Upper bounds for moments of $\zeta'(\rho)$

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Abstract

Assuming the Riemann hypothesis, we obtain an upper bound for the $2k$th moment of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$ for every positive integer k. Our bounds are nearly as sharp as the conjectured asymptotic formulae for these moments.

1. Introduction and statement of the main results

Let $\zeta(s)$ denote the Riemann zeta-function. This article is concerned with estimating discrete moments of the following form:

$$J_k(T) = \frac{1}{N(T)} \sum_{0 < \gamma \leqslant T} \left| \zeta'(\rho) \right|^{2k}, \tag{1.1}$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, the function

$$N(T) = \sum_{0 < \gamma \leqslant T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.2}$$

denotes the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity.

It is an open problem to determine the behavior of $J_k(T)$ as k varies. Independently, Gonek [7] and Hejhal [10] have conjectured that

$$J_k(T) \asymp (\log T)^{k(k+2)} \tag{1.3}$$

for fixed $k \in \mathbb{R}$ as $T \to \infty$. Though widely believed for positive values of k, there is evidence to suggest that this conjecture is false for $k \leqslant -3/2$.

Until recently, estimates in agreement with (1.3) were only known in a few cases. Assuming the Riemann hypothesis (which asserts that $\beta = \frac{1}{2}$ for each non-trivial zero of $\zeta(s)$), Gonek [5] has shown that $J_1(T) \sim \frac{1}{12}(\log T)^3$ and Ng [17] has proved that $J_2(T) \asymp (\log T)^8$. Confirming a conjecture of Conrey and Snaith [1, Section 7.1], the author [14] has calculated the lower-order terms in the asymptotic expression for $J_1(T)$. Under the additional assumption that the zeros of $\zeta(s)$ are simple, Gonek [7] has shown that $J_{-1}(T) \gg (\log T)^{-1}$ and conjectured [9] that $J_{-1}(T) \sim (6/\pi^2)(\log T)^{-1}$. In addition, there are a few related unconditional results where the sum in (1.1) is restricted to the simple zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$. See, for instance, [3, 4, 13, 21].

By modeling the behavior of the Riemann zeta-function and its derivative on the critical line using the characteristic polynomials of random matrices, Hughes, Keating and O'Connell [12] have refined Gonek's and Hejhal's conjecture in (1.3). In particular, they conjectured a precise constant $\mathcal{D}_k$ such that $J_k(T) \sim \mathcal{D}_k (\log T)^{k(k+2)}$ as $T \to \infty$ for fixed $k \in \mathbb{C}$ with $\Re k > -3/2$. Their conjecture is consistent with the results mentioned above.

Very little is known about the moments $J_k(T)$ when $k > 2$. However, assuming the Riemann hypothesis, one may deduce from well-known results of Littlewood ([**23**, Theorems 14.14 A-B]) that, for $\sigma \geqslant 1/2$ and $t \geqslant 10$, the estimate

$$\zeta'(\sigma + it) \ll \exp\left(\frac{C \log t}{\log \log t}\right)$$

holds for some constant $C > 0$. It immediately follows that

$$J_k(T) \ll \exp\left(\frac{2kC \log T}{\log \log T}\right)$$

for any $k \geqslant 0$. The goal of this paper is to improve this estimate by obtaining a conditional upper bound for $J_k(T)$ (when $k \in \mathbb{N}$) very near the conjectured order of magnitude. In particular, we prove the following result.

THEOREM 1.1.   *Assume the Riemann hypothesis. Let $k \in \mathbb{N}$ and let $\varepsilon > 0$ be arbitrary. Then for sufficiently large T we have*

$$\frac{1}{N(T)} \sum_{0 < \gamma \leqslant T} \left|\zeta'(\rho)\right|^{2k} \ll (\log T)^{k(k+2)+\varepsilon},$$

where the implied constant depends on k and ε.

Under the assumption of the generalized Riemann hypothesis for Dirichlet L-functions, Ng and the author [**15**] have shown that $J_k(T) \gg (\log T)^{k(k+2)}$ for each fixed $k \in \mathbb{N}$. Combining this result with Theorem 1.1 lends strong support for the conjecture in (1.3) when k is a positive integer.

Our proof of Theorem 1.1 is based upon a recent method of Soundararajan [**22**] that provides upper bounds for the frequency of large values of $|\zeta(\frac{1}{2} + it)|$. His method relies on obtaining an inequality for $\log|\zeta(\frac{1}{2} + it)|$ involving the real part of a 'short' Dirichlet polynomial which is a smoothed approximation to the Dirichlet series for $\log \zeta(s)$. Using mean-value estimates for high powers of this Dirichlet polynomial, he deduces upper bounds for the measure of the following set:

$$\{t \in [0, T] : \log|\zeta(\tfrac{1}{2} + it)| \geqslant V\},$$

and from this, for arbitrary positive values of k and ε, is able to conclude that

$$\frac{1}{T} \int_0^T \left|\zeta(\tfrac{1}{2} + it)\right|^{2k} dt \ll_{k,\varepsilon} (\log T)^{k^2+\varepsilon}. \tag{1.4}$$

Soundararajan's techniques build upon the work of Selberg [**18**–**20**] who studied the distribution of values of $\log \zeta(\frac{1}{2} + it)$ in the complex plane.

Since $\log \zeta'(s)$ does not have a Dirichlet series representation, it is not clear that the function $\log|\zeta'(\frac{1}{2}+it)|$ can be approximated by a Dirichlet polynomial.[†] For this reason, we do not study the distribution of the values of $\zeta'(\rho)$ directly, but instead examine the frequency of large values of $|\zeta(\rho + \alpha)|$, where $\alpha \in \mathbb{C}$ is a small shift away from a zero ρ of $\zeta(s)$. This requires deriving an inequality for $\log|\zeta(\sigma + it)|$ involving a short Dirichlet polynomial that holds uniformly for values of σ in a small interval to the right of, and including, $\sigma = \frac{1}{2}$. Using a result of Gonek (Lemma 4.1 below), we estimate high power moments of this Dirichlet polynomial averaged over the zeros of the zeta-function and are able to derive upper bounds for the frequency of large values of $|\zeta(\rho + \alpha)|$. Using this information we prove the following theorem.

[†]Hejhal [**10**] studied the distribution of $\log|\zeta'(\frac{1}{2} + it)|$ by a method that does not directly involve the use of Dirichlet polynomials.

THEOREM 1.2. *Assume the Riemann hypothesis. Let $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$ and $|\Re\alpha| \leqslant (\log T)^{-1}$. Let $k \in \mathbb{R}$ with $k > 0$ and let $\varepsilon > 0$ be arbitrary. Then, for sufficiently large T, the inequality*

$$\frac{1}{N(T)} \sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \ll_{k,\varepsilon} (\log T)^{k^2 + \varepsilon}$$

holds uniformly in α.

Our argument, modified in a straightforward manner, actually implies that the constant ε appearing in power of $\log T$ in the inequalities in Theorems 1.1 and 1.2 is $O(1/\log\log\log T)$. Comparing the result of Theorem 1.2 with the estimate in (1.4), we see that our theorem provides essentially the same upper bound (up to the implied constant) for discrete averages of the Riemann zeta-function near its zeros as can be obtained for continuous moments of $|\zeta(\frac{1}{2} + it)|$ by using the methods in [**22**]. There has been some previous work on discrete mean-value estimates of the zeta-function that are of a form that is similar to the sum appearing in Theorem 1.2. For instance, see the results of Fujii [**2**], Gonek [**5**], and Hughes [**11**].

We deduce Theorem 1.1 from Theorem 1.2 since, by Cauchy's integral formula, we can use bounds for $\zeta(s)$ near its zeros to recover bounds on the values for $\zeta'(\rho)$. For a precise statement of this idea, see Lemma 7.1 below. Our proof only allows us to establish Theorem 1.1 when k is a positive integer despite the fact that Theorem 1.2 holds for all $k > 0$.

2.   An inequality for $\log |\zeta(\sigma + it)|$ when $\sigma \geqslant \frac{1}{2}$

Throughout the remainder of this article, we use $s = \sigma + it$ to denote a complex variable and use p to denote a prime number. We let $\lambda_0 = 0.5671...$ be the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0$. Also, we put $\sigma_\lambda = \sigma_{\lambda,x} = 1/2 + \lambda/\log x$ and let

$$\log^+ |x| = \begin{cases} 0 & \text{if } |x| < 1, \\ \log |x| & \text{if } |x| \geqslant 1. \end{cases}$$

As usual, we denote by $\Lambda(\cdot)$ the arithmetic function defined by $\Lambda(n) = \log p$ when $n = p^k$ and $\Lambda(n) = 0$ when $n \neq p^k$. The main result of this section is the following lemma.

LEMMA 2.1.   *Assume the Riemann hypothesis. Let $\tau = |t| + 3$. Then, for $\lambda_0 \leqslant \lambda \leqslant \frac{1}{4}\log x$ and $3 \leqslant x \leqslant \tau^2$, the estimate*

$$\log^+ \left| \zeta(\sigma + it) \right| \leqslant \left| \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{\sigma_\lambda + it} \log n} \frac{\log x/n}{\log x} \right| + \frac{(1 + \lambda)}{2} \frac{\log \tau}{\log x} + O(1) \tag{2.1}$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \sigma_\lambda \leqslant \frac{3}{4}$.

In [**22**], Soundararajan proved an inequality similar to Lemma 2.1 for $\log \left| \zeta(\frac{1}{2} + it) \right|$. In his case, when $\zeta(\frac{1}{2} + it) \neq 0$, an inequality slightly stronger than (2.1) holds with the constant λ_0 replaced by $\delta_0 = 0.4912...$, where δ_0 is the unique positive real number satisfying $e^{-\delta_0} = \delta_0 + \frac{1}{2}\delta_0^2$. Our proof of the above lemma is a modification of his argument.

Proof of Lemma 2.1.   We assume that $|\zeta(\sigma + it)| \geqslant 1$, as otherwise the lemma holds for a trivial reason. In particular, we are assuming that $\zeta(\sigma + it) \neq 0$. Assuming the Riemann hypothesis, we denote a non-trivial zero of $\zeta(s)$ as $\rho = \frac{1}{2} + i\gamma$ and define a function $F(s)$ as follows:

$$F(s) = \Re \sum_\rho \frac{1}{s - \rho} = \sum_\rho \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + (t - \gamma)^2}.$$

Note that $F(s) \geqslant 0$ whenever $\sigma \geqslant 1/2$ and $s \neq \rho$. The partial fraction decomposition of $\zeta'(s)/\zeta(s)$ (see [**23**, equation (2.12.7)]) says that, for $s \neq 1$ and s not coinciding with a zero of $\zeta(s)$, we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}s+1\right) - \frac{1}{s-1} + B, \tag{2.2}$$

where the constant $B = -\Re \sum_{\rho} \frac{1}{\rho} = \log 2\pi - 1 - \frac{1}{2}\gamma_0$, in which γ_0 denotes Euler's constant. Since

$$\Re\left(\frac{1}{s-1} \right) < 0$$

when $\sigma < 1$, by taking the real part of each term in (2.2), we find that

$$-\Re \frac{\zeta'}{\zeta}(s) \leqslant \Re \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}s+1\right) - F(s) \tag{2.3}$$

for $\sigma \leqslant \frac{3}{4}$, say. Stirling's asymptotic formula for the gamma function implies that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(|s|^{-2}\right) \tag{2.4}$$

for $\delta > 0$ fixed, $|\arg s| < \pi - \delta$ and $|s| > \delta$ (cf. [**16**, Theorem C.1]). By combining (2.3) and (2.4) with the observation that $F(s) \geqslant 0$, we find that

$$-\Re \frac{\zeta'}{\zeta}(s) \leqslant \frac{1}{2} \log \tau - F(s) + O(1)$$

$$\leqslant \frac{1}{2} \log \tau + O(1) \tag{2.5}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$. Consequently, the inequality

$$\log |\zeta(\sigma+it)| - \log |\zeta(\sigma_\lambda+it)| = \Re \int_{\sigma}^{\sigma_\lambda} \left[-\frac{\zeta'}{\zeta}(u+it) \right] du$$

$$\leqslant (\sigma_\lambda - \sigma)\left(\frac{1}{2} \log \tau + O(1) \right)$$

$$\leqslant \left(\sigma_\lambda - \frac{1}{2} \right)\left(\frac{1}{2} \log \tau + O(1) \right) \tag{2.6}$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \sigma_\lambda \leqslant \frac{3}{4}$. Here, while using the inequality (2.5) in the second line of (2.6), we have implicitly assumed that $\lambda \leqslant \frac{1}{4} \log x$ to ensure that $\sigma_\lambda \leqslant \frac{3}{4}$.

To complete the proof of the lemma, we require an upper bound for $\log |\zeta(\sigma_\lambda+it)|$ which, in turn, requires an additional identity for $\zeta'(s)/\zeta(s)$. Specifically, for $s \neq 1$ and s not coinciding with a zero of $\zeta(s)$, we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n \leqslant x} \frac{\Lambda(n)}{n^s} \frac{\log(x/n)}{\log x} + \frac{1}{\log x}\left(\frac{\zeta'}{\zeta}(s) \right)' + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s}}{(\rho-s)^2}$$

$$- \frac{1}{\log x} \frac{x^{1-s}}{(1-s)^2} + \frac{1}{\log x} \sum_{k=1}^{\infty} \frac{x^{-2k-s}}{(2k+s)^2}. \tag{2.7}$$

This identity is due to Soundararajan ([**22**, Lemma 1]). Integrating over σ from σ_λ to ∞ and using the assumption that $3 \leqslant x \leqslant \tau^2$, we deduce from the above identity that

$$\log |\zeta(\sigma_\lambda+it)| = \Re \sum_{n \leqslant x} \frac{\Lambda(n)}{n^{\sigma_\lambda+it} \log n} \frac{\log x/n}{\log x} - \frac{1}{\log x} \Re \frac{\zeta'}{\zeta}(\sigma_\lambda+it)$$

$$+ \frac{1}{\log x} \sum_{\rho} \Re \int_{\sigma_\lambda}^{\infty} \frac{x^{\rho-s}}{(\rho-s)^2} d\sigma + O\left(\frac{1}{\log x} \right). \tag{2.8}$$

We now estimate the second and third terms on the right-hand side of this expression. Using the first inequality in (2.5), it follows that

$$-\Re\,\frac{\zeta'}{\zeta}(\sigma_\lambda + it) \leqslant \frac{1}{2}\log\tau - F(\sigma_\lambda + it) + O(1) \tag{2.9}$$

for $0 \leqslant \lambda \leqslant \frac{1}{4}\log x$. Also, by observing that

$$\sum_\rho \Big|\int_{\sigma_\lambda}^\infty \frac{x^{\rho-s}}{(\rho-s)^2}d\sigma\Big| \leqslant \sum_\rho \int_{\sigma_\lambda}^\infty \frac{x^{1/2-\sigma}}{|\rho-s|^2}d\sigma$$

$$\leqslant \sum_\rho \frac{x^{1/2-\sigma_\lambda}}{|\rho-\sigma_\lambda-it|^2\log x} = \frac{x^{1/2-\sigma_\lambda}F(\sigma_\lambda+it)}{(\sigma_\lambda-1/2)\log x}, \tag{2.10}$$

and then combining (2.9) and (2.10) with (2.8), we see that

$$\log|\zeta(\sigma_\lambda+it)| \leqslant \Re \sum_{n\leqslant x} \frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n}\frac{\log x/n}{\log x} + \frac{1}{2}\frac{\log\tau}{\log x}$$

$$+ \frac{F(\sigma_\lambda+it)}{\log x}\left(\frac{x^{1/2-\sigma_\lambda}}{(\sigma_\lambda-1/2)\log x}-1\right) + O\Big(\frac{1}{\log x}\Big).$$

If $\lambda \geqslant \lambda_0$, then the term on the right-hand side involving $F(\sigma_\lambda+it)$ is less than or equal to zero, and hence omitting it does not change the inequality. Thus, when $\lambda \geqslant \lambda_0$, we have

$$\log|\zeta(\sigma_\lambda+it)| \leqslant \Re\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n}\frac{\log x/n}{\log x} + \frac{1}{2}\frac{\log\tau}{\log x} + O\Big(\frac{1}{\log x}\Big). \tag{2.11}$$

Since we have assumed that $|\zeta(\sigma+it)| \geqslant 1$, the lemma now follows by combining the inequalities in (2.6) and (2.11) and then taking absolute values. □

3. A variation of Lemma 2.1

In this section, we prove a version of Lemma 2.1 in which the sum over n on the right-hand side of the inequality is restricted just to the primes. A sketch of the proof of the lemma appearing below has been given previously by Soundararajan (see [**22**, Lemma 2]). Our proof is different and the details are provided for completeness.

LEMMA 3.1.   *Assume the Riemann hypothesis. Consider* $\tau = |t| + 100$. *Then*

$$\left|\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{\sigma+it}\log n}\frac{\log x/n}{\log x} - \sum_{p\leqslant x}\frac{1}{p^{\sigma+it}}\frac{\log x/n}{\log x}\right| = O\big(\log\log\log\tau\big)$$

uniformly for $\sigma \geqslant \frac{1}{2}$ *and* $|t| \geqslant 1$. *As a consequence, for any* λ *satisfying* $\lambda_0 \leqslant \lambda \leqslant \frac{1}{4}\log x$ *and* $3 \leqslant x \leqslant \tau^2$, *the estimate*

$$\log^+|\zeta(\sigma+it)| \leqslant \left|\sum_{p\leqslant x}\frac{1}{p^{\sigma_\lambda+it}}\frac{\log x/p}{\log x}\right| + \frac{(1+\lambda)}{2}\frac{\log\tau}{\log x} + O\big(\log\log\log\tau\big)$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \sigma_\lambda$ *and* $|t| \geqslant 1$.

Proof. First, for $\sigma \geqslant \frac{1}{2}$, we observe that,

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n^s \log n} \frac{\log x/n}{\log x} - \sum_{p\leqslant x} \frac{1}{p^s} \frac{\log x/p}{\log x} = \frac{1}{2} \sum_{p\leqslant\sqrt{x}} \frac{1}{p^{2s}} \frac{\log \sqrt{x}/n}{\log \sqrt{x}} + O(1).$$

$$= \frac{1}{2} \sum_{n\leqslant\sqrt{x}} \frac{\Lambda(n)}{n^{2s} \log n} \frac{\log \sqrt{x}/n}{\log \sqrt{x}} + O(1).$$

Thus, if we let $w = u + iv$ and $\nu = |v| + 100$, then the lemma will follow if we can show that

$$\sum_{n\leqslant z} \frac{\Lambda(n)}{n^w \log n} \frac{\log z/n}{\log z} = O\left(\log\log\log\nu\right) \tag{3.1}$$

uniformly for $u \geqslant 1$ and $2 \leqslant z \leqslant \nu$. In what follows, we can assume that $z \geqslant (\log\nu)^2$, as otherwise

$$\sum_{n\leqslant z} \frac{\Lambda(n)}{n^w \log n} \frac{\log z/n}{\log z} \ll 1 + \sum_{p<\log^2\nu} \frac{1}{p} \ll \log\log\log\nu.$$

Let $c = \max(2, 1 + u)$. Then, by expressing $\zeta'(s + w)/\zeta(s + w)$ as a Dirichlet series and interchanging the order of summation and integration (which is justified by absolute convergence), it follows that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[-\frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+w}} \right] z^s \frac{ds}{s^2}$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^w} \int_{c-i\infty}^{c+i\infty} \left(\frac{z}{n}\right)^s \frac{ds}{s^2}$$

$$= \sum_{n\leqslant z} \frac{\Lambda(n)}{n^w} \log(z/n).$$

Here we have made use of the following standard identity:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s^2} = \begin{cases} \log x & \text{if } x \geqslant 1, \\ 0 & \text{if } 0 \leqslant x < 1, \end{cases}$$

which is valid for $c > 0$. By moving the line of integration in the integral left to $\Re s = \sigma = \frac{3}{4} - u$, we find (by the calculus of residues) that

$$\sum_{n\leqslant z} \frac{\Lambda(n)}{n^w} \log(z/n) = -(\log z) \frac{\zeta'}{\zeta}(w) - \left(\frac{\zeta'}{\zeta}(w)\right)' + \frac{z^{1-w}}{(w-1)^2}$$

$$+ \frac{1}{2\pi i} \int_{\frac{3}{4}-u-i\infty}^{\frac{3}{4}-u+i\infty} \left[-\frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2}. \tag{3.2}$$

That there are no residues obtained from poles of the integrand at the non-trivial zeros of $\zeta(s)$ follows from the Riemann hypothesis. To estimate the integral on the right-hand side of the above expression, we use [**23**, Theorem 14.5], namely, that if the Riemann hypothesis is true, then

$$\left| \frac{\zeta'}{\zeta}(\sigma + it) \right| \ll (\log\tau)^{2-2\sigma} \tag{3.3}$$

uniformly for $\frac{5}{8} \leqslant \sigma \leqslant \frac{7}{8}$, say. Using (3.3), it immediately follows that

$$\int_{\frac{3}{4}-u-i\infty}^{\frac{3}{4}-u+i\infty} \left[-\frac{\zeta'}{\zeta}(s+w) \right] z^s \frac{ds}{s^2} \ll z^{3/4-u} \sqrt{\log\nu}.$$

Inserting this estimate into equation (3.2) and dividing by $\log z$, it follows that

$$\sum_{n \leqslant z} \frac{\Lambda(n)}{n^w} \frac{\log(z/n)}{\log z} = -\frac{\zeta'}{\zeta}(w) - \frac{1}{\log z}\left(\frac{\zeta'}{\zeta}(w)\right)' + \frac{z^{1-w}}{(w-1)^2 \log z} + O\left(\frac{z^{3/4-u}}{\log z}\sqrt{\log \nu}\right). \quad (3.4)$$

Integrating the expression in (3.4) from ∞ to u (along the line $\sigma + i\nu$, $u \leqslant \sigma < \infty$), we find that

$$\sum_{n \leqslant z} \frac{\Lambda(n)}{n^w \log n} \frac{\log(z/n)}{\log z} = \log \zeta(w) + \frac{1}{\log z} \frac{\zeta'}{\zeta}(w) + O\left(\frac{z^{1-u}}{\nu^2 (\log z)^2} + \frac{z^{3/4-u}}{(\log z)^2}\sqrt{\log \nu}\right).$$

Assuming the Riemann hypothesis, we can estimate the terms on the right-hand side of the above expression by invoking the bounds

$$|\log \zeta(\sigma + it)| \ll \log \log \log \tau \quad \text{and} \quad \left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \ll \log \log \tau, \quad (3.5)$$

which hold uniformly for $\sigma \geqslant 1$ and $|t| \geqslant 1$. For a discussion of these (and other similar) estimates, see [16, Section 13.2]. Using the estimates in (3.5) and recalling that we are assuming that $u \geqslant 1$ and $z \geqslant (\log \nu)^2$, we find that

$$\sum_{n \leqslant z} \frac{\Lambda(n)}{n^w \log n} \frac{\log(z/n)}{\log z} \ll \log \log \log \nu + \frac{\log \log \nu}{\log z} + \frac{z^{1-u}}{\nu^2 (\log z)^2} + z^{-1/4} \frac{\sqrt{\log \nu}}{(\log z)^2}$$

$$\ll \log \log \log \nu.$$

This establishes (3.1) and, thus, the lemma. $\qquad \square$

4. A sum over the zeros of $\zeta(s)$

In this section we prove an estimate for the mean-square of a Dirichlet polynomial averaged over the zeros of $\zeta(s)$. Our estimate follows from the Landau–Gonek explicit formula.

LEMMA 4.1.   Let $x, T > 1$ and let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then

$$\sum_{0 < \gamma \leqslant T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O\big(x \log(2xT) \log \log(3x)\big)$$

$$+ O\left(\log x \min\left(T, \frac{x}{\langle x \rangle}\right)\right) + O\left(\log(2T) \min\left(T, \frac{1}{\log x}\right)\right),$$

where $\langle x \rangle$ denotes the distance from x to the nearest prime power other than x itself; $\Lambda(x) = \log p$ if x is a positive integral power of a prime p, and $\Lambda(x) = 0$ otherwise.

Proof.   This is due to Gonek [6, 8]. $\qquad \square$

LEMMA 4.2.   Assume the Riemann hypothesis and let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. For any sequence of complex numbers $\mathscr{A} = \{a_n\}_{n=1}^\infty$ and for $\xi \geqslant 1$, we define

$$m_\xi = m_\xi(\mathscr{A}) = \max_{1 \leqslant n \leqslant \xi} \big(1, |a_n|\big).$$

Then, for $3 \leqslant \xi \leqslant T(\log T)^{-1}$ and any complex number α with $\Re \alpha \geqslant 0$, we have

$$\sum_{0 < \gamma \leqslant T} \left|\sum_{n \leqslant \xi} \frac{a_n}{n^{\rho + \alpha}}\right|^2 \ll m_\xi T \log T \sum_{n \leqslant \xi} \frac{|a_n|}{n}, \quad (4.1)$$

where the implied constant is absolute (and independent of α).

Proof. Assuming the Riemann hypothesis, we note that $1 - \rho = \bar{\rho}$ for any non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. This implies that

$$\left| \sum_{n \leqslant \xi} \frac{a_n}{n^{\rho+\alpha}} \right|^2 = \sum_{m \leqslant \xi} \sum_{n \leqslant \xi} \frac{a_m}{m^{\rho+\alpha}} \frac{\overline{a_n}}{n^{1-\rho+\bar{\alpha}}},$$

and, moreover, that

$$\sum_{0 < \gamma \leqslant T} \left| \sum_{n \leqslant \xi} \frac{a_n}{n^{\rho+\alpha}} \right|^2 = N(T) \sum_{n \leqslant \xi} \frac{|a_n|^2}{n^{1+2\Re\alpha}} + 2\Re \sum_{m \leqslant \xi} \frac{a_m}{m^\alpha} \sum_{m < n \leqslant \xi} \frac{\overline{a_n}}{n^{1+\bar{\alpha}}} \sum_{0 < \gamma \leqslant T} \left(\frac{n}{m} \right)^\rho,$$

where $N(T) \sim \frac{T}{2\pi} \log T$ denotes the number of zeros ρ with $0 < \gamma \leqslant T$. Since $\Re\alpha \geqslant 0$, it follows that

$$N(T) \sum_{n \leqslant \xi} \frac{|a_n|^2}{n^{1+2\Re\alpha}} \ll T \log T \sum_{n \leqslant \xi} \frac{|a_n|^2}{n} \ll m_\xi T \log T \sum_{n \leqslant \xi} \frac{|a_n|}{n}.$$

Appealing to Lemma 4.1, we find that

$$\sum_{m \leqslant \xi} \frac{a_m}{m^\alpha} \sum_{n < m} \frac{\overline{a_n}}{n^{1+\bar{\alpha}}} \sum_{0 < \gamma \leqslant T} \left(\frac{n}{m} \right)^\rho = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,$$

where

$$\Sigma_1 = -\frac{T}{2\pi} \sum_{m \leqslant \xi} \frac{a_m}{m^\alpha} \sum_{m < n \leqslant \xi} \frac{\overline{a_n}}{n^{1+\bar{\alpha}}} \Lambda\left(\frac{n}{m} \right),$$

$$\Sigma_2 = O\left(\log T \log \log T \sum_{m \leqslant \xi} \frac{|a_m|}{m^{1+\Re\alpha}} \sum_{m < n \leqslant \xi} \frac{|a_n|}{n^{\Re\alpha}} \right),$$

$$\Sigma_3 = O\left(\sum_{m \leqslant \xi} \frac{|a_m|}{m^{1+\Re\alpha}} \sum_{m < n \leqslant \xi} \frac{|a_n|}{n^{\Re\alpha}} \frac{\log(m/n)}{\langle m/n \rangle} \right)$$

and

$$\Sigma_4 = O\left(\log T \sum_{m \leqslant \xi} \frac{|a_m|}{m^{\Re\alpha}} \sum_{m < n \leqslant \xi} \frac{|a_n|}{n^{1+\Re\alpha} \log(n/m)} \right).$$

We estimate Σ_1 first. Making the substitution $n = mk$, we rewrite our expression for Σ_1 as

$$-\frac{T}{2\pi} \sum_{m \leqslant \xi} \frac{a_m}{m^\alpha} \sum_{k \leqslant \frac{\xi}{m}} \frac{\overline{a_{mk}} \cdot \Lambda(k)}{(mk)^{1+\bar{\alpha}}} = -\frac{T}{2\pi} \sum_{m \leqslant \xi} \frac{a_m}{m^{1+2\Re\alpha}} \sum_{k \leqslant \frac{\xi}{m}} \frac{\overline{a_{mk}} \cdot \Lambda(k)}{k^{1+\bar{\alpha}}}.$$

Again using the assumption that $\Re\alpha \geqslant 0$, we find that

$$\Sigma_1 \ll m_\xi T \sum_{n \leqslant \xi} \frac{|a_n|}{n} \sum_{m \leqslant \xi/n} \frac{\Lambda(m)}{m} \ll m_\xi T \log T \sum_{n \leqslant \xi} \frac{|a_n|}{n}.$$

Here we have made use of the standard estimate $\sum_{m \leqslant \xi} \Lambda(m)/m \ll \log \xi$. We can replace $\Re\alpha$ by 0 in each of the sums Σ_i (for $i = 2, 3$ or 4), as doing so will only make the corresponding estimates larger. Thus, using the assumption that $3 \leqslant \xi \leqslant T/\log T$, it follows that

$$\Sigma_2 \ll m_\xi \log T \log \log T \sum_{n \leqslant \xi} \frac{|a_n|}{n} \sum_{m < n \leqslant \xi} 1 \ll m_\xi T \log T \sum_{n \leqslant \xi} \frac{|a_n|}{n}.$$

Next, turning to Σ_3, we find that

$$\Sigma_3 \ll m_\xi \sum_{m \leqslant \xi} \frac{|a_m|}{m} \sum_{m < n \leqslant \xi} \frac{\log(n/m)}{\langle n/m \rangle}.$$

Writing n as $qm + \ell$ with $-m/2 < \ell \leqslant m/2$, we have

$$\Sigma_3 \ll m_\xi \sum_{m \leqslant \xi} \frac{|a_m|}{m} \sum_{q \leqslant \lfloor \xi/m \rfloor + 1} \sum_{-m/2 < \ell \leqslant m/2} \frac{\log\left(q + \ell/m\right)}{\langle q + \ell/m \rangle},$$

where, as usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Now $\langle q + \ell/m \rangle = |\ell|/m$ if q is a prime power and $\ell \neq 0$, otherwise $\langle q + \ell/m \rangle$ is at least $\frac{1}{2}$. Using the estimate $\sum_{n \leqslant \xi} \Lambda(n) \ll \xi$, we now find that

$$\Sigma_3 \ll m_\xi \sum_{m \leqslant \xi} \frac{|a_m|}{m} \sum_{q \leqslant \lfloor \xi/m \rfloor + 1} \Lambda(q) \sum_{1 \leqslant \ell \leqslant m/2} \frac{m}{\ell} + m_\xi \sum_{m \leqslant \xi} \frac{|a_m|}{m} \sum_{q \leqslant \lfloor \xi/m \rfloor + 1} \log(q+1) \sum_{1 \leqslant \ell \leqslant m/2} 1$$

$$\ll m_\xi \sum_{m \leqslant \xi} |a_m| \log m \sum_{q \leqslant \lfloor \xi/m \rfloor + 1} \Lambda(q) + m_\xi \sum_{m \leqslant \xi} |a_m| \sum_{q \leqslant \lfloor \xi/m \rfloor + 1} \log(q+1)$$

$$\ll m_\xi (\xi \log \xi) \sum_{m \leqslant \xi} \frac{|a_m|}{m}$$

$$\ll m_\xi T \log T \sum_{m \leqslant \xi} \frac{|a_m|}{m}.$$

It remains to consider the contribution from Σ_4, which is much less than

$$m_\xi \log T \sum_{m \leqslant \xi} |a_m| \sum_{m < n \leqslant \xi} \frac{1}{n \log(n/m)} \ll m_\xi \log T \sum_{m \leqslant \xi} \frac{|a_m|}{m} \sum_{m < n \leqslant \xi} \frac{1}{\log(n/m)},$$

since $1/m > 1/n$ if $n > m$. Writing $n = m + \ell$, we see that

$$\sum_{m < n \leqslant \xi} \frac{1}{\log(n/m)} = \sum_{1 \leqslant \ell \leqslant \xi - m} \frac{1}{\log\left(1 + \ell/m\right)} \ll \sum_{1 \leqslant \ell \leqslant \xi - m} \frac{m}{\ell} \ll m \log \xi \ll \xi \log \xi.$$

Consequently, we have

$$\Sigma_4 \ll m_\xi T \log T \sum_{m \leqslant \xi} \frac{|a_m|}{m}.$$

Now, by combining the estimates, we obtain the lemma. $\qquad\square$

5. The frequency of large values of $|\zeta(\rho + \alpha)|$

Our proof of Theorem 1.2 requires the following lemma concerning the distribution of values of $|\zeta(\rho + \alpha)|$, where ρ is a zero of $\zeta(s)$ and $\alpha \in \mathbb{C}$ is a small shift. In what follows, $\log_3(\cdot)$ stands for $\log \log \log(\cdot)$.

LEMMA 5.1.   *Assume the Riemann hypothesis. Let T be large, let $V \geqslant 3$ be a real number and let $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$ and $0 \leqslant \Re \alpha \leqslant (\log T)^{-1}$. Consider the set*

$$\mathcal{S}_\alpha\left(T; V\right) = \left\{ \gamma \in (0, T] : \log |\zeta(\rho + \alpha)| \geqslant V \right\},$$

where $\rho = \frac{1}{2} + i\gamma$ denotes a non-trivial zero of $\zeta(s)$. Then, the following inequalities for $\#\mathcal{S}_\alpha(T;V)$, the cardinality of $\mathcal{S}_\alpha(T;V)$, hold.

(i) When $\sqrt{\log\log T} \leqslant V \leqslant \log\log T$, we have

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\frac{V^2}{\log\log T}\left(1 - \frac{4}{\log_3 T}\right)\right).$$

(ii) When $\log\log T \leqslant V \leqslant \frac{1}{2}(\log\log T)\log_3 T$, we have

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\frac{V^2}{\log\log T}\left(1 - \frac{4V}{(\log\log T)\log_3 T}\right)\right).$$

(iii) Finally, when $V > \frac{1}{2}(\log\log T)\log_3 T$, we have

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\exp\left(-\frac{V}{201}\log V\right).$$

Here, as usual, the function $N(T) \sim (T/2\pi)\log T$ denotes the number of zeros ρ of $\zeta(s)$ with $0 < \gamma \leqslant T$.

Proof. Since $\lambda_0 < 3/5$, by taking $x = (\log\tau)^{2-\varepsilon}$ in Lemma 3.1 (where $\varepsilon > 0$ arbitrary) and estimating the sum over primes trivially, we find that

$$\log^+|\zeta(\sigma + i\tau)| \leqslant \left(\frac{1+\lambda_0}{4} + o(1)\right)\frac{\log\tau}{\log\log\tau} \leqslant \frac{2}{5}\frac{\log\tau}{\log\log\tau}$$

for $1/2 \leqslant \sigma \leqslant \lambda_0/\log x$ and $|\tau|$ sufficiently large. Therefore, we may suppose that $V \leqslant (2/5)(\log T/\log\log T)$, for otherwise the set $\mathcal{S}_\alpha(T;V)$ is empty.

We define a parameter

$$A = A(T,V) = \begin{cases} \dfrac{1}{2}\log_3(T) & \text{if } V \leqslant \log\log T, \\[2mm] \dfrac{1}{2V}\log\log T\log_3(T) & \text{if } \log\log T < V \leqslant \dfrac{1}{2}(\log\log T)\log_3 T, \\[2mm] 1 & \text{if } V > \dfrac{1}{2}(\log\log T)\log_3 T, \end{cases}$$

set $x = T^{A/V}$ and put $z = x^{1/\log\log T}$. We now observe that since $\lambda_0 \geqslant \frac{1}{2}$, $x \leqslant T^{A/V} \leqslant T^{1/2}$ and $0 \leqslant \Re\alpha \leqslant (\log T)^{-1}$, it follows that

$$\frac{1}{2} \leqslant \Re(\rho + \alpha) \leqslant \frac{1}{2} + \frac{1}{\log T} \leqslant \frac{1}{2} + \frac{\lambda_0}{\log x}$$

for any non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leqslant T$. Thus, if we let

$$S_1(s) = \sum_{p \leqslant z}\frac{1}{p^{s+\lambda_0/\log x}}\frac{\log(x/p)}{\log x} \quad \text{and} \quad S_2(s) = \sum_{z < p \leqslant x}\frac{1}{p^{s+\lambda_0/\log x}}\frac{\log(x/p)}{\log x},$$

Lemma 3.1 implies that

$$\log^+|\zeta(\rho + \alpha)| \leqslant |S_1(\rho)| + |S_2(\rho)| + \frac{(1+\lambda_0)}{2A}V + O\left(\log_3 T\right)$$

for any non-trivial zero ρ of $\zeta(s)$. Since $\lambda_0 < 3/5$, it follows that

$$\log^+|\zeta(\rho + \alpha)| \leqslant |S_1(\rho)| + |S_2(\rho)| + \frac{4}{5}\frac{V}{A} + O\left(\log_3 T\right).$$

Therefore, if $\rho \in \mathcal{S}_\alpha(T; V)$, then either

$$|S_1(\rho)| \geqslant V\left(1 - \frac{9}{10A}\right) \quad \text{or} \quad |S_2(\rho)| \geqslant \frac{V}{10A}.$$

For simplicity, we put $V_1 = V\left(1 - 9/10A\right)$ and $V_2 = V/10A$.

Let $N_1(T; V)$ be the number of ρ with $0 < \gamma \leqslant T$ such that $|S_1(\rho)| \geqslant V_1$ and let $N_2(T; V)$ be the number of ρ with $0 < \gamma \leqslant T$ such that $|S_2(\rho)| \geqslant V_2$. We prove the lemma by obtaining upper bounds for the size of the sets $N_i(T; V)$ for $i = 1$ and 2 using the inequality

$$N_i(T; V) \cdot V_i^{2k} \leqslant \sum_{0 < \gamma \leqslant T} |S_i(\rho)|^{2k}, \tag{5.1}$$

which holds for any positive integer k. With some restrictions on the size of k, we can use Lemma 4.2 to estimate the sums appearing on the right-hand side of this inequality.

We first turn our attention to estimating $N_1(T; V)$. If we define the sequence $\alpha_k(n) = \alpha_k(n, x, z)$ by

$$\sum_{n \leqslant z^k} \frac{\alpha_k(n)}{n^s} = \left(\sum_{p \leqslant z} \frac{1}{p^s} \frac{\log x/p}{\log x}\right)^k,$$

then it is easily seen that $|\alpha_k(n)| \leqslant k!$. Thus, Lemma 4.2 implies that the estimate

$$\sum_{0 < \gamma \leqslant T} |S_1(\rho)|^{2k} \ll N(T)\, k!\, \left(\sum_{p \leqslant z} \frac{1}{p} \frac{\log(x/p)}{\log x}\right)^k$$

$$\ll N(T)\, k!\, \left(\sum_{p \leqslant z} \frac{1}{p}\right)^k$$

$$\ll N(T)\sqrt{k}\left(\frac{k \log \log T}{e}\right)^k$$

holds for any positive integer k with $z^k \leqslant T(\log T)^{-1}$ and T sufficiently large. Using (5.1), we deduce from this estimate that

$$N_1(T; V) \ll N(T)\sqrt{k}\left(\frac{k \log \log T}{eV_1^2}\right)^k. \tag{5.2}$$

It is now convenient to consider separately the case when $V \leqslant (\log \log T)^2$ and the case $V > (\log \log T)^2$. When $V \leqslant (\log \log T)^2$ we choose $k = \lfloor V_1^2 / \log \log T \rfloor$ where, as before, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. To see that this choice of k satisfies $z^k \leqslant T(\log T)^{-1}$, we note from the definition of A that

$$VA \leqslant \max\left(V,\ \tfrac{1}{2}(\log \log T) \log_3 T\right).$$

Therefore, we find that

$$z^k \leqslant z^{V_1^2 / \log \log T} = \exp\left(\frac{VA \log T}{(\log \log T)^2}\left(1 - \frac{9}{10A}\right)^2\right)$$

$$\leqslant \exp\left(\log T \left(1 - \frac{9}{10A}\right)^2\right)$$

$$\leqslant T/\log T.$$

Thus, by (5.2), we see that for $V \leqslant (\log \log T)^2$ and T large we have

$$N_1(T; V) \ll N(T)\frac{V}{\sqrt{\log \log T}} \exp\left(-\frac{V_1^2}{\log \log T}\right). \tag{5.3}$$

When $V > (\log \log T)^2$ we choose $k = \lfloor 10V \rfloor$. This choice of k satisfies $z^k \leqslant T(\log T)^{-1}$ since $z^{10V} = T^{10/\log \log T} \leqslant T(\log T)^{-1}$ for large T. With this choice of k, we conclude from (5.2) that

$$N_1(T; V) \ll N(T) \exp \left(\frac{1}{2} \log V - 10V \log \left(\frac{eV}{1000 \log \log T} \right) \right)$$
$$\ll N(T) \exp \left(- 10V \log V + 11V \log_3(T) \right) \tag{5.4}$$

for T sufficiently large. Since $V > (\log \log T)^2$, we have that $\log V \geqslant 2 \log_3(T)$ and thus it follows from (5.4) that

$$N_1(T; V) \ll N(T) \exp \left(- 4V \log V \right). \tag{5.5}$$

By combining (5.3) and (5.5), for any choice of V, we have shown that,

$$N_1(T; V) \ll N(T) \frac{V}{\sqrt{\log \log T}} \exp \left(- \frac{V_1^2}{\log \log T} \right) + N(T) \exp \left(- 4V \log V \right). \tag{5.6}$$

We now turn our attention to estimating $N_2(T; V)$. If we define the sequence $\beta_k(n) = \beta_k(n, x, z)$ by

$$\sum_{n \leqslant x^k} \frac{\beta_k(n)}{n^s} = \left(\sum_{z < p \leqslant x} \frac{1}{p^s} \frac{\log x/p}{\log x} \right)^k,$$

then it can be seen that $|\beta_k(n)| \leqslant k!$. Thus, Lemma 4.2 implies that

$$\sum_{0 < \gamma \leqslant T} |S_2(\rho)|^{2k} \ll N(T) \, k! \left(\sum_{z < p \leqslant x} \frac{1}{p} \frac{\log(x/p)}{\log x} \right)^k$$
$$\ll N(T) \, k! \left(\sum_{z < p \leqslant x} \frac{1}{p} \right)^k$$
$$\ll N(T) \, k! \left(\log_3(T) + O(1) \right)^k$$
$$\ll N(T) \, k! \left(2 \log_3(T) \right)^k$$
$$\ll N(T) \left(2k \log_3(T) \right)^k \tag{5.7}$$

for any natural number k with $x^k \leqslant T/\log T$ and T sufficiently large. The choice of $k = \lfloor V/A - 1 \rfloor$ satisfies $x^k \leqslant T/\log T$ when T is large. To see why, recall that $A \geqslant 1$, $x = T^{A/V}$ and $V \leqslant (2/5)(\log T/\log \log T)$. Therefore, we have

$$x^k \leqslant x^{(V/A - 1)} \leqslant T^{1 - A/V} \leqslant T^{1 - 1/V} = T(\log T)^{-5/2} \leqslant T(\log T)^{-1}.$$

Also, observing that $A \leqslant \frac{1}{2} \log_3(T)$ and recalling that $V \geqslant \sqrt{\log \log T}$, with this choice of k and T large, it follows from (5.1) that

$$N_2(T; V) \ll N(T) \left(\frac{10A}{V} \right)^{2k} \left(2k \log_3(T) \right)^k$$
$$\ll N(T) \exp \left(- 2k \log \left(\frac{V}{10A} \right) + k \log(2k \log_3(T)) \right)$$
$$\ll N(T) \exp \left(- 2 \frac{V}{A} \log \left(\frac{V}{10A} \right) + 2 \log \frac{V}{10A} + \frac{V}{A} \log \left(\frac{2V}{A} \log_3(T) \right) \right)$$
$$\ll N(T) \exp \left(- \frac{V}{2A} \log V \right). \tag{5.8}$$

Using our estimates for $N_1(T;V)$ and $N_2(T;V)$ we can now complete the proof of the lemma by checking the various ranges of V. By combining (5.6) and (5.8), we see that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V_1^2}{\log\log T}\Big) + N(T)\exp\Big(-4V\log V\Big)$$
$$+ N(T)\exp\Big(-\frac{V}{2A}\log V\Big). \tag{5.9}$$

If $\sqrt{\log\log T} \leqslant V \leqslant \log\log T$, then $A = \frac{1}{2}\log_3(T)$ and, for T sufficiently large, (5.9) implies that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1-\frac{9}{5\log_3 T}\Big)^2\Big)$$
$$\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1-\frac{4}{\log_3 T}\Big)\Big). \tag{5.10}$$

If $\log\log T < V \leqslant \frac{1}{2}(\log\log T)\log_3(T)$, then $A = (\log\log T/2V)\log_3(T)$ and we deduce from (5.9) that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1-\frac{9V}{5(\log\log T)\log_3 T}\Big)^2\Big)$$
$$+ N(T)\exp\Big(-\frac{V^2\log V}{(\log\log T)\log_3 T}\Big) + N(T)\exp\Big(-4V\log V\Big). \tag{5.11}$$

For V in this range, $\log V/(\log\log T)\log_3 T > 1/\log\log T$ and $V/\log V < \log\log T$, and hence (5.11) implies that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1-\frac{9V}{5(\log\log T)\log_3 T}\Big)^2\Big)$$
$$\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{\log\log T}\Big(1-\frac{4V}{(\log\log T)\log_3 T}\Big)\Big). \tag{5.12}$$

Finally, if $V \geqslant \frac{1}{2}(\log\log T)\log_3 T$, then $A = 1$ and we deduce from (5.9) that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\exp\Big(\log V - \frac{V^2}{100\log\log T}\Big) + N(T)\exp\Big(-\frac{V}{2}\log V\Big). \tag{5.13}$$

Certainly, if $V \geqslant \frac{1}{2}(\log\log T)\log_3 T$, then we have that $V^2/100\log\log T - \log V > (1/201)V\log V$ for T sufficiently large and hence it follows from (5.13) that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\exp\Big(-\frac{V}{201}\log V\Big). \tag{5.14}$$

The lemma now follows from the estimates in (5.10), (5.12) and (5.14). $\qquad\square$

6. Proof of Theorem 1.2

Using Lemma 5.1, we first prove Theorem 1.2 in the case where $|\alpha| \leqslant 1$ and $0 \leqslant \Re\alpha \leqslant (\log T)^{-1}$. Then, from this result, the case when $-(\log T)^{-1} \leqslant \Re\alpha < 0$ can be deduced from the functional equation for $\zeta(s)$ and Stirling's formula for the gamma function. In what follows, $k \in \mathbb{R}$ is fixed and we let $\varepsilon > 0$ be an arbitrarily small positive constant that may not be the same at each occurrence.

First, we partition the real axis into the intervals $I_1 = (-\infty, 3]$, $I_2 = (3, 4k\log\log T]$ and $I_3 = (4k\log\log T, \infty)$ and set

$$M_i = \sum_{\nu \in I_i \cap \mathbb{Z}} e^{2k\nu} \cdot \#\mathcal{S}_\alpha(T, \nu)$$

for $i = 1, 2$ and 3. Then we observe that

$$\sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \leqslant \sum_{\nu \in \mathbb{Z}} e^{2k\nu} \left[\#\mathcal{S}_\alpha(T, \nu) - \#\mathcal{S}_\alpha(T, \nu - 1) \right] \leqslant M_1 + M_2 + M_3. \tag{6.1}$$

Using the trivial bound $\#\mathcal{S}_\alpha(T, \nu) \leqslant N(T)$, which holds for every $\nu \in \mathbb{Z}$, we find that $M_1 \leqslant e^{6k} N(T)$. To estimate M_2, we use the bound

$$\#\mathcal{S}_\alpha(T, \nu) \ll N(T)(\log T)^\varepsilon \exp\left(\frac{-\nu^2}{\log \log T} \right),$$

which follows from the first two cases of Lemma 5.1 when $\nu \in I_2 \cap \mathbb{Z}$. From this, it follows that

$$\begin{aligned} M_2 &\ll N(T)(\log T)^\varepsilon \int_3^{4k \log \log T} \exp\left(2ku - \tfrac{u^2}{\log \log T} \right) du \\ &\ll N(T)(\log T)^\varepsilon \int_0^{4k} (\log T)^{u(2k-u)} \, du \\ &\ll N(T)(\log T)^{k^2 + \varepsilon} \end{aligned}$$

When $\nu \in I_3 \cap \mathbb{Z}$, the second two cases of Lemma 5.1 imply that

$$\#\mathcal{S}_\alpha(T, \nu) \ll N(T)(\log T)^\varepsilon e^{-4k\nu}.$$

Thus, we have

$$M_3 \ll N(T)(\log T)^\varepsilon \int_{4k \log \log T}^\infty e^{-2ku} \, du \ll N(T)(\log T)^{-8k^2 + \varepsilon}.$$

In light of (6.1), by collecting estimates, we see that

$$\sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \ll N(T)(\log T)^{k^2 + \varepsilon} \tag{6.2}$$

for every $k > 0$ when $|\alpha| \leqslant 1$ and $0 \leqslant \Re\alpha \leqslant (\log T)^{-1}$.

The functional equation for the zeta-function states that $\zeta(s) = \chi(s)\zeta(1 - s)$, where $\chi(s) = 2^s \pi^{s-1} \Gamma(1 - s) \sin(\pi s / 2)$. Stirling's asymptotic formula for the gamma function (cf. [16, Theorem C.1]) can be used to show that

$$\left| \chi(\sigma + it) \right| = \left(\frac{|t|}{2\pi} \right)^{1/2 - \sigma} \left(1 + O\left(\frac{1}{|t|} \right) \right)$$

uniformly for $-1 \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$. Using the Riemann hypothesis, we see that

$$\begin{aligned} \left| \zeta(\rho + \alpha) \right| &= \left| \chi(\rho + \alpha)\zeta(1 - \rho - \alpha) \right| \\ &= \left| \chi(\rho + \alpha)\zeta(\bar{\rho} - \alpha) \right| \\ &= \left| \chi(\rho + \alpha)\zeta(\rho - \bar{\alpha}) \right| \\ &\leqslant C \left| \zeta(\rho - \bar{\alpha}) \right| \end{aligned}$$

for some absolute constant $C > 0$ when $|\alpha| \leqslant 1$, $|\Re\alpha| \leqslant (\log T)^{-1}$ and $0 < \gamma \leqslant T$. Consequently, for $-(\log T)^{-1} \leqslant \Re\alpha < 0$, we have

$$\sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \leqslant C^{2k} \cdot \sum_{0 < \gamma \leqslant T} \left| \zeta(\rho - \bar{\alpha}) \right|^{2k}. \tag{6.3}$$

Applying the inequality in (6.2) to the right-hand side of (6.3) we see that

$$\sum_{0<\gamma\leqslant T} \left|\zeta(\rho+\alpha)\right|^{2k} \ll_k N(T)(\log T)^{k^2+\varepsilon} \tag{6.4}$$

for every $k > 0$ when $|\alpha| \leqslant 1$ and $-(\log T)^{-1} \leqslant \Re\alpha < 0$. The theorem now follows from the estimates in (6.2) and (6.4).

7.   *Theorem 1.2 implies Theorem 1.1*

Theorem 1.1 can now be established as a simple consequence of Theorem 1.2 and the following lemma.

LEMMA 7.1.   *Assume the Riemann hypothesis. Let $k, \ell \in \mathbb{N}$ and let $R > 0$ be arbitrary. Then we have*

$$\sum_{0<\gamma\leqslant T} \left|\zeta^{(\ell)}(\rho)\right|^{2k} \leqslant \left(\frac{\ell!}{R^\ell}\right)^{2k} \cdot \left[\max_{|\alpha|\leqslant R} \sum_{0<\gamma\leqslant T} \left|\zeta(\rho+\alpha)\right|^{2k}\right]. \tag{7.1}$$

Proof.   Since the function $\zeta^{(\ell)}(s)$ is real when $s \in \mathbb{R}$, it follows that $\zeta^{(\ell)}(\bar{s}) = \overline{\zeta^{(\ell)}(s)}$. Hence, assuming the Riemann hypothesis, the identity

$$\left|\zeta^{(\ell)}(1-\rho+\alpha)\right| = \left|\zeta^{(\ell)}(\bar{\rho}+\alpha)\right| = \left|\zeta^{(\ell)}(\rho+\overline{\alpha})\right| \tag{7.2}$$

holds for any non-trivial zero ρ of $\zeta(s)$ and any $\alpha \in \mathbb{C}$. For each positive integer k, let $\vec{\alpha}_k = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})$ and define

$$\mathcal{Z}\left(s;\vec{\alpha}_k\right) = \prod_{i=1}^{k} \zeta(s+\alpha_i)\zeta(1-s+\alpha_{i+k}).$$

If we suppose that each $|\alpha_i| \leqslant R$ for $i = 1, \ldots, 2k$ and apply Hölder's inequality in the form

$$\left|\sum_{n=1}^{N}\left(\prod_{i=1}^{2k} f_i(s_n)\right)\right| \leqslant \prod_{i=1}^{2k}\left(\sum_{n=1}^{N} |f_i(s_n)|^{2k}\right)^{1/2k},$$

then we see that (7.2) implies that

$$\left|\sum_{0<\gamma\leqslant T}\mathcal{Z}\left(\rho;\vec{\alpha}_k\right)\right| \leqslant \prod_{i=1}^{k}\left(\sum_{0<\gamma\leqslant T}\left|\zeta(\rho+\alpha_i)\right|^{2k}\right)^{1/2k}\left(\sum_{0<\gamma\leqslant T}\left|\zeta(\rho+\overline{\alpha_{k+i}})\right|^{2k}\right)^{1/2k}$$

$$\leqslant \max_{|\alpha|\leqslant R}\sum_{0<\gamma\leqslant T}\left|\zeta(\rho+\alpha)\right|^{2k}. \tag{7.3}$$

In order to prove the lemma, we first rewrite the left-hand side of equation (7.1) using the function $\mathcal{Z}(s;\vec{\alpha}_k)$ and then apply the inequality in (7.3). By Cauchy's integral formula and another application of (7.2), we see that

$$\sum_{0<\gamma\leqslant T}\left|\zeta^{(\ell)}(\rho)\right|^{2k} = \sum_{0<\gamma\leqslant T}\left(\prod_{i=1}^{k}\zeta^{(\ell)}(\rho)\zeta^{(\ell)}(1-\rho)\right)$$

$$= \frac{(\ell!)^{2k}}{(2\pi i)^{2k}}\int_{\mathscr{C}_1}\cdots\int_{\mathscr{C}_{2k}}\left(\sum_{0<\gamma\leqslant T}\mathcal{Z}\left(\rho;\vec{\alpha}_k\right)\right)\prod_{i=1}^{2k}\frac{d\alpha_i}{\alpha_i^{\ell+1}}, \tag{7.4}$$

where, for each $i = 1, \ldots, 2k$, the contour $\mathscr{C}_i$ denotes the positively oriented circle in the complex plane centered at 0 with radius R. Now, combining (7.3) and (7.4) we find that

$$
\begin{aligned}
\sum_{0 < \gamma \leqslant T} \left| \zeta^{(\ell)}(\rho) \right|^{2k} &\leqslant \left(\frac{\ell!}{2\pi} \right)^{2k} \cdot \left[\max_{|\alpha| \leqslant R} \sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \right] \cdot \int_{\mathscr{C}_1} \cdots \int_{\mathscr{C}_{2k}} \prod_{i=1}^{2k} \frac{d\alpha_i}{|\alpha_i|^{\ell+1}} \\
&\leqslant \left(\frac{\ell!}{2\pi} \right)^{2k} \cdot \left[\max_{|\alpha| \leqslant R} \sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \right] \cdot \left(\frac{2\pi}{R^\ell} \right)^{2k} \\
&\leqslant \left(\frac{\ell!}{R^\ell} \right)^{2k} \cdot \left[\max_{|\alpha| \leqslant R} \sum_{0 < \gamma \leqslant T} \left| \zeta(\rho + \alpha) \right|^{2k} \right],
\end{aligned}
$$

as claimed. $\qquad\square$

Proof of Theorem 1.1. Let $k \in \mathbb{N}$ and set $R = (\log T)^{-1}$. Then, it follows from Theorem 1.2 and Lemma 7.1 that

$$
\frac{1}{N(T)} \sum_{0 < \gamma \leqslant T} \left| \zeta^{(\ell)}(\rho) \right|^{2k} \ll_{k,\ell,\varepsilon} (\log T)^{k(k+2\ell)+\varepsilon} \tag{7.5}
$$

for any $\ell \in \mathbb{N}$ and for $\varepsilon > 0$ arbitrary. Theorem 1.1 now follows by setting $\ell = 1$. $\qquad\square$

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