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MULTIPLICATIVELY BADLY APPROXIMABLE MATRICES IN FIELDS OF POWER SERIES

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ABSTRACT. We study the notion of multiplicatively badly approximable matrices in the field of Laurent series with coefficients in a field K. We prove a transference principle in this setting, and show that such matrices exist when K is infinite.

1. INTRODUCTION

Let K be a field and let $L = K((x^{-1}))$ be the field of formal Laurent series with coefficients in K. That is, the nonzero elements of L consist of all formal Laurent series

$$F(x) = \sum_{m=-\infty}^{M} a(m)x^m, \tag{1}$$

where each coefficient $a(m)$ is in K, $a(M) \neq 0_K$, and M is an arbitrary integer. We write 0_L for the identically zero Laurent series, which is also in L. Addition and multiplication in L are defined in the obvious way. Then we define $| \ | : L \to [0, \infty)$ by $|0_L| = 0$, and

$$|F| = e^M, \tag{2}$$

where $F(x) \neq 0_L$ in L is given by (1). It follows that $| \ | : L \to [0, \infty)$ is a discrete, non-archimedean absolute value on L, and the resulting metric space $\big(L, | \ |\big)$ is complete. In this situation we define the subring of L-adic integers

$$\mathcal{O}_L = \big\{F \in L : |F| \leq 1\big\},$$

and its unique maximal ideal

$$\mathcal{M}_L = \big\{F \in \mathcal{O}_L : |F| < 1\big\}.$$

Clearly the residue class field $\mathcal{O}_L/\mathcal{M}_L$ is isomorphic to K. We recall that $\mathcal{O}_L$ is compact if and only if the residue class field $\mathcal{O}_L/\mathcal{M}_L$ is finite (see [5, Chapter 4, Corollary to Lemma 1.5].) Obviously $\mathcal{M}_L \subseteq \mathcal{O}_L$ is the principal ideal generated by the element x^{-1}.

It is clear that the polynomial ring $K[x]$ can be embedded in L by simply regarding a polynomial

$$P(x) = \sum_{m=0}^{M} \xi(m)x^m$$

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as a Laurent series with $\xi(m) = 0_K$ for integers $m \le -1$. In what follows we will always identify $K[x]$ with its image in L. Let $\varphi : L \to \mathcal{M}_L$ be the map defined by $\varphi(0_L) = 0_L$, and defined on nonzero elements F in L by

$$\varphi(F) = \varphi\biggl(\sum_{m=-\infty}^{M} a(m)x^m\biggr) = \sum_{m=-\infty}^{-1} a(m)x^m.$$

It is easy to check that φ is a surjective homomorphism of additive groups, and

$$\ker\{\varphi\} = K[x]. \tag{3}$$

We put $\mathbf{T} = L/K[x]$. Then φ induces an isomorphism of additive groups

$$\overline{\varphi} : \mathbf{T} \to \mathcal{M}_L. \tag{4}$$

The subset $K[x] \subseteq L$ is clearly discrete with respect to the metric topology induced by the absolute value $|\ |$. In particular, if $P(x) \ne Q(x)$ are polynomials in $K[x]$, then we have

$$1 \le |P - Q|. \tag{5}$$

Next we define $\|\ \| : \mathbf{T} \to [0, 1)$ on the additive group $\mathbf{T}$ by

$$\|F\| = \min\bigl\{|F - P| : P \in K[x]\bigr\}.$$

Alternatively, $\|F\|$ is the distance in L from F to the nearest polynomial. It is trivial to check that $\|\ \| : \mathbf{T} \to [0, 1)$ is a norm on $\mathbf{T}$ in the sense of Kaplansky [9, Appendix 1]. Therefore the map

$$(F, G) \mapsto \|F - G\|$$

defines a metric in $\mathbf{T}$, and so induces a metric topology in this quotient group.

There is a natural $K[x]$-module structure on the additive group $\mathbf{T}$. If $\bigl(P(x), F(x)+K[x]\bigr)$ is an element of $K[x] \times \mathbf{T}$ we define the product

$$P(x)\bigl(F(x) + K[x]\bigr) = P(x)F(x) + K[x]. \tag{6}$$

Again it is easy to check that (6) does give the additive group $\mathbf{T}$ the structure of a $K[x]$-module.

We can formulate Diophantine approximation problems in this setting, with $K[x]$, $K(x)$, L, and $\mathbf{T}$, playing roles analogous to $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and the quotient group $\mathbf{R}/\mathbf{Z}$, respectively. Indeed, Davenport and Lewis [6] studied the analog of Littlewood's conjecture in L and proved that this analog is actually *false* when K is infinite. Explicit counterexamples were later given by Baker [2]. In the case where K is finite, the analog of the Littlewood conjecture in L is believed to be true, but still remains an open problem (see [1]).

In [3] Bugeaud introduced the notion of multiplicatively badly approximable matrices, which is a strengthening of the usual notion of badly approximable matrices. Let $A = (\alpha_{mn})$ be an $M \times N$ matrix with entries in $\mathbf{R}/\mathbf{Z}$. Recall that A is *badly approximable* if there exists a positive constant $\beta(A)$ such that

$$\beta(A) \le \left(\max_{1\le m\le M} \left\|\sum_{n=1}^{N} \alpha_{mn}\xi_n\right\|\right)^M \left(\max_{1\le n\le N} |\xi_n|\right)^N \tag{7}$$

for all $\boldsymbol{\xi} \in \mathbf{Z}^N \setminus \{\mathbf{0}\}$. Here $\|\alpha\|$ denotes the distance from α to the nearest integer. It is known that a matrix A is badly approximable if and only if its transpose A^t is badly approximable. Furthermore, badly approximable matrices abound: the set of all $M \times N$ badly approximable matrices has full Hausdorff dimension in $(\mathbf{R}/\mathbf{Z})^{MN}$, (see [12]).

We say that A is *multiplicatively badly approximable* if there exists a positive constant $\gamma(A)$ such that

$$\gamma(A) \le \left(\prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\|\right) \left(\prod_{n=1}^{N} (|\xi_n| + 1)\right) \tag{8}$$

for all $\boldsymbol{\xi} \in \mathbf{Z}^N \setminus \{\mathbf{0}\}$. Clearly if A is multiplicatively badly approximable, then it is badly approximable. If $M = N = 1$ and $A = (\alpha)$, then A is multiplicatively badly approximable if α is a badly approximable number. If $M = 2$, $N = 1$ and $A = (\alpha\ \beta)$, then A is multiplicatively badly approximable if and only if there is a positive constant γ such that

$$\gamma \le \|k\alpha\|\|k\beta\|(|k| + 1)$$

for all $k \in \mathbf{Z} \setminus \{0\}$. That is, (α, β) is a counterexample to Littlewood's conjecture. Since Littlewood's conjecture is widely believed to be true, it is likely that 2×1 (and *a fortiori*, $M \times N$ for all $M + N \ge 3$) multiplicatively badly approximable matrices do not exist.

Bugeaud [3, Section 2.2] suggested a *transference principle* for multiplicatively badly approximable matrices, namely that A is multiplicatively badly approximable if and only if its transpose A^t is multiplicatively badly approximable. This transference principle was proved by German [7]. In [10], we gave another proof of the transference principle. We also found a connection between multiplicatively badly approximable matrices and certain inequalities involving sums of fractional parts of linear forms.

In this note, we consider the notion of multiplicatively badly approximable matrices with entries in $\mathbf{T}$. We prove an explicit criterion for multiplicatively badly approximable matrices in this setting, from which the transference principle readily follows. We also show that if K is infinite, then multiplicatively badly approximable matrices *do exist*.

Consider an $M \times N$ matrix $A = (\alpha_{mn})$ with entries in $\mathbf{T}$. We say that A is *multiplicatively badly approximable* if there exists a constant $C_1 = C_1(A) > 0$ such that

$$e^{-C_1} < \left(\prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\|\right) \left(\prod_{n=1}^{N} \max(|\xi_n|, 1)\right) \tag{9}$$

for all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in K[x]^N \setminus \{\mathbf{0}\}$. As $\|F\| \le e^{-1}$, it is clear that if C_1 exists then it is necessarily greater than M.

For each nonnegative integer U we denote the set of all elements $\xi \in K[x]$ with $|\xi| < e^U$ by $\mathbf{G}_U$. Obviously $\mathbf{G}_U$ is a vector subspace of $K[x]$ of dimension U. Given nonnegative integers $U_1, \ldots, U_N$, and $V_1, \ldots, V_M$, let us consider the set of all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ in $K[x]^N$ satisfying the conditions

$$|\xi_n| < e^{U_n} \text{ for each } n = 1, \ldots, N, \text{ and} \tag{10}$$

$$\left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\| < e^{-V_m} \text{ for each } m = 1, \ldots, M. \tag{11}$$

Clearly the solutions to the systems (10) and (11) form a vector subspace of

$$\mathbf{G}_{U_1} \times \cdots \times \mathbf{G}_{U_N} \subset K[x]^N,$$

which we denote by $\mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M)$. Furthermore, the conditions (11) determine $V_1 + \cdots + V_M$ equations whose unknowns are the coefficients of $\xi_1, \ldots, \xi_N$. Thus we immediately have the inequality

$$\dim \mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M) \ge \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M). \tag{12}$$

One may regard this as a generalization of Dirichlet's theorem.

2. A first characterization

We now prove our first characterization of multiplicatively badly approximable matrices in $\mathbf{T}$.

Proposition 1. *Let $A = (\alpha_{mn})$ be an $M \times N$ matrix with entries in $\mathbf{T}$, where $m = 1, 2, \ldots, M$ indexes rows and $n = 1, 2, \ldots, N$ indexes columns. Then A is multiplicatively badly approximable if and only if there exists a constant $C_2 \geq 0$ such that for all nonnegative integers $U_1, \ldots, U_N$ and $V_1 \ldots, V_M$, we have*

$$\dim \mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M) \leq \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M) + C_2. \tag{13}$$

In other words, A is multiplicatively badly approximable if and only if the inequality (12) is essentially best possible. One could regard Proposition 1 as an analog of [10, Lemma 4.2].

Proof. Suppose A is multiplicatively badly approximable. That is, there is a constant C_1 such that (9) holds for all $\boldsymbol{\xi} \in K[x]^N \setminus \{\mathbf{0}\}$. We first claim that

$$\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M) = \{\mathbf{0}\}$$

whenever

$$W_1 + \cdots + W_N \leq V_1 + \cdots + V_M + M - C_1.$$

Indeed, suppose there is $\boldsymbol{\xi}$ in $\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M)$, and $\boldsymbol{\xi} \neq \mathbf{0}$. By definition of $\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M)$, we have

$$\max(|\xi_n|, 1) \leq e^{W_n} \text{ for each } n = 1, \ldots, N, \text{ and} \tag{14}$$

$$\left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\| \leq e^{-V_m - 1} \text{ for each } m = 1, \ldots, M. \tag{15}$$

Therefore,

$$\left(\prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\| \right) \left(\prod_{n=1}^{N} \max(|\xi_n|, 1) \right) \leq e^{W_1 + \cdots + W_N - V_1 - \cdots - V_M - M} \tag{16}$$

This contradicts (9) if $W_1 + \cdots + W_N \leq V_1 + \cdots + V_M + M - C_1$.

We now show that (13) holds for

$$C_2 := C_1 - M, \tag{17}$$

(note that $C_2 > 0$). If $U_1 + \cdots + U_N \leq V_1 + \cdots + V_M - C_2$, then (13) is automatically true by what we have just proved. Suppose that

$$U_1 + \cdots + U_N \geq V_1 + \cdots + V_M - C_2.$$

We can find integers $0 \leq W_n \leq U_n$ such that

$$W_1 + \cdots + W_N = \max(V_1 + \cdots + V_M - C_2, 0).$$

Then $\mathbf{G}_{W_1} \times \cdots \times \mathbf{G}_{W_N}$ and $\mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M)$ are two vector subspaces of $\mathbf{G}_{U_1} \times \cdots \times \mathbf{G}_{U_N}$, whose intersection is $\mathcal{V}(W_1, \ldots, W_N; V_1, \ldots, V_M) = \{\mathbf{0}\}$. It follows that

$$\begin{aligned}\dim \mathcal{V}(U_1, \ldots, U_N; V_1, \ldots, V_M) &\leq U_1 + \cdots + U_N - (W_1 + \cdots + W_N)\\ &= U_1 + \cdots + U_N - \max(V_1 + \cdots + V_M - C_2, 0)\\ &\leq U_1 + \cdots + U_N - V_1 - \cdots - V_M + C_2\\ &\leq \max(0, U_1 + \cdots + U_N - V_1 - \cdots - V_M) + C_2\end{aligned}$$

as desired.

For the reverse direction, let us assume that (13) holds for some constant C_2. Let $\boldsymbol{\xi}$ be an arbitrary element in $K[x]^N \setminus \{\mathbf{0}\}$. Our goal is to find a lower bound for

$$P = \left(\prod_{m=1}^{M} \left\| \sum_{n=1}^{N} \alpha_{mn}\xi_n \right\|\right) \left(\prod_{n=1}^{N} \max(|\xi_n|, 1)\right).$$

Put $\max(|\xi_n|, 1) = e^{U_n}$ for $n = 1, \ldots, N$. First we observe that $\left\|\sum_{n=1}^{N} \alpha_{mn}\xi_n\right\| \neq 0$ for each $m = 1, \ldots, M$. Indeed, suppose for a contradiction that $\left\|\sum_{n=1}^{N} \alpha_{i,n}\xi_n\right\| = 0$ for some $1 \leq i \leq M$. Let W be an integer greater than C_2. Then for $V \geq 0$, the vector space $\mathcal{V}(U_1 + W + 1, \ldots, U_N + W + 1; 0, \ldots, 0, V, 0 \ldots, 0)$ (where V is in the i-th position) has dimension at least W (since it contains $f\boldsymbol{\xi}$, for each $f \in \mathbf{G}_W$). On the other hand, in view of (13), its dimension cannot exceed C_2 if V is sufficiently large.

Let us put $\left\|\sum_{n=1}^{N} \alpha_{mn}\xi_n\right\| = e^{-V_m}$, where V_m are nonnegative integers. Then we want to find a lower bound for

$$\sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m. \tag{18}$$

We can assume that (18) is negative, otherwise, we are already done. Let W be the smallest integer such that

$$\sum_{n=1}^{N} (W + U_n) - \sum_{m=1}^{M} \max(0, V_m - W) \geq 0. \tag{19}$$

Clearly W exists, and $W \geq 1$. From (19) we immediately have

$$\sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m \geq -(M + N)W. \tag{20}$$

By minimality of W, we get

$$\sum_{n=1}^{N} (W - 1 + U_n) - \sum_{m=1}^{M} \max(0, V_m - W + 1) < 0.$$

Upon observing that $\max(0, V_m - W + 1) \leq \max(0, V_m - W) + 1$, we find that

$$\sum_{n=1}^{N} (W + U_n) - \sum_{m=1}^{M} \max(0, V_m - W) < M + N.$$

Let us now consider the space

$$\mathcal{V}(W + U_1, \ldots, W + U_N; \max(0, V_1 - W), \ldots, \max(0, V_M - W)).$$

On the one hand, by hypothesis, its dimension is at most

$$\max\left(0, \sum_{n=1}^{N}(W+U_n) - \sum_{m=1}^{M}\max(0, V_m - W)\right) + C_2 < (M+N) + C_2.$$

On the other hand, its dimension is at least W, since it contains $f\boldsymbol{\xi}$ for each $f \in \mathbf{G}_W$. Therefore,

$$W < (M+N) + C_2. \tag{21}$$

Combining (20) and (21), we see that (9) holds with

$$C_1 := (M+N)^2 + (M+N)C_2. \tag{22}$$

□

Remark 2.1. One can contrast the values of C_1 and C_2 given by (17) and (22). It is an interesting problem to determine if these values are tight.

3. A matrix interpretation

The pleasantness of working in L is that we can write out each element of L in terms of its coordinates and express Diophantine inequalities in terms of linear equations. For each element $\alpha = \sum_{i=1}^{\infty} a_i x^{-i} \in \mathbf{T}$, and nonnegative integers U and V, let us denote by $\mathcal{M}_{U,V}(\alpha)$ the matrix

$$\mathcal{M}_{U;V}(\alpha) = \begin{pmatrix} a_1 & a_2 & \cdots & a_U \\ a_2 & a_3 & \cdots & a_{U+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_V & a_{V+1} & \cdots & a_{U+V-1} \end{pmatrix}. \tag{23}$$

In particular, when $U = 0$ or $V = 0$, $\mathcal{M}_{U;V}(\alpha)$ is the empty matrix.

Given nonnegative integers $U_1, \dots, U_N, V_1, \dots, V_M$ and an $M \times N$ matrix $A = (\alpha_{mn})$ with entries in $\mathbf{T}$, the conditions (10) and (11) represent a system of $V_1 + \cdots + V_M$ linear equations in $U_1 + \cdots + U_N$ variables, which can be written out explicitly as follows. Suppose that

$$\alpha_{mn} = \sum_{i=1}^{\infty} a_i^{(mn)} x^{-i} \qquad (1 \le m \le M, 1 \le n \le N). \tag{24}$$

Let us write

$$\xi_n = \sum_{j=0}^{U_n - 1} t_j^{(n)} x^j \qquad (1 \le n \le N) \tag{25}$$

where we regard the $t_j^{(n)}$ as variables. Then the conditions (10), (11) amount to the system

$$\sum_{n=1}^{N} \sum_{j=0}^{U_n-1} t_j^{(n)} a_{i+j}^{(mn)} = 0$$

for all $i = 1, \dots, V_m$ and $m = 1, \dots, M$. It is straightforward to see that the matrix of this system is

$$\mathcal{M}_{U_1,\dots,U_N;V_1,\dots,V_M}(A) = \begin{pmatrix} \mathcal{M}_{U_1;V_1}(\alpha_{11}) & \mathcal{M}_{U_2;V_1}(\alpha_{12}) & \cdots & \mathcal{M}_{U_N;V_1}(\alpha_{1N}) \\ \mathcal{M}_{U_1;V_2}(\alpha_{21}) & \mathcal{M}_{U_2;V_1}(\alpha_{22}) & \cdots & \mathcal{M}_{U_N;V_1}(\alpha_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_{U_1;V_M}(\alpha_{M1}) & \mathcal{M}_{U_2;V_M}(\alpha_{M2}) & \cdots & \mathcal{M}_{U_N;V_M}(\alpha_{MN}) \end{pmatrix}. \tag{26}$$

We thus arrive at a further characterization of multiplicatively badly approximable matrices.

Proposition 2. *The $M \times N$ matrix A is multiplicatively badly approximable if and only if there is a constant $C_2 \geq 0$ such that for all nonnegative integers $U_1, \ldots, U_N, V_1, \ldots, V_M$, the matrix $\mathcal{M}_{U_1,\ldots,U_N;V_1,\ldots,V_M}(A)$ defined in (26) has rank at least*

$$\min\left(\sum_{n=1}^{N} U_n, \sum_{m=1}^{M} V_m\right) - C_2. \tag{27}$$

Proof. This follows from Proposition 1, the rank-nullity theorem, and the fact that

$$\sum_{n=1}^{N} U_n - \max\left(0, \sum_{n=1}^{N} U_n - \sum_{m=1}^{M} V_m\right) - C_2 = \min\left(\sum_{n=1}^{N} U_n, \sum_{m=1}^{M} V_m\right) - C_2. \tag{28}$$

□

It is easy to see that the transpose of $\mathcal{M}_{U_1,\ldots,U_N;V_1,\ldots,V_M}(A)$ is $\mathcal{M}_{V_1,\ldots,V_M;U_1,\ldots,U_N}(A^t)$. Thus from Proposition 2, we immediately have the following transference principle:

Theorem 1. *A matrix A with entries in $\mathbf{T}$ is multiplicatively badly approximable if and only if its transpose A^t is multiplicatively badly approximable.*

Remark 3.1. Propositions 1 and 2 show that, if A is multiplicatively badly approximable, we can choose $C_1(A^t) = (M+N)^2 + (M+N)C_1(A)$, where $C_1(A)$ is the constant defined in (9). One can compare this to the bound (8.30) in [10].

Given Proposition 2, we will establish the existence of multiplicatively badly approximable matrices when K is infinite. This follows from the following stronger statement.

Theorem 2. *Suppose K is infinite. Then for each M and N, there exists an $M \times N$ matrix $A = (\alpha_{mn})$ with entries in $\mathbf{T}$ satisfying the following property.*

($*$) *For all nonnegative integers $U_1, \ldots, U_N$, $V_1, \ldots, V_M$ with $\sum_{n=1}^{N} U_n = \sum_{m=1}^{M} V_m$, the square matrix $\mathcal{M}_{U_1,\ldots,U_N;V_1,\ldots,V_M}(A)$ is non-singular.*

For $1 \times N$ matrices, this is a result of Bumby [4]. Our argument is a generalization of his.

Proof. We prove Theorem 2 by induction on $M+N$. When $M=0$ or $N=0$, the theorem is vacuously true. Suppose $M, N \geq 0$ and we know already the existence of an $M \times N$ matrix A' satisfying ($*$). We will show how to add one column to A' such that the new $M \times (N+1)$ matrix retains this property. By symmetry, we can also add one row to A'. In this way we can create $M \times N$ matrices satisfying ($*$) for each M and N. (Note that when $M = N = 0$, then this process creates badly approximable elements in L.)

Suppose the $M \times N$ matrix $A' = (\alpha_{mn})$ satisfies ($*$). We will find $\alpha_{m,N+1} \in \mathbf{T}$ $(1 \leq m \leq M)$ such that the $M \times (N+1)$ matrix $A = (\alpha_{mn})$ satisfies ($*$).

Suppose

$$\alpha_{mn} = \sum_{i=1}^{\infty} a_i^{(mn)} x^{-i} \qquad (1 \leq m \leq M, 1 \leq n \leq N+1).$$

Our goal is to construct M sequences

$$\left(a_i^{(m,N+1)}\right)_{i=1}^{\infty}, \qquad (1 \le m \le M),$$

such that for integers $U_1, \ldots, U_N, U_{N+1}$ and $V_1, V_2, \ldots, V_M$, with

$$U_1 + \cdots + U_N + U_{N+1} = V_1 + \cdots + V_M,$$

the square matrix

$$\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)$$
$$= \begin{pmatrix} a_1^{(11)} & \cdots & a_{U_1}^{(11)} & \cdots & a_1^{(N+1,1)} & \cdots & a_{U_{N+1}}^{(N+1,1)} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{V_1}^{(11)} & \cdots & a_{U_1+V_1-1}^{(11)} & \cdots & a_{V_1}^{(N+1,1)} & \cdots & a_{U_{N+1}+V_1-1}^{(N+1,1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_1^{(M1)} & \cdots & a_{U_1}^{(M1)} & \cdots & a_1^{(N+1,M)} & \cdots & a_{U_{N+1}}^{(N+1,M)} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{V_M}^{(M1)} & \cdots & a_{U_1+V_M-1}^{(M1)} & \cdots & a_{V_M}^{(N+1,M)} & \cdots & a_{V_M+U_{N+1}-1}^{(N+1,M)} \end{pmatrix}$$

is non-singular.

To this end, we will construct the M-tuples $\left(a_L^{(m,N+1)}\right)_{1\le m\le M}$ indexed by L recursively. Let us refer to the quantity $\max_{1\le m\le M}(U_{N+1}+V_m-1)$ as the *order* of the matrix

$$\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A).$$

Suppose all the tuples $\left(a_\ell^{(m,N+1)}\right)_{1\le m\le M}$, with $1 \le \ell \le L-1$, are already determined in such a way that all matrices of order smaller than L are non-singular. We want to find $\left(a_L^{(m,N+1)}\right)_{1\le m\le M}$ such that all the matrices $\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)$ satisfying

(i) $U_1 + \cdots + U_{N+1} = V_1 + \cdots + V_M$, and
(ii) $\max_{1\le m\le M}(U_{N+1}+V_m-1) = L$,

have non-zero determinant.

It is clear that the number of such matrices is finite. For all such matrices, by expanding along the last column, the non-zero determinant condition corresponds to an equation of the form

$$\sum_{m=1}^{M} r_m a_L^{(m,N+1)} + r_0 \ne 0 \tag{29}$$

where $r_0, \ldots, r_M \in K$. For each $1 \le m \le M$, r_m is either 0 or the determinant of a matrix of lower order (it is 0 if $a_L^{(m,N+1)}$ is not present in $\mathcal{M}_{U_1,\ldots,U_N,U_{N+1};V_1,V_2,\ldots,V_M}(A)$), but at least one of them is nonzero. The number of equations of type (29) is finite. Since K is infinite, a choice of $\left(a_L^{(m,N+1)}\right)_{1\le m\le M}$ is certainly possible. $\square$

4. Explicit examples of multiplicatively badly approximable matrices

In this section, we consider the problem of giving explicit examples of multiplicatively badly approximable matrices. We also assume in this section that K has characteristic zero. As mentioned earlier, Baker [2] gave explicit counterexamples to Littlewood's conjecture in this case. He also pointed out that the same method can be used to show that each $1 \times N$ matrix of the form

$$A = \begin{pmatrix} e^{\lambda_1/x} & e^{\lambda_2/x} & \cdots & e^{\lambda_N/x} \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_N$ are distinct elements of K, is (in our terminology) multiplicatively badly approximable. Here for each $\lambda \in K$, $e^{\lambda/x}$ is the formal Laurent series

$$e^{\lambda/x} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} x^{-i}.$$

In a different context, Jager [8] studied the notion of *perfect systems* of power series (see the definition in [8, p. 196]). This notion was developed by Mahler [11] and inspired by Hermite's proof of the transcendence of e. By examining the underlying linear equations, it is easy to see that if the system $(f_1(x), \dots, f_M(x))$ is perfect (where each f_i is a power series of the form $f_i(x) = \sum_{j=0}^{\infty} a_j^{(i)} x^i$), then the matrix

$$A = \begin{pmatrix} f_1\left(x^{-1}\right) \\ f_2\left(x^{-1}\right) \\ \vdots \\ f_M\left(x^{-1}\right) \end{pmatrix}$$

is multiplicatively badly approximable.

Jager then gave some examples of perfect systems. If $\lambda_1, \dots, \lambda_N$ are distinct elements of K, then the system $(e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_N x})$ is perfect. Coupled with the transference principle, this recovers Baker's result with a simpler proof. He also showed that if $w_1, \dots, w_N$ are elements of K, no two of which differ by an integer, then the system $((1-x)^{w_1}, \dots, (1-x)^{w_N})$ is also perfect. Here $(1-x)^w$ is the formal power series

$$(1-x)^w = \sum_{i=0}^{\infty} (-1)^i \binom{w}{i} x^i.$$

This gives another example of $1 \times N$ (hence $N \times 1$) multiplicatively badly approximable matrices. However, it seems to us that neither Baker nor Jager's method can be extended to give explicit examples of $M \times N$ for *some* $M, N > 1$. It is therefore an interesting problem to give explicit examples of multiplicatively badly approximable matrices for arbitrary dimensions.

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