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A NOTE ON A CONJECTURE OF GONEK

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Abstract: We derive a lower bound for a second moment of the reciprocal of the derivative of the Riemann zeta-function over the zeros of $\zeta(s)$ that is half the size of the conjectured value. Our result is conditional upon the assumption of the Riemann Hypothesis and the conjecture that the zeros of the zeta-function are simple.

Keywords: Riemann zeta-function, mean-value thereoms, Gonek's conjecture.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. Using a heuristic method similar to Montgomery's study [13] of the pair-correlation of the imaginary parts of the non-trivial zeros of $\zeta(s)$, Gonek has made the following conjecture [7, 8].

Conjecture. *Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple. Then, as $T \to \infty$,*

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3}{\pi^3} T \tag{1.1}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

The assumption of the simplicity of the zeros of the zeta-function in the above conjecture is so that the sum over zeros on the right-hand side of (1.1) is well defined. While the details of Gonek's method have never been published, he announced his conjecture in [5]. More recently, a different heuristic method of Hughes, Keating, and O'Connell [10] based upon modeling the Riemann zeta-function and its derivative using the characteristic polynomials of random matrices has led to the same conjecture. Through the work of Ingham [11], Titchmarsh (Chapter 14 of [21]), Odlyzko and te Riele [17], Gonek (unpublished), and Ng [15],

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it is known that the behavior of this and related sums are intimately connected to the distribution of the summatory function

$$M(x) = \sum_{n \leqslant x} \mu(n)$$

where $\mu(\cdot)$, the Möbius function, is defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is divisible by k distinct primes, and $\mu(n) = 0$ if $n > 1$ is not square-free. See also [9] and [20] for connections between similar sums and other arithmetic problems.

In support of his conjecture, Gonek [5] has shown, assuming the Riemann Hypothesis and the simplicity of the zeros of $\zeta(s)$, that

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^2} \geqslant CT \tag{1.2}$$

for some constant $C > 0$ and T sufficiently large. In this note, we show that the inequality in (1.2) holds for any constant $C < \frac{3}{2\pi^3}$.

Theorem. *Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple. Then, for any fixed $\varepsilon > 0$,*

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^2} \geqslant \left(\frac{3}{2\pi^3} - \varepsilon\right) T \tag{1.3}$$

for T sufficiently large.

While our result differs from the conjectural lower bound by a factor of 2, any improvements in the strength of this lower bound have, thus far, eluded us. It would be interesting to investigate whether for $k > 0$ there is a constant $C_k > 0$ such that

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^{2k}} \geqslant C_k T (\log T)^{(k-1)^2} \tag{1.4}$$

for T sufficiently large. However, a lower bound of this form is probably not of the correct order of magnitude for all k. This is because it is expected that for each $\varepsilon > 0$ there are infinitely many zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ satisfying $|\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon}$. If such a sequence of zeros were to exist, it would then follow that

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^{2k}} = \Omega\left(T^{2k/3-\varepsilon}\right)$$

and the lower bound in (1.4) would be significantly weaker than this Ω-result when $k > \frac{3}{2}$.

2. Proof of Theorem

The method we use to prove our theorem is based on a recent idea of Rudnick and Soundararajan [18]. Let

$$\xi = T^{\vartheta} \tag{2.1}$$

where $0 < \vartheta < 1$ is fixed and define the Dirichlet polynomial

$$\mathcal{M}_\xi(s) = \sum_{n \leqslant \xi} \mu(n) n^{-s}$$

where μ is the Möbius function. Assuming the Riemann Hypothesis, for any non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$, we see that $\overline{\mathcal{M}_\xi(\rho)} = \mathcal{M}_\xi(1-\rho)$. From this observation and Cauchy's inequality it follows that

$$\sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^2} \geqslant \frac{|M_1|^2}{M_2} \tag{2.2}$$

where

$$M_1 = \sum_{0<\gamma\leqslant T} \frac{1}{\zeta'(\rho)} \mathcal{M}_\xi(1-\rho) \qquad \text{and} \qquad M_2 = \sum_{0<\gamma\leqslant T} \left|\mathcal{M}_\xi(\rho)\right|^2.$$

Our Theorem is a consequence of the following proposition.

Proposition. *Assume the Riemann Hypothesis and let $0 < \vartheta < 1$ be fixed. Then*

$$M_2 = \frac{3}{\pi^3} \left(\vartheta + \vartheta^2\right) T \log^2 T + O(T \log T). \tag{2.3}$$

If we further assume that the zeros of $\zeta(s)$ are all simple, then there exists a sequence $\mathcal{T} := \{\tau_n\}_{n=3}^{\infty}$ such that $n < \tau_n \leqslant n+1$ and for $T \in \mathcal{T}$ we have

$$M_1 = \frac{3\vartheta}{\pi^3} T \log T + O(T). \tag{2.4}$$

We now deduce our theorem from the above proposition.

Proof of Theorem. Let $T \geqslant 4$ and choose τ_n to satisfy $T - 1 \leqslant \tau_n < T$. Combining (2.2), (2.4), and (2.3) we see that

$$\begin{aligned} \sum_{0<\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^2} \geqslant \sum_{0<\gamma\leqslant \tau_n} \frac{1}{|\zeta'(\rho)|^2} &\geqslant \frac{\vartheta^2}{(\vartheta + \vartheta^2)} \frac{3}{\pi^3} \tau_n + o(\tau_n) \\ &\geqslant \frac{1}{(1 + \vartheta^{-1})} \frac{3}{\pi^3} T + o(T) \end{aligned} \tag{2.5}$$

under the assumption of the Riemann Hypothesis and the simplicity of the zeros of $\zeta(s)$. From (2.5), our theorem follows by letting $\vartheta \to 1^-$. ■

We could have just as easily estimated the sums M_1 and M_2 using a Dirichlet polynomial $\sum_{n\leqslant\xi} a_n n^{-s}$ for a large class of coefficients a_n in place of $\mathcal{M}_\xi(s)$. In the special case where

$$a_n = \mu(n) P\left(\tfrac{\log \xi/n}{\log \xi}\right)$$

for polynomials P, we can show that the choice $P = 1$ is optimal in the sense that it leads to largest lower bound in (1.3).

We prove the above proposition in the next two sections; the sum M_1 is estimated in section 3 and the sum M_2 is estimated in section 4. The evaluation of sums like M_1 dates back to Ingham's [11] important work on $M(x)$ in which he considered sums of the form

$$\sum_{0<\gamma<T} (T-\gamma)^k \zeta'(\rho)^{-1}$$

for $k \in \mathbb{R}$. The sum M_2 is of the form

$$\sum_{0<\gamma<T} |A(\rho)|^2 \qquad \text{where} \quad A(s) = \sum_{n \leqslant \xi} a_n n^{-s} \tag{2.6}$$

is a Dirichlet polynomial with $\xi \leqslant T$. Such sums have played an important role in various applications. For instance, results concerning the distribution of consecutive zeros of $\zeta(s)$ and discrete mean values of the zeta-function and its derivatives are proven in [1, 2, 3, 6, 12, 13, 16, 19]. In each of these articles, the evaluation of the discrete mean (2.6) either makes use of the Guinand-Weil explicit formula or of Gonek's uniform version [6] of Landau's formula

$$\sum_{\substack{0<\gamma<T \\ \zeta(\beta+i\gamma)=0}} x^{\beta+i\gamma} = -\frac{T}{2\pi}\Lambda(x) + E(x,T) \tag{2.7}$$

for $x, T > 1$ where $E(x,T)$ is an explicit error function uniform in x and T. A novel aspect of our approach is that it does not require the use of the Guinand-Weil explicit formula or of the Landau-Gonek explicit formula (2.7). Instead we evaluate M_2 using the residue theorem and a version of Montgomery and Vaughan's mean value theorem for Dirichlet polynomials [14]. Our approach is simpler and it is likely that it can be extended to evaluate the discrete mean (2.6) for a large class of coefficients a_n when $\xi \leqslant T$.

3. The estimation of M_1

To estimate M_1, we require the following version of Montgomery and Vaughan's mean value theorem for Dirichlet polynomials.

Lemma. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. For any real number $T > 0$, we have*

$$\begin{aligned} &\int_0^T \left(\sum_{n=1}^{\infty} a_n n^{-it} \right) \left(\sum_{n=1}^{\infty} b_n n^{it} \right) dt \\ &\qquad = T \sum_{n=1}^{\infty} a_n b_n + O\left(\Big(\sum_{n=1}^{\infty} n |a_n|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} n |b_n|^2 \Big)^{\frac{1}{2}} \right). \end{aligned} \tag{3.1}$$

Proof. This is Lemma 1 of Tsang [22]. The special case where $b_n = \overline{a_n}$ is originally due to Montgomery and Vaughan [14]. It turns out, as shown by Tsang, that this special case is equivalent to the more general case stated in the lemma. ■

Let $T \geqslant 4$ and set $c = 1 + (\log T)^{-1}$. It is well known (see Theorem 14.16 of Titchmarsh [21]) that assuming the Riemann Hypothesis there exists a sequence $\mathcal{T} = \{\tau_n\}_{n=3}^{\infty}$, $n < \tau_n \leqslant n+1$, and a fixed constant $A > 0$ such that

$$\left|\zeta(\sigma+i\tau_n)\right|^{-1} \ll \exp\left(\frac{A\log\tau_n}{\log\log\tau_n}\right) \tag{3.2}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$. We now prove the estimate (2.4) assuming that $T \in \mathcal{T}$. Recall that $|\gamma| > 1$ for every non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Thus, assuming that all the zeros of $\zeta(s)$ are simple, the residue theorem implies that

$$\begin{aligned} M_1 &= \frac{1}{2\pi i}\left(\int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+1} + \int_{1-c+i}^{c+i}\right) \frac{1}{\zeta(s)}\mathcal{M}_\xi(1-s)\,ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

say. Here we are using the fact that the residue of the function $1/\zeta(s)$ at $s = \rho$ equals $1/\zeta'(\rho)$ if ρ is a simple zero of $\zeta(s)$.

The main contribution to M_1 comes from the integral I_1; the remainder of the integrals contribute an error term. Observe that

$$I_1 = \frac{1}{2\pi}\int_1^T \sum_{m=1}^{\infty}\frac{\mu(m)}{m^{c+it}}\sum_{n\leqslant\xi}\frac{\mu(n)}{n^{1-c-it}}\,dt.$$

By (3.1) with $a_m = \mu(m)m^{-c}$ and $b_n = \mu(n)n^{-1+c}$ it follows that

$$I_1 = \frac{(T-1)}{2\pi}\sum_{n\leqslant\xi}\frac{\mu(n)^2}{n} + O\left(\left(\sum_{n=1}^{\infty}\frac{\mu(n)^2}{n^{2c-1}}\right)^{\frac{1}{2}}\left(\sum_{n\leqslant\xi}\mu(n)^2n^{2c-1}\right)^{\frac{1}{2}}\right).$$

Since

$$\sum_{n\leqslant\xi}\frac{\mu(n)^2}{n} = \frac{6}{\pi^2}\log\xi + O(1), \tag{3.3}$$

we conclude that

$$I_1 = \frac{3}{\pi^3}T\log\xi + O\Big(\xi\sqrt{\log T} + T\Big)$$

for our choice of c. Here we have used the fact that

$$\sum_{n=1}^{\infty}\frac{\mu(n)^2}{n^{2c-1}} \leqslant \zeta(2c-1) \ll \log T.$$

To estimate the contribution from the integral I_2, we recall the functional equation for the Riemann zeta-function which says that

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{3.4}$$

where

$$\chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

Stirling's asymptotic formula for the Gamma-function can be used to show that

$$\left|\chi(\sigma+it)\right| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} \left(1 + O\left(|t|^{-1}\right)\right) \tag{3.5}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$. Combining this estimate and (3.2), it follows that, for $T \in \mathcal{T}$,

$$\left|\zeta(\sigma + iT)\right|^{-1} \ll T^{\min(\sigma-1/2),0)} \exp\left(\frac{A \log T}{\log\log T}\right)$$

uniformly for $-1 \leqslant \sigma \leqslant 2$. In addition, we have the trivial bound

$$|M_\xi(\sigma + it)| \ll \xi^{1-\sigma}. \tag{3.6}$$

Thus, estimating the integral I_2 trivially, we find that

$$I_2 \ll \exp\left(\frac{A \log T}{\log\log T}\right) \int_{1-c}^{c} T^{\min(\sigma-1/2),0)} \xi^\sigma \, d\sigma \ll \xi \exp\left(\frac{A \log T}{\log\log T}\right).$$

To bound the contribution from the integral I_3, we notice that the functional equation for $\zeta(s)$ combined with the estimate in (3.5) implies that, for $1 \leqslant |t| \leqslant T$,

$$\left|\zeta(1-c+it)\right|^{-1} \ll |t|^{1/2-c} \left|\zeta(c-it)\right|^{-1} \ll |t|^{1/2-c} \zeta(c) \ll |t|^{-1/2} \log T.$$

It therefore follows that

$$I_3 \ll \log T \Big(\sum_{n \leqslant \xi} \frac{|\mu(n)|}{n^c} \Big) \int_1^T t^{-1/2} \, dt \ll \sqrt{T} (\log T) \log \xi.$$

Finally, since $1/\zeta(s)$ and $\mathcal{M}_\xi(1-s)$ are bounded on the interval $[1-c+i, c+i]$, we find that $I_4 \ll 1$. Hence, our combined estimates for I_1, I_2, I_3, and I_4 imply that

$$M_1 = \frac{3}{\pi^3} T \log \xi + O\Big(\xi \exp\left(\frac{A \log T}{\log\log T}\right) + T\Big).$$

From this and (2.1), the estimate in (2.4) follows.

4. The estimation of M_2

We now turn our attention to estimating the sum M_2. As before, let $T \geqslant 4$ and $c = 1 + (\log T)^{-1}$. Assuming the Riemann Hypothesis, we notice that

$$M_2 = \sum_{0 < \gamma \leqslant T} \mathcal{M}_\xi(\rho) \mathcal{M}_\xi(1-\rho).$$

Therefore, by the residue theorem, we see that

$$\begin{aligned} M_2 &= \frac{1}{2\pi i}\left(\int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+1} + \int_{1-c+i}^{c+i}\right) M_\xi(s)M_\xi(1-s)\frac{\zeta'}{\zeta}(s)\,ds \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned}$$

say. In order to evaluate the integrals over the horizontal part of the contour we shall impose some extra conditions on T. Without loss of generality, we may assume that T satisfies

$$\begin{aligned} |\gamma - T| &\gg \frac{1}{\log T} \qquad \text{for all ordinates } \gamma \text{ and} \\ \frac{\zeta'}{\zeta}(\sigma+iT) &\ll (\log T)^2 \qquad \text{uniformly for all } 1-c \leqslant \sigma \leqslant c. \end{aligned} \tag{4.1}$$

In each interval of length one such a T exists. This well known argument may be found in [4], page 108. Applying (3.6) we find that

$$\sum_{T<\gamma<T+1} |M_\xi(\rho)M_\xi(1-\rho)| \ll \xi(\log T).$$

Here we have used the standard estimate that there are $O(\log T)$ zeros of $\zeta(s)$ with ordinates in the interval $[T, T+1]$. Therefore our choice of T determines M_2 up to an error term $O(\xi \log T)$. First we estimate the horizontal portions of the contour. By (3.6) and (4.1), we have

$$J_2 = \frac{1}{2\pi}\int_c^{1-c} M_\xi(\sigma+it)M_\xi(1-\sigma-it)\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma \ll \xi(\log T)^2.$$

Similarly, it may be shown that $J_4 \ll \xi$. Next we relate J_3 to J_1. We have

$$\begin{aligned} J_3 &= \frac{1}{2\pi}\int_T^1 M_\xi(1-c+it)M_\xi(c-it)\frac{\zeta'}{\zeta}(1-c+it)\,dt \\ &= -\frac{1}{2\pi}\overline{\int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\zeta'}{\zeta}(1-c-it)\,dt} \end{aligned}$$

By differentiating (3.4), the functional equation, we find that

$$-\frac{\zeta'}{\zeta}(1-c-it) = -\frac{\chi'}{\chi}(1-c-it) + \frac{\zeta'}{\zeta}(c+it)$$

and hence that

$$\begin{aligned} J_3 = &-\frac{1}{2\pi}\overline{\int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\chi'}{\chi}(1-c-it)\,dt} \\ &+ \frac{1}{2\pi}\overline{\int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\zeta'}{\zeta}(c+it)\,dt}. \end{aligned}$$

By (3.4) and Stirling's formula it can be shown that

$$-\frac{\chi'}{\chi}(1-c-it) = \log\Big(\frac{|t|}{2\pi}\Big)(1+O(|t|^{-1}))$$

uniformly for $1 \leqslant |t| \leqslant T$. By (3.6), the term $O(|t|^{-1})$ contributes to J_3 an amount which is $O(\xi \log T)$ and, hence, it follows that

$$J_3 = K + \overline{J_1} + O\big(\xi(\log T)\big)$$

where

$$K = \int_1^T \log\Big(\frac{t}{2\pi}\Big) M_\xi(c+it)M_\xi(1-c-it)\,dt.$$

Collecting estimates, we deduce that

$$M_2 = K + 2\Re J_1 + O\big(\xi(\log T)^2\big). \tag{4.2}$$

To complete our estimation of M_2, it remains to evaluate K and then J_1. Integrating by parts, it follows that

$$\begin{aligned} K &= \frac{1}{2\pi}\log\Big(\frac{T}{2\pi}\Big)\int_1^T M_\xi(c+it)M_\xi(1-c-it)\,dt \\ &\quad - \frac{1}{2\pi}\int_1^T \Big(\int_1^t M_\xi(c+iu)M_\xi(1-c-iu)du\Big)\frac{dt}{t}. \end{aligned}$$

By (3.1), we have

$$\begin{aligned} \int_1^t M_\xi(c+iu)M_\xi(1-c-iu)\,du &= (t-1)\sum_{n\leqslant\xi}\frac{\mu(n)^2}{n} + O(\xi\sqrt{\log T}) \\ &= \frac{6}{\pi^2}t\log\xi + O(\xi\sqrt{\log T}+t) \end{aligned}$$

for $t > 1$. Substituting this estimate into the above expression for K, we see that

$$\begin{aligned} K &= \frac{3}{\pi^3}T\log\Big(\frac{T}{2\pi}\Big)\log\xi + O(T\log T) + O(T\log\xi) \\ &= \frac{3}{\pi^3}T\log\Big(\frac{T}{2\pi}\Big)\log\xi + O(T\log T). \end{aligned} \tag{4.3}$$

We finish by evaluating the integral J_1 which is similar to the evaluation of the integral I_1 in the previous section. By another application of (3.1), we find that

$$\begin{aligned} J_1 = -\frac{1}{2\pi}\int_1^T \sum_{n=1}^\infty \frac{\alpha_n}{n^{c+it}}\sum_{n\leqslant\xi}\frac{\mu(n)}{n^{1-c-it}}\,dt &= -\frac{(T-1)}{2\pi}\sum_{n\leqslant x}\frac{\alpha_n\mu(n)}{n} \\ &\quad + O\bigg(\Big(\sum_{n=1}^\infty\frac{\alpha_n^2}{n^{2c-1}}\Big)^{\frac12}\Big(\sum_{n\leqslant\xi}\frac{\mu(n)^2}{n^{1-2c}}\Big)^{\frac12}\bigg) \end{aligned}$$

where the coefficients α_n are defined by

$$\alpha_n = \sum_{\substack{k\ell=n \\ \ell\leqslant\xi}} \Lambda(k)\mu(\ell).$$

Observe that trivially $|\alpha_n| \leqslant \sum_{u|n} \Lambda(u) \leqslant \log n$. It follows that the error term in the above expression for J_1 is $\ll \zeta''(2c-1)^{\frac{1}{2}}\xi \ll \xi(\log T)^{\frac{3}{2}}$. Finally, we note that

$$\begin{aligned}
\sum_{n\leqslant x} \frac{\alpha_n\mu(n)}{n} &= \sum_{\ell\leqslant x} \frac{\mu(\ell)}{\ell} \sum_{k\leqslant \frac{x}{\ell}} \frac{\Lambda(k)\mu(k\ell)}{k} = \sum_{\ell\leqslant\xi} \frac{\mu(\ell)}{\ell} \sum_{\substack{p^j\leqslant\xi/\ell \\ p \text{ prime},\ j\geqslant 0}} \frac{\mu(p^j\ell)\log p}{p^j} \\
&= \sum_{\ell\leqslant\xi} \frac{\mu(\ell)}{\ell} \sum_{p\leqslant\xi/\ell} \frac{\mu(p\ell)\log p}{p} + O(\log\xi) \\
&= -\sum_{\ell\leqslant\xi} \frac{\mu(\ell)^2}{\ell} \sum_{p\leqslant\xi/\ell} \frac{\log p}{p} + O\Big(\log\xi + \sum_{\ell\leqslant\xi} \frac{1}{\ell} \sum_{p|\ell} \frac{\log p}{p}\Big)
\end{aligned}$$

since $\mu(p\ell) = -\mu(\ell)$ if $(p,\ell) = 1$ and $\mu(p\ell) = 0 = O(1)$ if $p|\ell$. The sum in the error term is

$$\sum_{\ell\leqslant\xi} \frac{1}{\ell} \sum_{p|\ell} \frac{\log p}{p} = \sum_{p\leqslant x} \frac{(\log p)}{p^2} \sum_{\ell'\leqslant\frac{\xi}{p}} \frac{1}{\ell'} \ll \log\xi.$$

Hence, by the elementary result $\sum_{p\leqslant\xi} \frac{\log p}{p} = \log\xi + O(1)$, (3.3), and partial summation, we deduce that

$$\sum_{n\leqslant x} \frac{\alpha_n\mu(n)}{n} = -\sum_{l\leqslant\xi} \frac{\mu(l)^2\log(\frac{\xi}{l})}{l} + O(\log\xi) = -\frac{3}{\pi^2}(\log\xi)^2 + O(\log\xi).$$

Therefore, combining formulae, we have

$$J_1 = -\frac{3}{2\pi^3} T(\log\xi)^2 + O(T\log T). \tag{4.4}$$

Finally (4.2), (4.3), and (4.4) imply that

$$M_2 = \frac{3}{\pi^3} T\log T\log\xi + \frac{3}{\pi^3} T(\log\xi)^2 + O(T\log T)$$

and, thus, by (2.1) we deduce (2.3).

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