ON A QUESTION OF SÁRKÖZY ON GAPS OF PRODUCT SEQUENCES

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Abstract. Motivated by a question of Sárközy, we study the gaps in the product sequence $\mathcal{B} = \mathcal{A} \cdot \mathcal{A} = \{b_1 < b_2 < \cdots\}$ of all products $a_i a_j$ with $a_i, a_j \in \mathcal{A}$ when $\mathcal{A}$ has upper Banach density $\alpha > 0$. We prove that there are infinitely many gaps $b_{n+1} - b_n \ll \alpha^{-3}$ and that for $t \geq 2$ there are infinitely many t-gaps $b_{n+t} - b_n \ll t^2\alpha^{-4}$. Furthermore we prove that these estimates are best possible.

We also discuss a related question about the cardinality of the quotient set $\mathcal{A}/\mathcal{A} = \{a_i/a_j,\ a_i, a_j \in \mathcal{A}\}$ when $\mathcal{A} \subset \{1, \ldots, N\}$ and $|\mathcal{A}| = \alpha N$.

1. Introduction

Let $\mathcal{A} = \{a_1 < a_2 < \ldots\}$ be an infinite sequence of positive integers. The lower and upper asymptotic densities of $\mathcal{A}$ are defined by

$$\underline{d}(\mathcal{A}) = \liminf_{N\to\infty} \frac{|\mathcal{A} \cap \{1, \ldots, N\}|}{N} \quad \text{and} \quad \overline{d}(\mathcal{A}) = \limsup_{N\to\infty} \frac{|\mathcal{A} \cap \{1, \ldots, N\}|}{N}.$$

The lower and upper Banach density of $\mathcal{A}$ are defined by

$$d_*(\mathcal{A}) = \liminf_{|I|\to\infty} \frac{|\mathcal{A} \cap I|}{|I|} \quad \text{and} \quad d^*(\mathcal{A}) = \limsup_{|I|\to\infty} \frac{|\mathcal{A} \cap I|}{|I|}$$

where I runs through all intervals. Clearly $d_*(\mathcal{A}) \leq \underline{d}(\mathcal{A}) \leq \overline{d}(\mathcal{A}) \leq d^*(\mathcal{A})$.

Sárközy considered the set

$$\mathcal{B} = \mathcal{A} \cdot \mathcal{A} = \{b_1 < b_2 < \ldots\}$$

of all products $a_i a_j$ with $a_i, a_j \in \mathcal{A}$ and asked the following question, stated as problem 22 in [4].

Question 1. *Is it true that for all $\alpha > 0$ there is a number $c = c(\alpha) > 0$ such that if $\mathcal{A} \subset \mathbb{N}$ is an infinite sequence with $\underline{d}(\mathcal{A}) > \alpha$, then $b_{n+1} - b_n \leq c$ holds for infinitely many n?*

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This question is not trivial, since for any $0 < \alpha < 1$ and $\epsilon > 0$ there is a sequence $\mathcal{A}$ such that $\underline{d}(\mathcal{A}) > \alpha > 0$ but $\bar{d}(\mathcal{B}) < \epsilon$, thus the gaps of $\mathcal{B}$ are greater than $\frac{1}{\epsilon}$ on average. See the construction in [1].

Bérczi [1] answered Sárközy's question in the affirmative by proving that we can take $c(\alpha) \ll \alpha^{-4}$. Sándor [3] improved it to $c(\alpha) \ll \alpha^{-3}$ even assuming the weaker hypothesis $\overline{d}(\mathcal{A}) > \alpha$.

In this work we consider Sárközy's question for the upper Banach density, that is to find a constant $c^*(\alpha)$ such that $b_{n+1} - b_n \le c^*(\alpha)$ infinitely often whenever $d^*(\mathcal{A}) > \alpha$. In this setting we can find the best possible value for $c^*(\alpha)$ up to a multiplicative constant.

Theorem 1. *For every $0 < \alpha < 1$ and every sequence $\mathcal{A}$ with $d^*(\mathcal{A}) > \alpha$, we have $b_{n+1} - b_n \ll \alpha^{-3}$ infinitely often.*

Theorem 2. *For every $0 < \alpha < 1$, there exists a sequence $\mathcal{A}$ with $d^*(\mathcal{A}) > \alpha$ and such that $b_{n+1} - b_n \gg \alpha^{-3}$ for every n.*

We observe that, since $d^*(\mathcal{A}) \ge \overline{d}(\mathcal{A})$, Theorem 1 is stronger than Sándor's result.

We also extend this question and study the difference $b_{n+t} - b_n$ for a fixed t, namely to find a constant $c^*(\alpha, t)$ such that $b_{n+t} - b_n \le c^*(\alpha, t)$ infinitely often. Theorems 1 and 2 above correspond to the case $t = 1$. For greater t the answer is perhaps surprising, in that the exponent of α involved in $c^*(\alpha, t)$ is -4, not -3 like in the case $t = 1$.

Theorem 3. *For every $0 < \alpha < 1$, every $t \ge 2$ and every sequence $\mathcal{A}$ with $d^*(\mathcal{A}) > \alpha$, we have $b_{n+t} - b_n \ll t^2\alpha^{-4}$ infinitely often.*

Theorem 4. *For every $0 < \alpha < 1$ and every $t \ge 2$, there is a sequence $\mathcal{A}$ such that $d^*(\mathcal{A}) > \alpha$ and $b_{n+t} - b_n \gg t^2\alpha^{-4}$ for every n.*

The method of proof for Theorems 1 and 3 is related to the Erdős-Turán method in Sidon sets theory. Sidon sets are also the main tool in the constructions involved in Theorems 2 and 4.

Notation. We will denote by $\lceil x \rceil$ the smallest integer greater or equal to x, $\lfloor x \rfloor$ the greatest integer small than or equal to x. For quantities A, B we write $A \ll B$, or $B \gg A$ if there is an absolute constant $c > 0$ such that $A \le cB$.

2. Proof of the results

In our proofs of Theorems 1, 3 we will use the following simple observation:

Lemma 1. *Let K be a positive integer, α a real number with $0 < \alpha < 1$. Then, if $d^*(\mathcal{A}) > \alpha$, there exist infinitely many pairwise disjoint intervals I of length K such that $|\mathcal{A} \cap I| \ge \alpha|I|$.*

Proof. Suppose for a contradiction, there exists at most a finite number of intervals I of length K with $|\mathcal{A} \cap I| \geq \alpha K$. Thus, there exists N such that if $I \cap [1, N] = \emptyset$ and $|I| = K$ then $|\mathcal{A} \cap I| < \alpha |I|$.

Any interval J can be written as an union of disjoint consecutive intervals

$$J = J_0 \cup J_1 \cup \cdots \cup J_r \cup J_{r+1},$$

where $J_0 = J \cap [1, N], \ |J_i| = K, \ i = 1, \dots, r$ and $|J_{r+1}| \leq K$.

We observe that

$$\begin{aligned} \frac{|\mathcal{A} \cap J|}{|J|} &= \frac{|\mathcal{A} \cap J_0| + |\mathcal{A} \cap J_1| + \cdots + |\mathcal{A} \cap J_r| + |\mathcal{A} \cap J_{r+1}|}{|J|} \\ &< \frac{N}{|J|} + \frac{\alpha(|J_1| + \cdots |J_r|)}{|J|} + \frac{K}{|J|} < \frac{N+K}{|J|} + \alpha. \end{aligned}$$

Since $\lim_{|J| \to \infty} \frac{N+K}{|J|} = 0$ we obtain that $d^*(\mathcal{A}) = \limsup_{|J| \to \infty} \frac{|\mathcal{A} \cap J|}{|J|} \leq \alpha$, a contradiction.

Finally, it is clear that if there exist infinitely many intervals I of length K with $|\mathcal{A} \cap I| \geq \alpha |I|$, there exist infinitely many of them which are pairwise disjoint. □

Proof of Theorem 1. Let $L = \lceil 2\alpha^{-1} \rceil$. Since $d^*(\mathcal{A}) > \alpha$, the above lemma with $K = L^2$ implies that there are infinitely many disjoint intervals I of length L^2 such that $|I \cap \mathcal{A}| \geq \alpha L^2$.

We divide each interval I into L subintervals of equal length L. For $i = 1, \dots, L$, let A_i be the number of elements of $\mathcal{A}$ in the i-th interval. We count the number of differences $a - a'$ where $0 < a' < a$ are in the same interval. On the one hand, it is

$$\begin{aligned} \sum_{1 \leq i \leq L} \binom{A_i}{2} &= \frac{1}{2} \sum_{1 \leq i \leq L} (A_i^2 - A_i) \geq \frac{1}{2} \left(\frac{1}{L} \left(\sum_{1 \leq i \leq L} A_i \right)^2 - \sum_{1 \leq i \leq L} A_i \right) \\ &= \frac{1}{2} \left(\frac{|\mathcal{A} \cap I|^2}{L} - |\mathcal{A} \cap I| \right) = \frac{|\mathcal{A} \cap I|}{2} \left(\frac{|\mathcal{A} \cap I|}{L} - 1 \right) \\ &\geq \frac{|\mathcal{A} \cap I|}{2} (\alpha L - 1) = \frac{|\mathcal{A} \cap I|}{2} \left(\alpha \lceil 2\alpha^{-1} \rceil - 1 \right) \\ &\geq \frac{|\mathcal{A} \cap I|}{2} \geq \frac{\alpha L^2}{2} \geq L. \end{aligned}$$

On the other hand, the number of their possible values is at most $L-1$. Thus we can find 2 couples $(a,a'),(a'',a''')$ such that $0<a-a'=a''-a'''<L$. Then

$$\begin{aligned} 0<|aa'''-a'a''| &= |a(a''+a'-a)-a'a''| \\ &= |(a-a')(a''-a)| \\ &\le (L-1)(L^2-1)=(L-1)^2(L+1) \\ &= (\lceil 2\alpha^{-1}\rceil-1)^2(\lceil 2\alpha^{-1}\rceil+1) \\ &\le 4\alpha^{-2}(2\alpha^{-1}+2) \\ &< 4\alpha^{-2}(2\alpha^{-1}+2\alpha^{-1})=16\alpha^{-3}. \end{aligned}$$

Thus, each interval I provides two distinct elements of $\mathcal{B}=\mathcal{A}\cdot\mathcal{A}$, say $b<b'$, with $b'-b<16\alpha^{-3}$. Since there are infinitely many such intervals and they are pairwise disjoint, we conclude that $b_{n+1}-b_n<16\alpha^{-3}$ infinitely often. □

Proof of Theorem 3. Let $L=\lceil 4t\alpha^{-2}\rceil$. Again, since $d^*(\mathcal{A})>\alpha$, we can apply Lemma 1 with $K=L$ to deduce that there exist infinitely many intervals I of length L which contain at least αL elements of $\mathcal{A}$.

For each interval I, the number of sums $a+a',\ a\le a',\ a,a'\in I\cap\mathcal{A}$ is greater than $(\alpha L)^2/2$ and they are all contained in an interval of length $2L$.

Since $\frac{(\alpha L)^2}{2}=2L\left(\frac{\alpha^2 L}{4}\right)=2L\left(\frac{\alpha^2\lceil 4t\alpha^{-2}\rceil}{4}\right)\ge 2Lt$, the pigeonhole principle implies that some sum s must be obtained in at least $t+1$ different ways,

$$s=a_1+a_1'=\cdots=a_{t+1}+a_{t+1}',\quad a_i,a_i'\in I\cap\mathcal{A},\quad a_j\ne a_i,a_i' \text{ for } i\ne j.$$

If $i\ne j$, since $a_j+a_j'=a_i+a_i'$, we have

$$0<|a_ia_i'-a_ja_j'|=|a_ia_i'-a_j(a_i+a_i'-a_j)|=|(a_i-a_j)(a_i'-a_j)|<L^2,$$

so the $t+1$ products a_ia_i' lie in an interval of length

$$L^2<(4t\alpha^{-2}+1)^2\le(5t\alpha^{-2})^2\le 25t^2\alpha^{-4}.$$

As in the proof of theorem 1, each interval I provides $t+1$ distinct elements of $\mathcal{B}=\mathcal{A}\cdot\mathcal{A}$, say $b_{i_0}<\cdots<b_{i_t}$, such that $b_{i_t}-b_{i_0}<25t^2\alpha^{-4}$. Since there are infinitely many such intervals and they are pairwise disjoint, we can conclude that $b_{n+t}-b_n<25t^2\alpha^{-4}$ infinitely many times. □

In the proofs of Theorems 2 and 4, we will take $\mathcal{A}$ to be a union of blocks sufficiently far apart from one another, so that small differences $b_{i+1}-b_i$ (or $b_{i+t}-b_i$) can only arise when the b_i in question are made up from elements in the same block. To make this precise let us make the following:

Definition 1. Given a positive value x_1 and an infinite sequence of finite sets of nonnegative integers $\mathcal{A}_1, \mathcal{A}_2, \dots,$ we define a sequence $\mathcal{A}$ associated to these inputs by

$$\mathcal{A} = \bigcup_{n=1}^{\infty} (x_n + \mathcal{A}_n), \tag{1}$$

where the sequence (x_n) is defined for $n \geq 2$ by

$$x_n = x_1 + M_n^2 + M_n(x_{n-1} + M_{n-1}) + (x_{n-1} + M_{n-1})^2 \tag{2}$$

and M_n is the largest element of $\mathcal{A}_n$.

Clearly all the sets $x_n + \mathcal{A}_n$ in (1) are disjoint. Let us now verify that small gaps in $\mathcal{B}$ can only come from products of elements in the same block $x_n + \mathcal{A}_n$.

Lemma 2. *Let $\mathcal{A}$ be defined as in (1). Then, all the nonzero differences $d = c_1c_2 - c_3c_4$, with $c_1, c_2, c_3, c_4 \in \mathcal{A}$ but not all c_i in the same $x_n + \mathcal{A}_n$, satisfy $|d| \geq x_1$.*

Proof. Let n be the largest integer such that $c_i \in x_n + \mathcal{A}_n$ for some $i = 1, 2, 3, 4$. We can assume that $c_1 \in \mathcal{A}_n$. Then there are many possibilities for c_2, c_3, c_4. It is a routine to check that the inequality $|d| \geq x_1$ holds in all these cases. We will use repeatedly the definition of x_n in (2) and the fact that if $c \in x_m + \mathcal{A}_m$ then $x_m \leq c \leq x_m + M_m$.

i) $c_2 \in x_n + \mathcal{A}_n$ and c_3 or $c_4 \notin x_n + \mathcal{A}_n$. In this case

$$\begin{aligned} |d| &\geq x_n^2 - |c_3c_4| \\ &\geq x_n^2 - (x_n + M_n)(x_{n-1} + M_{n-1}) \\ &= x_n(x_n - x_{n-1} - M_{n-1}) - M_n(x_{n-1} + M_{n-1}) \\ &\geq x_n - M_n(x_{n-1} + M_{n-1}) \geq x_1. \end{aligned}$$

ii) $c_2, c_3, c_4 \notin x_n + \mathcal{A}_n$. In this case

$$|d| \geq x_n - c_3c_4 \geq x_n - (x_{n-1} + M_{n-1})^2 \geq x_1.$$

iii) $c_3 \in x_n + \mathcal{A}_n$ and $c_2, c_4 \notin x_n + \mathcal{A}_n$.
In this case we write $c_1 = x_n + a_1$ and $c_3 = x_n + a_3$. Then

$$|d| = |x_n(c_2 - c_4) + a_1c_2 - a_3c_4|.$$

If $c_2 = c_4$, then $|d| = c_2|a_1 - a_3| \geq x_1$.
If $c_2 \neq c_4$, then

$$|d| \geq x_n - |a_1c_2 - a_3c_4| \geq x_n - M_n(x_{n-1} + M_{n-1}) \geq x_1,$$

since $|a_1c_2 - a_3c_4| \leq \max\{a_1c_2, a_3c_4\} \leq M_n(x_{n-1} + M_{n-1})$.

$\square$

In order to prove Theorems 2 and 4, we also need the following construction of Sidon sets due to Erdős and Turán [2]:

Lemma 3. *Let p be an odd prime number. Let*

$$\mathcal{S}_p = \{s_i = 2pi + (i^2)_p : i = 0, \dots, p-1\},$$

where $(x)_p \in [0, p-1]$ is the residue of x modulo p. Then $\mathcal{S}_p$ is a Sidon set in $[0, 2p^2)$ with p elements and $|s_i - s_j| \geq p$ for every $i \neq j$.

Proof. It is clear that

$$|s_i - s_j| \geq 2p|i-j| - |(i^2)_p - (j^2)_p| \geq p.$$

Suppose we have $s_i + s_j = s_k + s_l$ for some i, j, k, l. Then

$$2p(i + j - k - l) = (i^2)_p + (j^2)_p - (k^2)_p - (l^2)_p.$$

The left hand side is a multiple of $2p$ while the absolute value of the right hand side is strictly smaller than $2p$. Thus

$$i - k = l - j$$

and

$$(i^2)_p - (k^2)_p = (l^2)_p - (j^2)_p,$$

i.e.,

$$i^2 - k^2 \equiv l^2 - j^2 \pmod{p}.$$

Thus

$$(i-k)(i+k) = (i-k)(l+j) \equiv 0 \pmod{p}.$$

Either $i = k$ and $j = l$, or $i + k \equiv l + j \pmod{p}$, in which case $k = l$ and $i = j$. □

Proof of Theorem 2. We can assume that $\alpha < 1/16$. Otherwise it is clear that all the gaps in $\mathcal{A} \cdot \mathcal{A}$ are $\geq 1 \gg \alpha^{-3}$.

Let p be an odd prime such that $\frac{1}{8\alpha} < p < \frac{1}{4\alpha}$, $\mathcal{S}_p$ the Sidon set defined in Lemma 3 and $m = 2p^2$. We consider the sequence $\mathcal{A}$ defined in (1) with $x_1 = 4p^3$ and

$$\mathcal{A}_n = \bigcup_{k=1}^{n} (2km + \mathcal{S}_p). \tag{3}$$

First we observe that $\mathcal{A}_n$ is contained in the interval $I_n = [2m, 2mn + m)$ and then

$$d^*(\mathcal{A}) \geq \limsup_{n \to \infty} \frac{|\mathcal{A}_n|}{|I_n|} = \limsup_{n \to \infty} \frac{|np|}{|(2m-1)n|} > \frac{1}{4p} \geq \alpha.$$

Next we will prove that all the nonzero differences $d = c_1c_2 - c_3c_4$ with $c_1, c_2, c_3, c_4 \in \mathcal{A}$ satisfy $|d| \geq 4p^3$, and clearly $|d| \geq 2^{-7}\alpha^{-3}$.

By Lemma 2 this is true when not all c_i belong to the same $x_n + \mathcal{A}_n$. Suppose then that $c_i = x_n + a_i,\ i = 1, 2, 3, 4$. Then

$$\begin{aligned} d &= (x_n + a_1)(x_n + a_2) - (x_n + a_3)(x_n + a_4) \\ &= x_n(a_1 + a_2 - a_3 - a_4) + a_1a_2 - a_3a_4. \end{aligned}$$

- If $a_1 + a_2 \neq a_3 + a_4$ then
$$|d| \geq x_n - |a_1a_2 - a_3a_4| \geq x_n - M_n^2 \geq x_1 = 4p^3.$$
- If $a_1 + a_2 = a_3 + a_4$ then
$$\begin{aligned} |d| &= |a_1a_2 - a_3a_4| \\ &= |a_1a_2 - a_3(a_1 + a_2 - a_3)| \\ &= |(a_2 - a_3)(a_1 - a_3)|. \end{aligned}$$
Now we write $a_i = 2k_im + s_i,\ 1 \leq k_i \leq n,\ s_i \in \mathcal{S}_p$. The condition $a_1 + a_2 = a_3 + a_4$ implies
$$2m(k_1 + k_2 - k_3 - k_4) = s_3 + s_4 - s_1 - s_2.$$
Since $|s_1+s_2-s_3-s_4| < 2m$, we have $k_1+k_2 = k_3+k_4$ and $s_1+s_2 = s_3+s_4$. Now we use the fact that $\mathcal{S}_p$ is a Sidon set to conclude that $\{s_1, s_2\} = \{s_3, s_4\}$. We can assume that $s_1 = s_3$ and $s_2 = s_4$, Then
$$|d| = |2m(k_2 - k_3) + (s_2 - s_3)||2m(k_1 - k_3)|.$$
  - If $s_2 = s_3$, since $d \neq 0$ we have that
  $$|d| \geq (2m)^2 \geq 16p^4 > 4p^3.$$
  - If $s_2 \neq s_3$, by Lemma 3 we know that
  $$p \leq |s_2 - s_3| < m.$$
    * If $k_2 \neq k_3$ then $|d| \geq |2m - m||2m| = 2m^2 = 8p^4 > 4p^3$.
    * If $k_2 = k_3$ then $|d| \geq p(2m) = 4p^3$.

In any case $|d| \geq 4p^3$. □

Proof of Theorem 4. For $\alpha \geq 1/16$ we consider the sequence $\mathcal{A}$ defined in (1) with $x_1 = t^2$ and $\mathcal{A}_n = \{1, \ldots, n\}$. Clearly $d^*(\mathcal{A}) = 1 > \alpha$.

Next, let $c_0c'_0, \ldots, c_tc'_t$ be distinct elements in $\mathcal{A} \cdot \mathcal{A}$. We will prove that

$$|c_ic'_i - c_jc'_j| \geq t^2/36 \qquad (4)$$

for some $i, j,\ i \neq j$.

In view of Lemma 2, we need only to consider the case where all the c_i, c'_i belong to the same $x_n + \mathcal{A}_n$. Otherwise, $|c_ic'_i - c_jc'_j| \geq x_1 = t^2$.

The inequality (4) is obviously true for $2 \le t \le 6$. Suppose $t \ge 7$. We write

$$\begin{aligned} d_i = c_0c_0' - c_ic_i' &= (x_n + a_0)(x_n + a_0') - (x_n + a_i)(x_n + a_i') \\ &= x_n(a_0 + a_0' - a_i - a_i') + a_0a_0' - a_ia_i'. \end{aligned}$$

If the coefficient of x_n is non zero then $|d_i| \ge x_n - M_n^2 \ge x_1 = t^2$.

We suppose then that $a_0 + a_0' - a_i - a_i' = 0$ for all $i = 1, \dots, t$. This implies that $a_i \ne a_j$ if $i \ne j$ (since if not, $c_ic_i' = c_jc_j'$). Then we have

$$\begin{aligned} |c_0c_0' - c_ic_i'| &= |a_0a_0' - a_ia_i'| \\ &= |a_0a_0' - a_i(a_0 + a_0' - a_i)| \\ &= |(a_0' - a_i)(a_0 - a_i)|. \end{aligned}$$

Since all a_i are distinct and there are at most $2(1 + 2(t/6)) < t$ values of i for which $|a_0 - a_i| \le t/6$ or $|a_0' - a_i| \le t/6$, we obtain

$$|a_0' - a_i||a_0 - a_i| > (t/6)^2 \ge 2^{-22}t^2\alpha^{-4}$$

for some i.

For $0 < \alpha < 1/16$ we take the same sequence $\mathcal{A}$ used in the proof of Theorem 2 but with $x_1 = t^2p^4$. As we saw, this sequence has density $d^*(\mathcal{A}) \ge \alpha$. As in that proof, we apply Lemma 2 to see that if c_i, c_i', c_j, c_j' not in the same $x_n + \mathcal{A}_n$ for some $i \ne j$ then $|c_ic_i' - c_jc_j'| \ge x_1 = t^2p^4$ and we are done because $t^2p^4 \ge 2^{-12}t^2\alpha^{-4}$.

Therefore, if $c_0c_0', \dots, c_tc_t'$ are distinct elements of $\mathcal{A} \cdot \mathcal{A}$, we can assume that all c_i, c_i' belong to the same $x_n + \mathcal{A}_n$ and we write them as $c_i = x_n + a_i,\ a_i \in \mathcal{A}_n$. Then

$$d_i = c_0c_0' - c_ic_i' = x_n(a_0 + a_0' - a_i - a_i') + a_0a_0' - a_ia_i'$$

If $a_i + a_i' \ne a_0 + a_0'$ for some $i \ne 0$ then

$$|d_i| \ge x_n - M_n^2 \ge x_1 = t^2p^4.$$

So we assume that $a_i + a_i' = a_0 + a_0'$ for all $i = 0, \dots, t$. We write $a_i = 2mk_i + s_i$ and we can assume that $s_i \le s_i'$ for $i = 0, \dots, t$. The condition $a_i + a_i' = a_0 + a_0'$ for all $i = 0, \dots, t$ implies that $2m(k_i + k_i' - k_0 - k_0') = s_0 + s_0' - s_i - s_i'$ and since $|s_0 + s_0' - s_i - s_i'| < 2m$, we have $k_i + k_i' = k_0 + k_0'$ and $s_i + s_i' = s_0 + s_0'$.

Since $\mathcal{S}_p$ is a Sidon set and $s_i \le s_i'$ we have $s_i = s_0$ and $s_i' = s_0'$ for $i = 0, \dots, t$. Then

$$c_ic_i' - c_0c_0' = 2m(k_i - k_0)(2m(k_i - k_0') + s_0 - s_0').$$

We observe that all k_i are distinct and $k_i \ne 0$. (Otherwise, if $k_i = k_j$ then $k_i' = k_j'$ and then $c_ic_i' = c_jc_j'$.)

Suppose $k_i \ne k_0'$. Then

$$|c_ic_i' - c_0c_0'| = |2m(k_i - k_0)(2m(k_i - k_0') + s_0 - s_0')|$$

Since $|s_0 - s_0'| \le m$, we have

$$\begin{aligned} |c_i c_i' - c_0 c_0'| &\ge 2m|k_i - k_0||2m|k_i - k_0'| - m| \\ &\ge 2m^2|k_i - k_0||k_i - k_0'| \\ &\ge 8p^4|k_i - k_0||k_i - k_0'|. \end{aligned}$$

If $2 \le t \le 6$ we consider k_1 and k_2. One of them (or both) is distinct from k_0'. For that k_i we have $|c_0c_0' - c_ic_i'| \ge 8p^4 \ge 2^{-9}\alpha^{-4} \ge 2^{-14}t^2\alpha^{-4}$.

If $t \ge 7$ we observe that there are at most $2(1 + 2(t/6)) < t$ values of i such that $|k_0 - k_i| \le t/6$ or $|k_0' - k_i| \le t/6$. So there exists some i such that

$$|c_0c_0' - c_ic_i'| \ge 8p^4(t/6)^2 \ge 2^{-14}t^2\alpha^{-4}.$$

□

3. A related question

We do not know if the exponent -3 in Theorem 1 can be improved when $\overline{d}(\mathcal{A}) > \alpha$ or when $\underline{d}(\mathcal{A}) > \alpha$, which is the original problem of Sárközy. Clearly nothing better than -2 is possible. We present an alternative approach to this question, which gives the bound of G. Bérczi quickly.

Let $\mathcal{A} \subset \{1, \dots, N\}$ a set with αN elements. We consider the set

$$\mathcal{A}/\mathcal{A} = \{a/a',\ a < a',\ a, a' \in A\}.$$

What can we say about the cardinality of $\mathcal{A}/\mathcal{A}$ when N is large? Clearly $|\mathcal{A}/\mathcal{A}| \ll \alpha^2 N^2$. Probably it is the true order of magnitude but we do not know how to improve the theorem below

Theorem 5. *If $\mathcal{A} \subset \{1, \dots, N\}$ with $|\mathcal{A}| = \alpha N$, then $|\mathcal{A}/\mathcal{A}| \gg \alpha^4 N^2$.*

Proof. Let $(\mathcal{A} \times \mathcal{A})_d = \{(a, a') \in \mathcal{A} \times \mathcal{A} : a < a', \gcd(a, a') = d\}$. Then for every d, all the quotients a/a', $(a, a') \in (\mathcal{A} \times \mathcal{A})_d$ are distinct and contained in [0,1]. We first show that there exists d such that $|(\mathcal{A} \times \mathcal{A})_d| \ge \frac{\alpha^4}{9} N^2$. Let T be an integer to be chosen later. Then

$$\begin{aligned} (\alpha N)^2 \le |\mathcal{A}|^2 &= \sum_d |(\mathcal{A} \times \mathcal{A})_d| \\ &= \sum_{d \le T} |(\mathcal{A} \times \mathcal{A})_d| + \sum_{d > T} |(\mathcal{A} \times \mathcal{A})_d| \\ &\le T \max_{d \le T} |(\mathcal{A} \times \mathcal{A})_d| + \sum_{d > T} \left(\frac{N}{d}\right)^2 \\ &\le T \max_{d \le T} |(\mathcal{A} \times \mathcal{A})_d| + \frac{N^2}{T} \end{aligned}$$

Thus there exists $d \le T$ such that

$$|(\mathcal{A}\times\mathcal{A})_d| \ge N^2\left(\frac{\alpha^2}{T}-\frac{1}{T^2}\right).$$

If we choose $T = \lceil \frac{2}{\alpha^2} \rceil$ and observe that $T < \frac{3}{\alpha^2}$ when $\alpha < 1$ we obtain $\frac{\alpha^2}{T} - \frac{1}{T^2} \ge \frac{1}{T^2} \ge \frac{\alpha^4}{9}$. Thus for some d, $|(\mathcal{A}\times\mathcal{A})_d| \ge N^2\alpha^4/9$.

Finally we observe that $|\mathcal{A}/\mathcal{A}| \ge |(\mathcal{A}\times\mathcal{A})_d|$ for any d. □

We observe that if $\overline{d}(\mathcal{A}) > \alpha$ there exist infinitely many intervals $[1, N]$ such that $|\mathcal{A}\cap[1, N]| > \alpha$. Theorem above and the pigeon hole principle implies that there are $a/a', a''/a''' \in \mathcal{A}/\mathcal{A}$ such that

$$\left|\frac{a}{a'} - \frac{a''}{a'''}\right| \le 9\alpha^{-4}N^{-2},$$

so $|aa''' - a'a''| \le 9\alpha^{-4}$.

Theorem 5 motivates the following question of independent interest:

Question 2. *Let $\mathcal{A} \subset \{1, \dots, N\}$ with $|\mathcal{A}| = \alpha N$. Is it true that $|\mathcal{A}/\mathcal{A}| \gg \alpha^2 N^2$?*

Clearly an affirmative answer to Question 2 will answer Question 1 with $c(\alpha) \gg \alpha^{-2}$.

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