's for which (B) is not true. Following the advance in [I], a general result about ternary quadratic forms is proven in [D-SP] from which it follows that the list of exceptions in (C) is finite. The method does not give an effective bound for the largest possible exception. As for (D) for arithmetic hyperbolic surfaces, the equidistribution problem is all but settled in [Li] using ergodic theoretic methods.

The passage to such results is often based on approximations to GRH and GLH that have been proven. The basic approximations to GRH that are known are zero free regions (the best ones for zeta being based on I. Vinogradov's method (see [I-K] for example)), and zero density theorems which give upper bounds for the total number of zeros that a family of L-functions can have in a box $\sigma_0 \leq \sigma < 1$ and $|t| \leq T$, where $\sigma_0 \geq \frac{1}{2}$ and $T > 0$. The best known approximation of GRH for the L-functions $L(s,\chi)$ is the Bombieri-A.Vinogradov Theorem [Bo2], [V]. It gives sharp bounds for the average equidistribution of primes in progressions and it has often served as a complete substitute for GRH in applications. The approximations to GLH which are decisive in various applications (including (C) above) are the so-called subconvex estimates (see [I-S]) which are known in some generality. Neither the zero density theorems nor the subconvex estimates are known for general families of automorphic L-functions and establishing them would be an important achievement.

As pointed out in [Bo1] there is an analogue of RH for curves and more generally varieties defined over finite fields. These analogues in their full generality are known as the Weil Conjectures. They have had a major impact and they have been proven! There are at least two fundamental steps in the proof. The first is the linearization of the problem [A-G-V], that is the realization of the zeros of the corresponding zeta functions in terms of eigenvalues of a linear transformation (specifically the eigenvalues of Frobenius acting on an associated cohomology group). This is then used together with a second step [De] which involves deforming the given variety in a family. The family has a symmetry group (the monodromy group) which is used together with its high tensor power representations and a positivity argument to prove the Weil Conjectures for each member of the family at once (see [K] for an overview). Like GRH, the Weil Conjectures have found far-reaching and often unexpected applications.

The success of this analogue is for many a guide as to what to expect in the case of GRH itself. Apparently both Hilbert and Polya (see the letter from Polya to Odlyzko [Od1]) suggested that there might be a spectral interpretation of the zeros of the zeta function in which the corresponding operator (after a change of variable) is self-adjoint, hence rendering the zeros on the line $\sigma = \frac{1}{2}$. While the evidence for the spectral nature of the zeros of any $\Lambda(s,\pi)$ has grown dramatically in recent years, I don't believe that the self-adjointness idea is very likely. It is not the source of the proof of the Weil Conjectures. In fact, it is the deformation in families idea that lies behind much of the progress in the known approximations to GRH and GLH and which seems more promising.

The spectral nature of the zeros emerges clearly when studying the local statistical fluctuations of the high zeros of a given $\Lambda(s,\pi)$ or the low-lying zeros of a given family of L-functions. These statistical distributions are apparently dictated by random matrix ensembles [K-S]. For a given $\Lambda(s,\pi)$ the local fluctuations of high zeros are universal and follow the laws of fluctuations of the eigenvalue distribution for matrices in the Gaussian Unitary Ensemble (GUE). We call this the "Montgomery-Odlyzko Law." For low-lying zeros in a family there is a symmetry type that one can associate with the family from which the densities and fluctuations can be predicted. The analogues of these fluctuation questions are understood in the setting of varieties over finite fields [K-S], and this in fact motivated the developments for the $\Lambda(s,\pi)$'s. Numerical experimentations starting with [Od2] have given further striking confirmation of this statistical fluctuation phenomenon. Based on this and a more sophisticated analysis, Keating and Snaith have put forth very elegant conjectures for the asymptotics of the moments of $L(\frac{1}{2}+it,\pi)$ for $|t| \leq T$ a $T \to \infty$, as well as for $L(\frac{1}{2},\pi)$ as π varies over a family (see [C-F-K-S-R] and also [D-G-H], the latter being an approach based on multiple Dirichlet series). The numerical and theoretical confirmation of these conjectures is again very striking.

The above gives ample evidence for there being a spectral interpretation of the zeros of an L-function and indeed such interpretations have been given. One which has proven to be important is via Eisenstein series [La]. These are automorphic forms associated with boundaries of arithmetic quotients of semi-simple groups. These series have poles at the zeros of L-functions (hence the zeros can in this way be thought of as resonances for a spectral problem). Combining this with a positivity argument using the inner product formula for Eisenstein series ("Maass Selberg Formula") yields effective zero free regions in $\Re(s) \leq 1$ for all L-functions whose analytic continuation and functional equations are known (see [G-L-S]). As mentioned earlier, general conjectures of Langlands assert that the L-functions in question should be products of standard L-functions but this is far from being established. This spectral method for non-vanishing applies to L-functions for which the generalizations of the method of de la Vallée Poussin do not work and so at least for now, this method should be considered as the most powerful one towards GRH.

Connes [Co] defines an action of the idele class group $\mathbb{A}^*/\mathbb{Q}^*$ on a singular space of adeles (the action is basically multiplication over addition). In order to make sense of the space he uses various Sobolev spaces and shows that the decomposition of this abelian group action corresponds to the zeros of all the $\Lambda(s, \chi)$'s which lie on $\Re(s) = \frac{1}{2}$. This artificial use of function spaces in order to pick up only the eigenvalues of the action that lie on a line, makes it difficult to give a spectral interpretation. In a recent paper Meyer [Me] fixes this problem by using a much larger space of functions. He shows that the decomposition of the action on his space has eigenvalues corresponding to all the zeros of all the $\Lambda(s, \chi)$'s whether they lie on $\Re(s) = \frac{1}{2}$ or not. In this way he gives a spectral interpretation of the zeros of Dirichlet's L-functions. Taking traces of the action he derives (following Connes) the explicit formula of Weil-Guinand-Riemann [We2]. This derivation is quite different from the usual complex analytic one which is done by taking logarithmic derivatives, shifting contours and using the functional equation. Whether this spectral interpretation can be used to prove anything new about L-functions remains to be seen. The explicit formula itself has been very useful in the study of the distribution of the zeros of L-functions. In [Bo3] an in-depth investigation of the related Weil quadratic functional is undertaken.

There have been many equivalences of RH that have been found over the years. Without their having led to any new information about the zeta function and without being able to look into the future, it is difficult to know if such equivalences constitute any real progress. In an evolving paper, de Branges constructs a Hilbert space of entire functions from an L-function. He relates positivity properties of associated kernel functions to the zeros of the L-function. Whether this approach can be used to give any information about the zeros of the L-function is unclear. In any case, it should be noted that the positivity condition that he would like to verify in his recent attempts, is false since it implies statements about the zeros of the zeta function which are demonstrably false [C-L].

We end with some philosophical comments. We have highlighted GRH as the central problem. One should perhaps entertain the possibility that RH is true but GRH is false or even that RH is false. In fact Odlyzko, who has computed the 10^{23}-rd zero of zeta and billions of its neighbors, notes that the fact that the first 10^{13} zeros are on the line $\Re(s) = \frac{1}{2}$ (Gourdon 2004) should not by itself be taken as very convincing evidence for RH. One reason to be cautious is the following. The function $S(t)$ which measures the deviation of the number of zeros of height at most t from the expected number of such zeros, satisfies $|S(t)| < 1$ for $t < 280$ (and hence immediately all the zeros in this range are on the half line), while the largest observed value of $S(t)$ in the range that it has been computed, is about 3.2. On the other hand it is known that the mean-square of $S(t)$ is asymptotically $\frac{1}{2\pi^2}\log\log t$ ([Sel]), and hence $S(t)$ gets large but does so very slowly. One can argue that unless $S(t)$ is reasonably large (say 100), one has not as yet seen the true state of affairs as far as the behavior of the zeros of zeta. Needless to say, the height t for which $S(t)$ is of this size is far beyond the computational capabilities of present day machines. Returning to the expected scenario that RH is true, it is quite possible (even likely) that the first proof of RH that is given will not generalize to the $\Lambda(s,\pi)$ cases. Most people believe that any such proof for zeta would at least apply to the family of Dirichlet L-functions.* In fact it would be quite disappointing if it didn't since the juicy applications really start with this family. The millennium prize correctly focuses on the original basic case of zeta, however a closer reading of the fine-print of the rules shows them to be more guarded in the case that a disproof is given. As far as the issue of what might or might not be true, some may feel that GRH is true for the $\Lambda(s,\pi)$'s for which π is arithmetic in nature (in particular that the coefficients in the local factor $L(s,\pi_p)$ should be algebraic) but that for the more transcendental π's such as general Maass forms, that it may fail. I am an optimist and don't contemplate such a world.

To conclude, we point the reader to the interesting recent article by Conrey on the Riemann Hypothesis [Conr] as well as the comprehensive treatment by Iwaniec and Kowalski [I-K] of the modern tools, which are used in the study of many of the topics mentioned above.

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