

PARTITION REGULARITY AND THE PRIMES

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ABSTRACT. The purpose of this note is to point out, as a simple yet nice consequence of Green and Tao's program on counting linear patterns in the primes and Deuber's work on partition regularity, that if a system of equations is partition regular over the positive integers, then it is also partition regular over the sets $\{p - 1 : p \text{ prime}\}$ as well as $\{p + 1 : p \text{ prime}\}$. This answers a question of Li and Pan.

1. INTRODUCTION

Earliest results in arithmetic Ramsey theory include Schur's theorem and van der Waerden's theorem. In 1916 Schur proved that if the positive integers are finitely colored (or partitioned), then there are x, y, z in the same color class such that $x + y = z$. In 1927 van der Waerden proved that if the positive integers are finitely colored, then for any k , we can find a monochromatic arithmetic progression of length k . We can even require further that the step of the arithmetic progression belong to the same color class (Brauer [2]). More generally, given a $u \times v$ matrix with integer entries A , we say that A is (*kernel*) *partition regular* over a set $X \subset \mathbf{Z} \setminus \{0\}$ if under any finite coloring of X , we can always find a vector $\vec{x} \in X^v$, all of whose components are of the same color, such that $A\vec{x} = \vec{0}$.

The problem of determining all partition regular matrices over \mathbf{Z}^+ was settled by Rado [13]. He proved that A is partition regular over \mathbf{Z}^+ if and only if it satisfies the *columns condition*. Let the columns of A be $\vec{c}_1, \dots, \vec{c}_v$. Then A satisfies the columns condition if there exist $m \in \{1, 2, \dots, v\}$ and a partition $\{I_1, I_2, \dots, I_m\}$ of $\{1, 2, \dots, v\}$ into nonempty sets such that

- (1) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$.
- (2) for each $t \in \{2, \dots, m\}$, $\sum_{i \in I_t} \vec{c}_i$ is a linear combination over \mathbf{Q} of $\{\vec{c}_i : i \in \cup_{j=1}^{t-1} I_j\}$.

On the other hand, the closely related question of determining linear forms that can be found in color classes was addressed by Hindman and Leader [11]. They completely determined $v \times u$ matrices B such that under any finite coloring of \mathbf{Z}^+ we can always find a vector $\vec{y} \in \mathbf{Z}^u$ such that the components of $B\vec{y}$ are monochromatic. Such matrices are called *image partition regular* (over \mathbf{Z}^+). We refer to the survey of Hindman [10] in which problems on this theme are discussed in depth.

Let us call a set $X \subset \mathbf{Z}$ *large* if it contains a solution to all partition regular matrix over \mathbf{Z}^+ (that is, for any partition regular matrix A over \mathbf{Z}^+ , there exists a vector \vec{x} , all of whose components are in X , such that $A\vec{x} = \vec{0}$). Thus Rado's theorem says that under any finite partition of \mathbf{Z}^+ , one of the sets is large. Rado conjectured the following, which was confirmed by Deuber:

Theorem 1. *If a large set is partitioned into a finite number of sets, then again one of these sets is large.*

In his resolution of Rado's conjecture, Deuber introduced the notion of (m, p, c) -sets. Given positive integers m, p, c, x_1, \dots, x_m , the (m, p, c) -set generated by x_1, \dots, x_m is the set of all numbers of the form $cx_i + \lambda_{i+1}x_{i+1} + \dots + \lambda_mx_m$ where $1 \leq i \leq m, -p \leq \lambda_j \leq p$ for all $j = i + 1, \dots, m$. Thus an (m, p, c) -set is a generalization of an arithmetic progression, as well as the sumset of a finite set.

One of the ingredients of Deuber's resolution of Rado's conjecture is the following

Theorem 2. *A set $X \subset \mathbf{Z}$ is large if and only if it contains an (m, p, c) -set for every $m, p, c \in \mathbf{Z}^+$.*

We now connect these results to the primes. Let \mathbf{P} be the set of all primes. The following is a deep result of Green and Tao's program on counting linear patterns in the primes (Green-Tao [7, 8], Green-Tao-Ziegler [9]):

Theorem 3. *Let $\psi_1, \dots, \psi_l : \mathbf{Z}^r \rightarrow \mathbf{Z}^l$ be affine-linear forms with integer coefficients, $\psi_i(\vec{x}) = \sum_{j=1}^r m_{i,j}x_j + c_i$, no two of which are affinely dependent. Then there exists $\vec{n} \in \mathbf{Z}^r$ such that $\psi_1(\vec{n}), \dots, \psi_l(\vec{n})$ are all primes if and only if for any $k \in \mathbf{Z}, k \geq 2$, there exists an $\vec{x} \in \mathbf{Z}^r$ such that $\psi_1(\vec{x}), \dots, \psi_l(\vec{x})$ are all non-divisible by k .*

From this we easily deduce

Corollary 1. *The sets $\mathbf{P} - 1 = \{p - 1 : p \text{ prime}\}$ and $\mathbf{P} + 1 = \{p + 1 : p \text{ prime}\}$ contain an (m, p, c) -set for every $m, p, c \in \mathbf{Z}^+$.*

By Theorem 2, this implies that the sets $\mathbf{P} - 1$ and $\mathbf{P} + 1$ are large. By Theorem 1, this means that under any finite coloring of $\mathbf{P} - 1$ (or $\mathbf{P} + 1$), there exists a color class that is large. Thus we have proved the following:

Theorem 4. *If a matrix A is partition regular over \mathbf{Z}^+ , then it is partition regular over $\mathbf{P} - 1$ and $\mathbf{P} + 1$.*

As a consequence, we have

Corollary 2. *If the primes are finitely colored, then for any k we can find a monochromatic arithmetic progression of length k . Furthermore, if d is the step of the arithmetic progression, then we can require that $d + 1$ be prime, and of the same color as the arithmetic progression.*

Corollary 2 is Conjecture 1.1 in Li-Pan [12]. In the same paper, Li and Pan proved the case $k = 2$, based on Green's transference principle [5], coupled with a quantitative version of Schur's theorem. In the same spirit, one can use a coloring version of Green-Tao's transference principle in [6], which is readily available [3, Lemma 2.6], coupled with a quantitative version of Theorem 1, to prove a quantitative version of Theorem 4. It is more convenient to state this quantitative version in terms of image partition regular matrices.

Theorem 5. *Let $B : \mathbf{Z}^r \rightarrow \mathbf{Z}^l$ be an image partition regular matrix. For a vector $\vec{x} \in \mathbf{Z}^r$, write $B\vec{x} = \begin{pmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \\ \vdots \\ \psi_l(\vec{x}) \end{pmatrix}$. Then, under any finite coloring of $\mathbf{P} - 1$, there is a color class C such that*

$$\sum_{\vec{x} \in \mathbf{Z}^r, \|\vec{x}\|_\infty \leq N} f(\psi_1(\vec{x})) \cdots f(\psi_l(\vec{x})) \gg N^{r+l}$$

where $f(x) = 1_C(x)\Lambda(x+1)$ and Λ is the von Mangoldt function. Here the implied constant only depends on B and the number of colors used.

In other words, the number of monochromatic configurations $\psi_1(\vec{x}), \dots, \psi_l(\vec{x})$ is as large as one can hope for in order of magnitude. We won't prove Theorem 5 since it's only slightly stronger than Theorem 4, and utilizes the same machinery (specifically, the fact that (a variant of) the function Λ is very close to 1 in Gowers norms).

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