BOHR SETS IN SUMSETS I: COMPACT GROUPS

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Abstract. Let G be a compact abelian group and ϕ_1, ϕ_2, ϕ_3 be continuous endomorphisms on G. Under certain natural assumptions on the ϕ_i's, we prove the existence of Bohr sets in the sumset $\phi_1(A) + \phi_2(A) + \phi_3(A)$, where A is either a set of positive Haar measure, or comes from a finite partition of G. The first result generalizes theorems of Bogolyubov and Bergelson-Ruzsa. As a variant of the second result, we show that for any partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$, there exists an i such that $A_i - A_i + sA_i$ contains a Bohr set for any $s \in \mathbb{Z} \setminus \{0\}$. The latter is a step toward an open question of Katznelson and Ruzsa.

1. Introduction and statements of results

Let G be an abelian topological group. For a finite set Λ of characters (i.e. continuous homomorphisms from G to $S^1 := \{z \in \mathbb{C} : |z| = 1\}$) and $\eta > 0$, the set

$$B(\Lambda; \eta) := \{x \in G : |\gamma(x) - 1| < \eta \text{ for any } \gamma \in \Lambda\}$$

is called a *Bohr set* or a *Bohr neighborhood of* 0. We refer to η as the *radius* and $|\Lambda|$ as the *rank* (or *dimension*) of the Bohr set. The set $B(\Lambda; \eta)$ is also called a Bohr-$(|\Lambda|, \eta)$ set.

If $A, B \subset G$, the sumset and difference set of A and B are $A \pm B := \{a \pm b : a \in A, b \in B\}$. If $c \in \mathbb{Z}$, we define $cA := \{ca : a \in A\}$. The study of Bohr sets in sumsets started with the following important theorem of Bogolyubov [7][1].

Theorem 1.1 (Bogolyubov [7]). *If $A \subset \mathbb{Z}$ has positive upper Banach density, i.e.*

$$d^*(A) := \lim_{N \to \infty} \sup_{M \in \mathbb{Z}} \frac{|A \cap [M+1, M+N]|}{N} > 0,$$

then $A + A - A - A$ contains a Bohr set whose rank and radius depend only on $d^*(A)$.

While it originated from the study of almost periodic functions, Bogolyubov's theorem is now a standard tool in additive combinatorics. It was used in Ruzsa's proof of Freiman's theorem [26] and in Gowers' proof of Szemerédi's theorem [14]. See [5, 15] for a recent variant of Bogolyubov's theorem and its applications.

The more copies of A are involved, the more structured the sumset is. This reflects the fact that more convolutions result in smoother functions. Thus, a natural question is: What is the smallest number of copies of A that will guarantee the existence of a Bohr set? In $\mathbb{Z}$, it is known that $A - A$ does not necessarily contain a Bohr set, which is a result

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[1] This is reminiscent of Steinhaus' theorem, which says that if $A \subset \mathbb{R}$ has positive Lebesgue measure, then $A - A$ contains an open interval around 0.

of Kriz [22]. On the other hand, Følner [9] proved that there is a Bohr set B such that $(A-A)\setminus B$ has density 0.

Regarding three copies of A, Bergelson and Ruzsa [3] proved the following:

Theorem 1.2 (Bergelson-Ruzsa [3])**.** *Let r,s,t be non-zero integers satisfying $r+s+t=0$. If $A\subset\mathbb{Z}$ has positive upper Banach density, then $rA+sA+tA$ contains a Bohr set whose rank and radius depend only on r,s,t and $d^*(A)$.*

The condition $r+s+t=0$ is easily seen to be necessary, by taking $A=M\mathbb{Z}+1$ for some $M>|r|+|s|+|t|$, since any Bohr set must necessarily contain 0. In particular, one cannot expect $A+A-A$ to contain a Bohr set. When $(r,s,t)=(1,1,-2)$, Bergelson-Ruzsa's theorem generalizes Bogolyubov's, since $A+A-2A\subset A+A-A-A$.

1.1. **Partition results in $\mathbb{Z}$.** While the problem of finding Bohr sets in sumsets of sets having positive density has attracted much attention, the analogous question concerning partitions of $\mathbb{Z}$ was little studied until recently. Regarding the latter, there is a well-known problem in additive combinatorics and dynamical systems, which was popularized by Ruzsa [27, Chapter 5] and Katznelson [21].

Question 1.3. *If $\mathbb{Z}=\bigcup_{i=1}^r A_i$, must there exist $i\in\{1,2,\dots,r\}$ such that A_i-A_i contains a Bohr set?*

In terms of dynamical systems, Question 1.3 asks if any set of recurrence for minimal *topological systems* is also a set of recurrence for minimal *isometries* (also known as a set of *Bohr recurrence*). See [13] for a detailed account of the history of Question 1.3, as well as its many equivalent formulations.[2] While Question 1.3 remains open at the moment and only some partial results were obtained [13, 19], we do have a positive answer when three copies of A_i are involved.

Theorem 1.4. *Let $\mathbb{Z}=\bigcup_{i=1}^r A_i$ be a partition.*

(a) For any $s_1,s_2\in\mathbb{Z}\setminus\{0\}$, there exists $i\in\{1,2,\dots,r\}$ such that the set $s_1A_i-s_1A_i+s_2A_i$ contains a Bohr set whose rank and radius depend only on r and s_1,s_2.

(b) There exists $i\in\{1,2,\dots,r\}$ such that for any $s\in\mathbb{Z}\setminus\{0\}$, the set $A_i-A_i+sA_i$ contains a Bohr set.

Theorem 1.4 highlights the difference between partition and density since, as we mentioned earlier, there is a set $A\subseteq\mathbb{Z}$ of positive density such that $A-A+A$ does not contain a Bohr set.

The expression $s_1A_i-s_1A_i+s_2A_i$ is related to Rado's condition on partition regularity [24]. Recall that an equation $s_1x_1+s_2x_2+\cdots+s_\ell x_\ell=0$ with coefficients in $\mathbb{Z}\setminus\{0\}$ is *partition regular* if under any finite partition (or coloring) of $\mathbb{Z}\setminus\{0\}$, there exists a monochromatic solution $(x_1,x_2,\dots,x_\ell)$. Rado's theorem says that the equation $s_1x_1+s_2x_2+\cdots+s_\ell x_\ell=0$ is partition regular if and only if $\{s_1,\dots,s_\ell\}$ satisfies the following condition: There exists a nonempty set $J\subset\{1,\dots,\ell\}$ such that $\sum_{i\in J}s_i=0$. Using the

[2] In [13], what we call "Bohr set" is referred to as a "Bohr neighborhood of 0." Their Bohr sets are our Bohr sets translated by any element.

facts that $(s_1+\ldots+s_\ell)A \subseteq s_1A+\ldots+s_\ell A$, and a Bohr set must contain 0, Theorem 1.4(a) implies that for $\ell \geq 3$ and $s_1, \ldots, s_\ell \in \mathbb{Z} \setminus \{0\}$, the following are equivalent:

(1) For any partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$, there exists $i \in \{1, \ldots, r\}$ such that $s_1A_i + \ldots + s_\ell A_i$ contains a Bohr set.
(2) $\{s_1, \ldots, s_\ell\}$ satisfies Rado's condition above.

A novelty of Theorem 1.4(b) is that it guarantees a single set A_i that works for every coefficient s (on the other hand, we do lose control on the rank and radius of the Bohr set). When s is very large, the set sA_i is small and so its contribution to the sum diminishes. While there is no consensus on what the answer to Question 1.3 should be, Theorem 1.4(b) provides evidence that the answer to Question 1.3 is either positive or very delicate.

In [13, Table 1, p. 8], Glasscock-Koutsogiannis-Richter summarized results on Bohr sets in sumsets, pertaining to both density and partition. Our Theorem 1.4 fills in the blank on the Syndeticity[3] column and $rA + sA + tA$ row of their table.

1.2. **Results in compact groups.** Bogolyubov's theorem has been generalized to other groups as well (in more general groups, the upper Banach density d^* can be defined in terms of *Følner sequences* or *invariant means*). Følner [9, 10] extended Bogolyubov's theorem to all abelian groups. Answering a question of Hegyvári-Ruzsa [20], Björlund-Griesmer [6] proved that in any countable discrete abelian group G, for any $A \subset G$ with $d^*(A) > 0$, for "many" $a \in A$, the set $A+A-A-a$ contains a Bohr set whose rank and radius depend only on $d^*(A)$. Very recently, Griesmer [18] generalized Theorem 1.2 to all countable discrete abelian groups, though his proof does not give effective bounds for the rank and radius of the Bohr set in question.

Bergelson-Ruzsa and Bogolyubov first proved their theorems in the cyclic group $\mathbb{Z}_N$, and the statements in $\mathbb{Z}$ follow from a compactness argument. Likewise, in Björlund-Griesmer [6] and Griesmer [18], certain compact groups (namely *Bohr compactifications* and *Kronecker factors*) play a prominent role. In view of this "compact first" strategy, the main goal of this paper is, in fact, to study the existence of Bohr sets in sumsets of compact groups. Under this investigation, Theorem 1.4 arises as an application of our general method. In a subsequent paper, we will study the existence of Bohr sets in arbitrary discrete groups by transferring our results from compact groups.

Another feature of our work is the consideration of continuous homomorphisms $\phi : G \to G$ and the image $\phi(A)$ rather than just dilations cA. This point of view leads to a wider range of applications, for example, linear maps on vector spaces and multiplication by an element in a ring (see Theorems 1.7 and 1.8 below). This new perspective was also adopted in recent work of Ackelsberg-Bergelson-Best [1] on Khintchine-type recurrence for actions of an abelian group (Theorem 1.10 below).

Our main result on Bohr sets in sumsets arising from partitions is as follows.

Theorem 1.5. *Let G be a compact abelian group with normalized Haar measure μ and let $\phi_1, \phi_2 : G \to G$ be continuous homomorphisms satisfying*

[3] A subset A of a group G is called *syndetic* if G can be covered by finitely many translated of A.

(a) ϕ_1, ϕ_2 *are commuting, and*
(b) $\phi_1(G), \phi_2(G)$ *have finite indices in* G.

Let $G = \bigcup_{i=1}^r A_i$ *be a partition of* G *into measurable sets. Then for some* $1 \leq i \leq r$,

$$\phi_1(A_i) - \phi_1(A_i) + \phi_2(A_i)$$

contains a Bohr-(k, η) *set, where* k *and* η *depend only on* r, $[G : \phi_1(G)]$ *and* $[G : \phi_2(G)]$.

Remark 1.

- If $\mu(A) > 0$, then $A + A - A$ is not guaranteed to contain a Bohr set. For a counterexample, take $G = \mathbb{Z}_{2N}$ for some large N and $A = \{a \in \mathbb{Z}_{2N} : a \text{ is odd}\}$. In particular, the analogous version of Theorem 1.5 for sets of positive measure fails.
- We do not know if the commuting condition can be removed entirely, though it can be slightly relaxed (see Theorem 3.5). For example, commutativity is not required when ϕ_1 or ϕ_2 is an automorphism (see Remark 5).
- The finite index condition on $\phi_1(G)$ cannot be removed, by taking for example $\phi_1 = 0$ and $\phi_2(x) = x$. On the other hand, we do not know whether the finite index condition on $\phi_2(G)$ can be removed. If we let $\phi_2 = 0$, then the situation amounts to Question 1.3 itself.

We now turn our attention to density results. In compact abelian groups, the Haar measure plays the role of the upper Banach density.

Theorem 1.6. *Let* G *be a compact abelian group with normalized Haar measure* μ *and* $\phi_1, \phi_2, \phi_3 : G \to G$ *be continuous homomorphisms satisfying*
(a) $\phi_1 + \phi_2 + \phi_3 = 0$,
(b) ϕ_1, ϕ_2, ϕ_3 *are commuting,*
(c) $\phi_1(G), \phi_2(G), \phi_3(G)$ *have finite indices in* G.

Let $A \subseteq G$ *be a measurable subset with* $\mu(A) = \delta > 0$. *Then*

$$\phi_1(A) + \phi_2(A) + \phi_3(A)$$

contains a Bohr-(k, η) *set, where* k *and* η *depend only on* δ *and the indices* $[G : \phi_i(G)]$ $(1 \leq i \leq 3)$.

Remark 2.

- The condition $\phi_1 + \phi_2 + \phi_3 = 0$ cannot be removed. For a counterexample, take $G = \mathbb{Z}_N$ for some large N and $A = \{1, \cdots, \lfloor N/10 \rfloor\}$. Then $A + A + A$ does not contain 0, and hence not a Bohr set.
- We do not know if the condition on commutativity can be removed entirely, though it can be weakened (see Theorem 4.4). For example, commutativity is not required when one of ϕ_1, ϕ_2 and ϕ_3 is an automorphism (see Remark 6).
- The finite index condition cannot be removed. Indeed, we can take $G = \mathbb{F}_2^n$ for some large n, $\phi_1(x) = x$, $\phi_2(x) = -x, \phi_3(x) = 0$. In this setting, Bohr sets are simply vector subspaces. A construction of Green [17, Theorem 9.4] gives a set A of size $\geq |G|/4$ such that any subspace contained in $A - A$ must have codimension $\geq \sqrt{n}$.

As an application, Theorem 1.6 can be used to obtain an effective version of the aforementioned result of Griesmer [18]. We plan to pursue this idea in a subsequent paper.

1.3. **Number-theoretic consequences.** As mentioned earlier, the fact that we accommodate homomorphisms in Theorem 1.5 and Theorem 1.6 enables us to generalize Theorem 1.2 and Theorem 1.4 to number fields and function fields.

In the following, for a subset A of a ring R and $c \in R$, we write

$$cA = \{ca : a \in A\} \tag{1}$$

and

$$A/c = \{b \in A : bc \in A\}. \tag{2}$$

The next theorem is true for any number field, but we only state for $\mathbb{Z}[i]$ for simplicity.

Theorem 1.7. *Let $s_1, s_2, s_3 \in \mathbb{Z}[i] \setminus \{0\}$ such that $s_1 + s_2 + s_3 = 0$.*

(a) If a set $A \subseteq \mathbb{Z}[i]$ has positive upper density, i.e.

$$\overline{d}(A) := \limsup_{N\to\infty} \frac{|A \cap [-N, N]^d|}{(2N+1)^d} = \delta > 0,$$

then $s_1A + s_2A + s_3A$ contains a (k, η)-Bohr set in $\mathbb{Z}[i]$, where k and η depend only on s_1, s_2, s_3 and δ.

(b) If $\mathbb{Z}[i] = \bigcup_{j=1}^{r} A_j$, then for some $j \in \{1, 2, \dots, r\}$, $s_1A_j - s_1A_j + s_2A_j$ contains a (k, η)-Bohr set in $\mathbb{Z}[i]$, where k and η depend only on s_1, s_2 and r.

(c) If $\mathbb{Z}[i] = \bigcup_{j=1}^{r} A_j$, then there exists $j \in \{1, 2, \dots, r\}$ such that $A_j - A_j + sA_j$ contains a Bohr set for any $s \in \mathbb{Z}[i] \setminus \{0\}$.

Here, as a group, we identify $\mathbb{Z}[i]$ with $\mathbb{Z}^2$.

Our next result deals with the ring $\mathbb{F}_q[t]$ of polynomials over a finite field $\mathbb{F}_q$.

Theorem 1.8. *Let $s_1, s_2, s_3 \in \mathbb{F}_q[t] \setminus \{0\}$ such that $s_1 + s_2 + s_3 = 0$.*

(a) If a set $A \subseteq \mathbb{F}_q[t]$ has positive upper density, i.e.

$$\overline{d}(A) := \limsup_{N\to\infty} \frac{|\{x \in \mathbb{F}_q[t] : \deg x < N\}|}{q^N} = \delta > 0,$$

then $s_1A + s_2A + s_3A$ contains a $\mathbb{F}_q$-vector subspace of finite codimension in $\mathbb{F}_q[t]$, where the codimension depends only on s_1, s_2, s_3 and δ.

(b) If $\mathbb{F}_q[t] = \bigcup_{i=1}^{r} A_i$, then for some $i \in \{1, \dots, r\}$, $s_1A_i - s_1A_i + s_2A_i$ contains a $\mathbb{F}_q$-vector subspace of finite codimension of $\mathbb{F}_q[t]$, where the codimension depends only on s_1, s_2 and δ.

(c) If $\mathbb{F}_q[t] = \bigcup_{i=1}^{r} A_i$, then there exists $i \in \{1, 2, \dots, r\}$ such that $A_i - A_i + sA_i$ contains an $\mathbb{F}_q$-vector subspace of finite codimension of $\mathbb{F}_q[t]$ for any $s \in \mathbb{F}_q[t] \setminus \{0\}$.

We remark that the special case $s_1, s_2, s_3 \in \mathbb{F}_q \setminus \{0\}$ of Theorem 1.8(a) is essentially Corollary 1.4 in [18].

1.4. **Counting linear patterns.** Similarly to the proofs of Bogolyubov [7] and Bergelson-Ruzsa [3]'s theorems, we deduce Theorem 1.6 from a lower bound (of correct order of magnitude) for the number of certain linear patterns in G. This is straightforward in Bogolyubov's case, but less so in Bergelson and Ruzsa's. Bergelson and Ruzsa had to count the number of generalized Roth patterns $\{x, x+ry, x+sy\}$ (where $r, s \in \mathbb{Z}$) and they deduced this from Szemerédi's theorem [30] and Varnavides' argument [32]. For us, we need to count the number of patterns $\{x, x+\phi(y), x+\psi(y)\}$ (where ϕ and ψ are homomorphisms). This is accomplished by generalizing a Fourier-analytic argument of Bourgain [8]. Bourgain's argument, in essence an arithmetic regularity lemma, allows us to obtain the following Khintchine-type result.

Theorem 1.9 (Khintchine-Roth theorem in compact abelian groups)**.** *Let G be a compact abelian group with probability Haar measure μ and $\phi, \psi : G \to G$ be continuous homomorphisms such that $[G : \phi(G)], [G : \psi(G)]$ and $[G : (\phi - \psi)(G)]$ are finite. Let $f : G \to [0,1]$ be a measurable function with $\int_G f \, d\mu = \delta > 0$.*

Then for any $\epsilon > 0$, there exists a constant $c_1 > 0$ that depends only on δ, ϵ and the indices above such that the set

$$B = \left\{ y \in G : \int_G f(x) f(x+\phi(y)) f(x+\psi(y)) \, d\mu(x) > \delta^3 - \epsilon \right\}$$

has measure at least c_1. Consequently,

$$\iint_{G^2} f(x) f(x+\phi(y)) f(x+\psi(y)) \, d\mu(x) d\mu(y) \geq c_2 \tag{3}$$

for some positive constant c_2 depending only on δ and the indices above.

Theorem 1.9 was proved independently by Berger-Sah-Sawhney-Tidor [4], under the hypothesis that ϕ, ψ and $\phi - \psi$ are automorphisms, using a very similar argument. Our execution is slightly different from theirs, in that we follow Bergelson-Host-McCutcheon-Parreau [2]'s elaboration of Bourgain's argument, while they follow Tao [31]'s.

Theorem 1.9 is markedly similar to the following result of Ackelsberg, Bergelson and Best:

Theorem 1.10 ([1, Theorem 1.10])**.** *Let G be a countable discrete abelian group, and $\phi, \psi : G \to G$ be homomorphisms such that $[G : \phi(G)], [G : \psi(G)]$ and $[G : (\phi - \psi)(G)]$ are finite. For any ergodic system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$, any $\epsilon > 0$, and any $A \in \mathcal{B}$, the set*

$$B = \left\{ g \in G : \mu(A \cap T_{\phi(g)}^{-1} A \cap T_{\psi(g)}^{-1} A) > \mu(A)^3 - \epsilon \right\}$$

is syndetic in G.

As discussed in [1, Section 10], the finite index condition in Theorem 1.10 is necessary. The following result of Fox-Sah-Sawhney-Stoner-Zhao [11], improving on an earlier result of Mandache [23], shows that the finite index condition is also necessary in Theorem 1.9.

Example 1. Let $\ell < 4$ be arbitrary and $\delta > 0$ be sufficiently small in terms of l. Let $G = \mathbb{F}_2^n \times \mathbb{F}_2^n$ where n is sufficiently large, $\phi(u,v) = (u, 0), \psi(u,v) = (0, u)$. Then the left hand side of (3) counts the number of "corners" $\{(a,b), (a+u, b), (a, b+u)\}$ in $\mathbb{F}_2^n \times \mathbb{F}_2^n$.

[11, Corollary 1.3] states that there exists a set $A \subset G$ of size $\geq \delta|G|$ such that for any $u \in \mathbb{F}_2^n \setminus \{0\}$, we have

$$\#\{(a,b) \in G : (a,b), (a+u,b), (a,b+u) \in A\} < \delta^\ell |G|.$$

Hence, the set B in Theorem 1.9 has to be $\{0\} \times \mathbb{F}_2^n$. But the measure of this set in G goes to 0 as n goes to infinity.

Regarding Theorem 1.5, we deduce it from the following result, which counts the number of monochromatic configurations under finite partitions of G.

Theorem 1.11. *Let G be a compact abelian group with probability Haar measure μ and let $\psi, \phi_1, \ldots, \phi_k : G \to G$ be continuous homomorphisms satisfying:*

(a) $\psi, \phi_1, \ldots, \phi_k$ are commuting, and

(b) $\psi(G), \phi_1(G), \ldots, \phi_k(G)$ have finite indices in G.

Suppose $G = \bigcup_{i=1}^r A_i$ is a partition of G into measurable sets. Then

$$\sum_{i=1}^r \iint_{G^2} 1_{A_i}(\psi(y)) 1_{A_i}(x) 1_{A_i}(x+\phi_1(y)) \cdots 1_{A_i}(x+\phi_k(y))\, d\mu(x) d\mu(y) \geq c_3 \tag{4}$$

for some positive constant c_3 depending only on r, k and the indices above.

Remark 3.

- By taking $\psi = 0$, we see that the condition $[G : \psi(G)]$ is finite cannot be removed. However, we do not know whether the condition $[G : \phi_i(G)] < \infty$ is necessary or not.
- Our proof relies heavily on the commuting condition and we do not know if it can be removed.

When ψ and ϕ are dilations, the configuration $\{\psi(y), x+\phi_1(y), \ldots, x+\phi_k(y)\}$ becomes the *Brauer configuration* $\{y, x, x+y, \ldots, x+ky\}$. Results on counting such monochromatic configurations have been established by Serra-Vena [29, Theorem 1.3] for finite abelian groups of bounded torsion. Thus, besides the fact that it allows for more general homomorphisms, Theorem 1.11 has the advantage of being uniform over all groups. On the other hand, our finite index condition is certainly related, and in a sense, dual to Serra-Vena's bounded exponent condition [29].

We remark that despite the apparent similarity between (3) and (4), their proofs are very different. The proof of Theorem 1.11 is "Fourier-free" and its main ingredient is the Hales-Jewett theorem. Thus, our approach in proving this theorem is also genuinely different from Serra-Vena's, which relies on a removal lemma for groups.

On the quantitative side, our bounds leave much to be desired. Since the proof of Theorem 1.9 relies on the regularity lemma (Proposition 4.2), in Theorem 1.6, the dependence of k and η on δ and $[G : \phi_i(G)]$ is of tower type. Likewise, since the proof of Theorem 1.11 uses the Hales-Jewett theorem, the bounds for k and η in Theorem 1.5 are even worse. It is an interesting problem to obtain good bounds for Theorems 1.6 and 1.5, even in special classes of groups such as $\mathbb{F}_p^n$. Indeed, Sanders [28, Theorem A.1] obtained a near optimal bound for Bogolyubov's theorem in $\mathbb{F}_p^n$.

Outline of the paper. In Section 2, we set up notation and collect some basic facts about Bohr sets, kernels and homomorphisms in compact abelian groups. Section 3 is devoted to proving results involving partitions, especially, Theorems 1.5 and 1.11. Theorems 1.6, 1.9 and related density results will be proved in Section 4. Section 5 contains proofs of results in $\mathbb{Z}$, number fields and function fields, i.e. Theorems 1.4, 1.7 and 1.8. Lastly, we present some related open questions in Section 6.

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2. Preliminaries

In this section, we gather some background on Bohr sets, kernels and homomorphisms in compact abelian groups. Most of the results are well-known or resemble known theorems. We include proofs for the results that we cannot pinpoint precisely in the literature.

2.1. **Notation.** We write $[N]$ for the set $\{1, \ldots, N\}$. If A and B are two quantities, we write $A = O(B)$ or $A \ll B$ if there is a constant C such that $|A| \leq CB$. We write $e(x)$ for $e^{2\pi i x}$.

Throughout this paper, G is a Hausdorff compact abelian group with probability Haar measure μ and Γ is the dual of G, written additively. The relevance of homomorphisms is that if $\gamma \in \Gamma$ and $\phi : G \to G$ is a continuous homomorphism, then $\gamma \circ \phi$ is also an element of Γ.

If $f : G \to \mathbb{C}$ is a function, for $t \in G$ we define the function $f_t(x) = f(x+t)$. For $f \in L^1(G)$, the Fourier transform of f is the function

$$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, d\mu(x) \qquad \text{for } \gamma \in \Gamma.$$

For $f, g \in L^2(G)$, we then have Parseval's formula

$$\int_G f(x)\overline{g(x)}\, d\mu(x) = \sum_{\gamma \in \Gamma} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}$$

and Plancherel's formula

$$\int_G |f(x)|^2\, d\mu(x) = \sum_{\gamma \in \Gamma} \left|\widehat{f}(\gamma)\right|^2.$$

2.2. **Bohr sets.** For $\Lambda, \Lambda_1, \Lambda_2 \subseteq \Gamma$ and $\eta_1, \eta_2 > 0$, it follows from the definition of Bohr sets that

$$B(\Lambda_1; \eta_1) \cap B(\Lambda_2; \eta_2) \supset B(\Lambda_1 \cup \Lambda_2; \min(\eta_1, \eta_2))$$

and

$$B(\Lambda; \eta_1) + B(\Lambda; \eta_2) \subset B(\Lambda; \eta_1 + \eta_2).$$

Lemma 2.1. *Suppose $f_1, \ldots, f_k \in L^\infty(G)$, $\|f_i\|_\infty \leq 1$ for all $i = 1, \ldots, k$. Let $\phi_1, \ldots, \phi_k$ be homomorphisms $G \to G$. Then for any $\eta > 0$, the set*

$$B = \{t \in G : \|\widehat{f_i} - \widehat{f_{i,\phi_i(t)}}\|_\infty < \eta \text{ for } i = 1, \ldots, k\}$$

contains a Bohr set $B(\Lambda; \eta)$ where $|\Lambda| \leq \frac{4k}{\eta^2}$.

Proof. Note that if $\|\widehat{f}_i - \widehat{f_{i,\phi_i(t)}}\|_\infty \geq \eta$, then for some $\gamma \in \Gamma$,

$$|\widehat{f}_i(\gamma) - \widehat{f_{i,\phi_i(t)}}(\gamma)| = |1-\gamma(\phi_i(t))||\widehat{f}_i(\gamma)| \geq \eta.$$

This implies that $|1-\gamma(\phi_i(t))| \geq \eta$ and $\gamma \in \Lambda_i := \{\lambda \in \Gamma : |\widehat{f}_i(\lambda)| > \eta/2\}$.

We have thus shown that

$$B(\bigcup_{i=1}^k \Lambda_i \circ \phi_i; \eta) \subset B,$$

where $\Lambda_i \circ \phi_i := \{\gamma \circ \phi_i : \gamma \in \Lambda_i\} \subset \Gamma$. By Plancherel's formula,

$$\left(\frac{\eta}{2}\right)^2 |\Lambda_i| \leq \sum_{\lambda \in \Lambda_i} \left|\widehat{f}_i(\lambda)\right|^2 \leq 1.$$

Therefore, $|\Lambda_i| \leq \frac{4}{\eta^2}$ and $|\bigcup_{i=1}^k \Lambda_i \circ \phi_i| \leq \frac{4k}{\eta^2}$. □

Next lemma is needed in 5.

Lemma 2.2. *Let H be a locally compact abelian group, K be a closed subgroup of finite index m. Then K is a Bohr-$(m, |e(1/m)-1|)$ set in H.*

Proof. Let $\chi_1, \ldots, \chi_m$ be all characters on H/K. For any $x \in H/K$ and $1 \leq i \leq m$, we have $|\chi_i(x)|^m = 1$, so either $\chi_i(x) = 1$ or $|\chi_i(x) - 1| \geq |e(1/m) - 1|$. If $\chi_i(x) = 1$ for all i then $x = 0$. Hence

$$\{0\} = B(\chi_1, \ldots, \chi_m; |e(1/m) - 1|).$$

The characters χ_i lift to characters $\tilde{\chi_i}$ on H by $\tilde{\chi_i}(h) = \chi_i(h + K)$. Therefore,

$$H = B(\tilde{\chi_1}, \ldots, \tilde{\chi_m}; |e(1/m) - 1|),$$

as desired. □

We will also need Bogolyubov's theorem for compact abelian groups.

Lemma 2.3 (Bogolyubov for compact abelian groups, see [26, Lemma 2.1])**.** *Let G be a compact abelian group with Haar measure μ and let $A \subseteq G$ of positive measure. Then $A - A + A - A$ contains a Bohr-(k, η) set where k, η depends only on $\mu(A)$.*

2.3. **Kernels.** A kernel on G is a non-negative continuous function that satisfies $\int_G K \, d\mu = 1$. Specifically, we will utilize the kernels whose Fourier transforms are also non-negative and supported on given Bohr sets. For a kernel K, we write $\|\widehat{K}\|_1$ to denote $\sum_{\gamma \in \Gamma} |\widehat{K}(\gamma)|$.

Lemma 2.4 (cf. [2, Lemma 4.3])**.** *Given a finite set $\Lambda \subset \Gamma$ and $\eta \in (0, 1/2]$, there exists a kernel K satisfying the following:*

(1) $K \geq 0, \widehat{K} \geq 0$ and $\int_G K \, d\mu = \|K\|_1 = 1$,
(2) $\|\widehat{K}\|_1 = \|K\|_\infty \leq 1/(C_0\eta)^{|\Lambda|}$ for some absolute constant $C_0 > 0$, and
(3) K vanishes outside the Bohr set $B(\Lambda; \eta)$.

Consequently,

$$\mu(B(\Lambda; \eta)) \geq (C_0\eta)^{|\Lambda|}. \tag{5}$$

We remark that the bound (5) can also be obtained from an elementary covering argument (see [31]).

Proof. First, for each $\lambda \in \Lambda$, there exists a kernel $K_\lambda : G \to [0,\infty)$ satisfying the following properties:

(1) $\|K_\lambda\|_1 = 1$,
(2) $\widehat{K}_\lambda \geq 0$,
(3) K_λ is supported on $B(\{\lambda\};\eta) = \{x \in G : |\lambda(x) - 1| < \eta\}$,
(4) $\|K_\lambda\|_\infty = K_\lambda(0) \leq 1/(C_0\eta)$ for some absolute constant C_0.

Indeed, let $B = B(\{\lambda\}; \frac{\eta}{2})$ and let $K_\lambda = \frac{1_B}{\mu(B)} * \frac{1_B}{\mu(B)}$. Clearly the first and second properties are satisfied. Additionally, K_λ is supported on $B(\{\lambda\}; \frac{\eta}{2}) + B(\{\lambda\}; \frac{\eta}{2}) \subset B(\{\lambda\};\eta)$.

Concerning the last property, we have for every $x \in G$,

$$K_\lambda(x) = \sum_{\gamma \in \Gamma} \widehat{K}_\lambda(\gamma)\gamma(x)$$

and so

$$|K_\lambda(x)| \leq \sum_{\gamma \in \Gamma} \widehat{K}_\lambda(\gamma) = K_\lambda(0).$$

Therefore, $\|K_\lambda\|_\infty = K_\lambda(0) = \frac{1}{\mu(B)}$. Since λ is continuous, its image $\lambda(G)$ is a closed subgroup of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and so it is either S^1 or $\{z \in \mathbb{C} : z^q = 1\}$ for some $q \in \mathbb{N}$. Since λ is a homomorphism, it is measure-preserving (see Lemma 2.7 below). Hence $\mu(B)$ is equal to the normalized Haar measure of the set

$$\left\{z \in S^1 : |z - 1| < \frac{\eta}{2}\right\}$$

in the group $\lambda(G)$. In either case, where $\lambda(G) = S^1$ or $\{z \in \mathbb{C} : |z|^q = 1\}$, we find that $\mu(B) \geq C_0\eta$ for some absolute constant C_0. Therefore, $\|K_\lambda\|_\infty \leq 1/(C_0\eta)$.

We now define

$$\widetilde{K} = \prod_{\lambda \in \Lambda} K_\lambda.$$

It follows that $\widetilde{K} \geq 0$ and $\widetilde{K}$ is supported on $B(\Lambda;\eta)$. Repeatedly using the fact that $\widehat{fg}(\gamma) = \sum_{\lambda \in \Gamma} \widehat{f}(\lambda)\widehat{g}(\gamma - \lambda)$ for all $f, g \in L^\infty(G)$, we have $\widehat{\widetilde{K}} \geq 0$. Likewise, since $\widehat{K}_\lambda(0) = \|K_\lambda\|_1 = 1$, we have $\|\widetilde{K}\|_1 = \widehat{\widetilde{K}}(0) \geq 1$. Upon defining

$$K = \widetilde{K}/\|\widetilde{K}\|_1$$

we obtain the desired kernel. $\square$

2.4. **Homomorphisms.**

We will often make use of the following facts about homomorphisms $G \to G$.

Lemma 2.5. *Let $\phi : G \to G$ be a continuous homomorphism such that $[G : \phi(G)] = m$ is finite. Then for any $\gamma \in \Gamma$, there are at most m elements $\chi \in \Gamma$ such that $\gamma = \chi \circ \phi$.*

Proof. It is easy to see that for each $\gamma \in \Gamma$, the set $S_\gamma := \{\chi \in \Gamma : \gamma = \chi \circ \phi\}$ is either empty, or a coset of the group S_0. On the other hand, S_0 is the annihilator of the group $\phi(G)$, so by [25, Theorem 2.1.2], it is isomorphic to $G/\phi(G)$, and hence has cardinality m. $\square$

Lemma 2.6. *Let $\phi, \psi : G \to G$ be homomorphisms such that $[G : \phi(G)] = m$ and $[G : \psi(G)] = \ell$ are finite. Then $[G : \phi(\psi(G))] \leq m\ell$ is finite.*

Proof. We have $[G : \phi(\psi(G))] = [G : \phi(G)][\phi(G) : \phi(\psi(G))]$. It suffices to show that $[\phi(G) : \phi(\psi(G))] \leq \ell$.

Let $x_1+\psi(G), \dots, x_\ell+\psi(G)$ be all cosets of $\psi(G)$ in G. Then $\phi(x_1)+\phi(\psi(G)), \dots, \phi(x_\ell)+\phi(\psi(G))$ are all cosets of $\phi(\psi(G))$ in $\psi(G)$ (these are not necessarily distinct, so the actual number of cosets may be less than ℓ), proving the desired claim. □

Lemma 2.7. *Let G, H be compact abelian groups and μ, ν be the normalized Haar measures of G and H, respectively. Suppose $\phi : G \to H$ is a continuous surjective homomorphism. Then $\phi_*\mu = \nu$ (i.e. $\nu(B) = \mu(\phi^{-1}(B))$ for any Borel set $B \subset H$).*

Proof. Let $\nu_0 = \phi_*\mu$. By the uniqueness of the normalized Haar measure, it suffices to show that ν_0 is a translation-invariant probability measure on H. First, ν_0 is a probability measure because $\nu_0(H) = \mu(\phi^{-1}(H)) = \mu(G) = 1$. Now let $B \subset H$ be a Borel set and $h_0 \in H$ be arbitrary. Since ϕ is surjective, there exists $g_0 \in G$ such that $\phi(g_0) = h_0$. For any $g \in \phi^{-1}(B + h_0)$, we have

$$\phi(g - g_0) = \phi(g) - \phi(g_0) \in B + h_0 - h_0 = B.$$

Therefore, $\phi^{-1}(B + h_0) \subseteq \phi^{-1}(B) + g_0$. On the other hand,

$$\phi(\phi^{-1}(B) + g_0) \subseteq B + h_0$$

and so $\phi^{-1}(B + h_0) = \phi^{-1}(B) + g_0$. Since μ is translation-invariant on G, it follows that

$$\nu_0(B + h_0) = \mu(\phi^{-1}(B + h_0)) = \mu(\phi^{-1}(B) + g_0) = \mu(\phi^{-1}(B)) = \nu_0(B).$$

Thus ν_0 is translation-invariant on H and so $\nu_0 = \nu$. □

Lemma 2.8. *Let $\phi : G \to G$ be a continuous homomorphism such that $[G : \phi(G)] = m$ is finite. Then for any measurable set $A \subset G$, we have*

$$\mu(A) \leq m\mu(\phi(A)) \tag{6}$$

and

$$\mu(\phi^{-1}(A)) \leq m\mu(A). \tag{7}$$

Consequently, if $f \in L^1(G)$ is nonnegative, then

$$\int_G f(x)\, d\mu(x) \geq \frac{1}{m} \int_G f(\phi(x))\, d\mu(x).$$

Proof. First, since ϕ is continuous and G is compact, $\phi(G)$ is a compact subgroup of G. Since G is Hausdorff, $\phi(G)$ is closed. In other words, $\phi(G)$ is a closed subgroup of G.

For each Borel set $B \subset \phi(G)$, $\lambda(B) = m\mu(B)$ defines a probability measure on $\phi(G)$. Since this measure is translation invariant, it is equal to the normalized Haar measure on $\phi(G)$. By Lemma 2.7, $\lambda = \phi_*\mu$. This means that for any Borel set $B \subset \phi(G)$, we have $\mu(\phi^{-1}(B)) = m\mu(B)$.

Let A be any Borel set in G. Since $A \subset \phi^{-1}(\phi(A))$, we have hence $\mu(A) \leq \mu(\phi^{-1}(\phi(A)) = m\mu(\phi(A))$, and the first assertion is proved. Applying the first assertion to the set $\phi^{-1}(A)$, we get the second assertion.

The third assertion follows from the second one, and the fact that f can be approximated by functions of the form $\sum_{i=1}^{n} c_i 1_{A_i}$ for Borel sets A_i and $c_i \geq 0$. $\square$

The next lemmas deal with images and preimages of Bohr sets under homomorphisms.

Lemma 2.9. *Let $B \subset G$ be a Bohr-(k,η) set and $\phi : G \to G$ be a continuous homomorphism. Then $\phi^{-1}(B)$ is also a Bohr-(k,η) set.*

Proof. If $B = \{x \in G : |\gamma_i(x) - 1| < \eta \text{ for } i = 1, \dots, k\}$ is a Bohr-(k,η) set, then $\phi^{-1}(B) = \{x \in G : |\gamma_i \circ \phi(x) - 1| < \eta \text{ for } i = 1, \dots, k\}$ is also a Bohr-(k,η)-set. $\square$

The next lemma is more surprising.

Lemma 2.10 (cf. Griesmer [18, Lemma 1.7])**.** *Let $B \subset G$ be a Bohr-(k,η) set and $\phi : G \to G$ be a continuous homomorphism such that $[G : \phi(G)] = m < \infty$. Then $\phi(B)$ contains a Bohr-(k',η') set, where k', η' depend on k, η and m.*

Proof. Suppose $B = \{x \in G : |\gamma_i(x) - 1| < \eta \text{ for } 1 \leq i \leq k\}$ where $\gamma_i \in \Gamma$. Then

$$A = \{x \in G : |\gamma_i(x) - 1| < \eta/4 \text{ for } 1 \leq i \leq k\}$$

satisfies $A - A + A - A \subseteq B$. The bound (5) implies that $\mu(A) \geq (C_0\eta/4)^k$ for some absolute constant $C_0 > 0$.

In view of Lemma 2.8, $\mu(\phi(A)) \geq \mu(A)/m \geq \frac{(C_0\eta)^k}{4^k m}$. Therefore, by Lemma 2.3, the set $\phi(B) \supseteq \phi(A) - \phi(A) + \phi(A) - \phi(A)$ is a Bohr-(k',η') set where k', η' depend only on $\mu(\phi(A))$, which is bounded below by $\frac{(C_0\eta)^k}{4^k m}$. $\square$

2.5. **Counting lemmas.**

Lemma 2.11 (cf. [8, Lemma 2])**.** *Let $\phi, \psi : G \to G$ be continuous homomorphisms such that $\phi(G), \psi(G)$ have finite indices in G. Then for $f_1, f_2, f_3 \in L^\infty(G)$ and $K \in L^1(G)$ such that $\widehat{K} \in L^1(\Gamma)$, we have*

$$\left| \iint_{G^2} f_1(x) f_2(x + \phi(y)) f_3(x + \psi(y)) K(y) \, d\mu(x) d\mu(y) \right| \ll \|\widehat{f_1}\|_\infty \|f_2\|_2 \|f_3\|_2 \|\widehat{K}\|_1 \tag{8}$$

where the implied constant depends only on the indices of $\phi(G)$ and $\psi(G)$ in G.

Proof. Since linear combinations of characters are dense in $L^2(G)$, without loss of generality, we can assume f_1, f_2, f_3 and K are equal to their Fourier series. For $x \in G$, write $g(x) = \int_G f_2(x + \phi(y)) f_3(x + \psi(y)) K(y) \, d\mu(y)$.

By Plancherel's formula,

$$\left| \int_G f_1(x) g(x) \, d\mu(x) \right| = \left| \sum_{\gamma \in \Gamma} \widehat{f_1}(\gamma) \widehat{g}(\overline{\gamma}) \right| \leq \|\widehat{f_1}\|_\infty \cdot \|\widehat{g}\|_1.$$

Thus

$$\begin{aligned}
g(x) &= \int_G f_2(x+\phi(y))f_3(x+\psi(y))K(y)\,d\mu(y) \\
&= \int_G \left(\sum_{\gamma_2,\gamma_3,\gamma_0\in\Gamma} \widehat{f_2}(\gamma_2)\gamma_2(x+\phi(y))\widehat{f_3}(\gamma_3)\gamma_3(x+\psi(y))\widehat{K}(\gamma_0)\gamma_0(y) \right) d\mu(y) \\
&= \int_G \left(\sum_{\gamma_2,\gamma_3,\gamma_0\in\Gamma} \widehat{f_2}(\gamma_2)\widehat{f_3}(\gamma_3)\widehat{K}(\gamma_0)(\gamma_2+\gamma_3)(x)(\gamma_2\circ\phi+\gamma_3\circ\psi+\gamma_0)(y) \right) d\mu(y) \\
&= \sum_{\substack{\gamma_2,\gamma_3,\gamma_0\in\Gamma,\\ \gamma_2\circ\phi+\gamma_3\circ\psi+\gamma_0=0}} \widehat{f_2}(\gamma_2)\widehat{f_3}(\gamma_3)\widehat{K}(\gamma_0)(\gamma_2+\gamma_3)(x)
\end{aligned}$$

Consequently,

$$\widehat{g}(\gamma) = \sum_{\substack{\gamma_2,\gamma_3,\gamma_0\in\Gamma,\\ \gamma_2\circ\phi+\gamma_3\circ\psi+\gamma_0=0,\\ \gamma_2+\gamma_3=\gamma}} \widehat{f_2}(\gamma_2)\widehat{f_3}(\gamma_3)\widehat{K}(\gamma_0)$$

and

$$\|\widehat{g}\|_1 \le \sum_{\substack{\gamma_2,\gamma_3,\gamma_0\in\Gamma,\\ \gamma_2\circ\phi+\gamma_3\circ\psi+\gamma_0=0}} |\widehat{f_2}(\gamma_2)|\cdot|\widehat{f_3}(\gamma_3)|\cdot|\widehat{K}(\gamma_0)|.$$

Therefore, it suffices to show that for each $\gamma_0\in\Gamma$, we have

$$\sum_{\substack{\gamma_2,\gamma_3\in\Gamma,\\ \gamma_2\circ\phi+\gamma_3\circ\psi+\gamma_0=0}} |\widehat{f_2}(\gamma_2)|\cdot|\widehat{f_3}(\gamma_3)| \ll \|f_2\|_2\cdot\|f_3\|_2.$$

By the Cauchy-Schwarz inequality and the Plancherel's formula, the left hand side is at most

$$\begin{aligned}
& \|f_2\|_2\cdot\left(\sum_{\gamma_2}\left(\sum_{\gamma_3\circ\psi=-\gamma_0-\gamma_2\circ\phi}|\widehat{f_3}(\gamma_3)|\right)^2\right)^{1/2} \\
\ll\; & \|f_2\|_2\cdot\left(\sum_{\gamma_2}\sum_{\gamma_3\circ\psi=-\gamma_0-\gamma_2\circ\phi}|\widehat{f_3}(\gamma_3)|^2\right)^{1/2} \qquad (9)\\
\ll\; & \|f_2\|_2\cdot\left(\sum_{\gamma_3}|\widehat{f_3}(\gamma_3)|^2\right)^{1/2} \qquad (10)\\
=\; & \|f_2\|_2\cdot\|f_3\|_2.
\end{aligned}$$

In (9), we use the fact that for each $\xi\in\Gamma$, there are at most $[G:\psi(G)]$ values of γ_3 such that $\gamma_3\circ\psi=\xi$. Likewise, in (10), we use the fact that for each $\xi\in\Gamma$, there are at most $[G:\phi(G)]$ values of γ_2 such that $\gamma_2\circ\phi=\xi$. Both of these facts follow from Lemma 2.5. $\square$

Remark 4. Lemma 2.11 is not true without the assumption on finite indices. As a counterexample, we let $\phi(x)=x,\psi(x)=2x$ and $G=\mathbb{F}_2^k$ for some large k. Let $n=|G|=2^k$. For each $i=1,\dots,n$, define

- $\widehat{f_1}(\gamma_i)=\widehat{f_3}(\gamma_i)=1$, so $f_1(x)=f_3(x)=n\cdot 1_{x=0}$.

- $\widehat{f_2}(\gamma_i) = a_i$, where $a_i \geq 0$.
- $\widehat{K}(\gamma_i) = b_i$, where $b_i \geq 0$.

Then (8) says that

$$n(a_1 b_1 + \cdots + a_n b_n)^2 \ll (a_1^2 + \cdots + a_n^2)(b_1 + \cdots + b_n)^2.$$

This is false by taking $a_1 = b_1 = 1$ and $a_i = b_i = 0$ for $i \neq 1$.

While the previous lemma involves the configuration $x, x+\phi(y), x+\psi(y)$, the next one is concerned with $x, x+\phi(y)$ and $\psi(y)$. Its proof is almost identical and so we only highlight the differences.

Lemma 2.12. *Let $\phi, \psi : G \to G$ be continuous homomorphism such that $\phi(G), \psi(G)$ have finite indices in G. Then for $f_1, f_2, f_3 \in L^\infty(G)$, we have*

$$\left| \iint_{G^2} f_1(x) f_2(x+\phi(y)) f_3(\psi(y)) \, d\mu(x) d\mu(y) \right| \ll \|\widehat{f_1}\|_\infty \|f_2\|_2 \|f_3\|_2 \tag{11}$$

where the implicit constant depends only on the indices of $\phi(G)$ and $\psi(G)$ in G.

Proof. Similar to the proof of Lemma 2.11, without loss of generality, we can assume f_1, f_2, f_3 are equal to their Fourier series. For $x \in G$, write $g(x) = \int_G f_2(x+\phi(y)) f_3(\psi(y)) \, d\mu(y)$ and then by Plancherel's formula,

$$\left| \int_G f_1(x) g(x) \, d\mu(x) \right| = \left| \sum_{\gamma \in \Gamma} \widehat{f_1}(\gamma) \widehat{g}(\overline{\gamma}) \right| \leq \|\widehat{f_1}\|_\infty \cdot \|\widehat{g}\|_1.$$

Moreover, we also have

$$\begin{aligned} g(x) &= \int_G \left(\sum_{\gamma_2, \gamma_3 \in \Gamma} \widehat{f_2}(\gamma_2) \gamma_2(x+\phi(y)) \widehat{f_3}(\gamma_3) \gamma_3(\psi(y)) \right) d\mu(y) \\ &= \sum_{\substack{\gamma_2, \gamma_3 \in \Gamma, \\ \gamma_2 \circ \phi + \gamma_3 \circ \psi = 0}} \widehat{f_2}(\gamma_2) \widehat{f_3}(\gamma_3) \gamma_2(x). \end{aligned}$$

As a consequence,

$$\widehat{g}(\gamma) = \sum_{\substack{\gamma_2, \gamma_3 \in \Gamma, \\ \gamma_2 \circ \phi + \gamma_3 \circ \psi = 0, \\ \gamma_2 = \gamma}} \widehat{f_2}(\gamma_2) \widehat{f_3}(\gamma_3) = \widehat{f_2}(\gamma) \sum_{\substack{\gamma_3 \in \Gamma, \\ \gamma \circ \phi + \gamma_3 \circ \psi = 0}} \widehat{f_3}(\gamma_3)$$

and so

$$\|\widehat{g}\|_1 \leq \sum_{\substack{\gamma_2, \gamma_3 \in \Gamma, \\ \gamma_2 \circ \phi + \gamma_3 \circ \psi = 0}} \left| \widehat{f_2}(\gamma_2) \right| \left| \widehat{f_3}(\gamma_3) \right|.$$

On the other hand, we have

$$
\begin{aligned}
\sum_{\substack{\gamma_2,\gamma_3\in\Gamma,\\ \gamma_2\circ\phi+\gamma_3\circ\psi=0}} \left|\widehat{f_2}(\gamma_2)\right|\left|\widehat{f_3}(\gamma_3)\right| &= \sum_{\gamma_2\in\Gamma}\left(\left|\widehat{f_2}(\gamma_2)\right| \sum_{\substack{\gamma_3\in\Gamma,\\ \gamma_3\circ\psi=-\gamma_2\circ\phi}} \left|\widehat{f_3}(\gamma_3)\right|\right) \\
&\le \left(\sum_{\gamma_2\in\Gamma}\left|\widehat{f_2}(\gamma_2)\right|^2\right)^{1/2} \left(\sum_{\gamma_2\in\Gamma}\left(\sum_{\substack{\gamma_3\in\Gamma,\\ \gamma_3\circ\psi=-\gamma_2\circ\phi}} \left|\widehat{f_3}(\gamma_3)\right|\right)^2\right)^{1/2} \\
&\ll \|f_2\|_2 \left(\sum_{\gamma_2\in\Gamma}\sum_{\substack{\gamma_3\in\Gamma,\\ \gamma_3\circ\psi=-\gamma_2\circ\phi}} \left|\widehat{f_3}(\gamma_3)\right|^2\right)^{1/2} \qquad (12) \\
&\ll \|f_2\|_2 \left(\sum_{\gamma_3\in\Gamma}\left|\widehat{f_3}(\gamma_3)\right|^2\right)^{1/2} \qquad (13) \\
&= \|f_2\|_2\|f_3\|_2.
\end{aligned}
$$

In (12), we use the fact that for each $\xi\in\Gamma$, there are $\le [G:\psi(G)]$ values of γ_3 such that $\gamma_3\circ\psi=\xi$ while (13) follows from the fact that there are $\le [G:\phi(G)]$ values of γ_2 such that $\gamma_2\circ\phi=\xi$. ☐

3. Bohr sets and partitions

3.1. **Monochromatic configurations.** We make some preparations before the proof of Theorem 1.11. In this section, we only need G to be a commutative semigroup with neutral element. Fix $k+1$ commuting (semigroup) homomorphisms $\psi,\phi_1,\dots,\phi_k : G\to G$. We write

$$\Phi_m = \{\psi^{i_0}\circ\phi_1^{i_1}\circ\cdots\circ\phi_1^{i_k} : 0\le i_0,i_1,\dots,i_k\le m\}\cup\{0\}$$

(where ϕ^i is the i-th composition of ϕ).

For *formal variables* $x_1,\dots,x_n$, we write

$$S_m(x_1,\dots,x_n) = \left\{\sum_{i=1}^n \xi_i(x_i) : \xi_i\in\Phi_m\right\}$$

and we refer to $S_m(x_1,\dots,x_n)$ as the *$S_{m,n}$-set* with *generators* $x_1,\dots,x_n$.

For an element $x=\sum_{i=1}^n\xi_i(x_i)\in S_m(x_1,\dots,x_n)$, by the *support* of x we mean the set $\{i\in[n]:\xi_i\ne 0\}$. The goal of this section is to prove the following:

Theorem 3.1. *For any $r>0$, there exist n and m such that under any r-coloring of $S_m(x_1,\dots,x_n)$, there is a monochromatic configuration*

$$\{\psi(y), x, x+\phi_1(y),\dots,x+\phi_k(y)\},$$

where x,y have nonempty and disjoint supports.

The fact that the supports of x and y are nonempty and disjoint will be crucial in our applications (Theorem 1.11 and Proposition 5.1). Theorem 3.1 follows from Proposition 3.3

below whose proof requires the multidimensional Hales-Jewett theorem (for a reference, see [16, Theorem 7, p.40]). We recall the theorem here for reader's convenience.

The set $[t]^N = \{(x_1, \dots, x_N) : x_i \in [t]\}$ is called a *cube* of dimension N over t elements. Let $[N] = A_0 \cup A_1 \dots A_m$ be any disjoint partition of $[N]$, where $A_i \neq \varnothing$ for $i \neq 0$ (A_0 may be empty), and $f : A_0 \to [t]$ be any map. Define a map $g : [t]^m \to [t]^N$ by assigning to each $(y_1, \dots, y_m) \in [t]^m$ the element $(x_1, \dots, x_N) \in [t]^N$, where

$$x_i = \begin{cases} f(i), & \text{if } i \in A_0 \\ y_j, & \text{if } i \in A_j \text{ for } j \in [m]. \end{cases} \tag{14}$$

A *combinatorial space* of dimension m is the image of g for some choice of $A_0, A_1, \dots, A_m$ and f. We can now state:

Theorem 3.2 (Multidimensional Hales-Jewett)**.** *For any r, t, m, there exists a number $N = HJ(t, m; r)$ such that whenever $[t]^N$ is r-colored, there must be a monochromatic combinatorial space of dimension m.*

Using this, we can prove the following proposition:

Proposition 3.3. *For any $r > 0$ and $\ell > 0$, there exist $n = n(k, \ell, r)$ and $m = m(k, \ell, r)$ such that under any r-coloring of $S_m(x_1, \dots, x_n)$, there are elements $y_1, \dots, y_\ell \in S_m(x_1, \dots, x_n)$ with nonempty and disjoint supports, such that for each $i \in [\ell]$, the elements*

$$\psi(y_i) + \sum_{1 \leq j \leq i-1} \xi_j(y_j) \qquad \text{where } \xi_j \in \{0, \psi, \phi_1, \dots, \phi_k\}$$

have the same color (i.e. their color depends only on i).

Proof. The number of colors r will be fixed throughout. We will proceed by induction on ℓ. When $\ell = 1$ the statement is obvious. Suppose the statement is true for ℓ, we will prove it is true for $\ell + 1$.

Write $n' = n(k, \ell, r), m' = m(k, \ell, r)$. We define $m = m(k, \ell, r) := |\Phi_{m'+1}| + 1$, $N := HJ(|\Phi_{m'+1}|, n'; r)$ and $n = n(k, \ell + 1, r) := 1 + N$.

Consider an arbitrary r-coloring of $S_m(x_1, \dots, x_n)$. An r-coloring of $S_m(x_1, \dots, x_n)$ induces an r-coloring of $\Phi_{m'+1}^N$ by assigning to $(a_1, \dots, a_N) \in (\Phi_{m'+1})^N$ the color of

$$\psi(x_n) + \sum_{i=1}^{N} \phi \circ a_i(x_i).$$

Since $N = HJ(|\Phi_{m'+1}|, n'; r)$, there is a disjoint partition

$$[N] = A_0 \cup A_1 \cdots \cup A_{n'}, \quad A_i \neq \varnothing \, \forall i \neq 0$$

and functions $f_i \in \Phi_{m'+1}$ for $i \in A_0$ such that when $\xi_1, \xi_2, \dots, \xi_{n'}$ range over $\Phi_{m'+1}$, all the elements

$$\psi(x_n) + \sum_{i=1}^{N} \phi \circ a_i(x_i),$$

with

$$a_i = \begin{cases} f_i, & \text{if } i \in A_0 \\ \zeta_j, & \text{if } i \in A_j \text{ for } j \in [n'] \end{cases}$$

have the same color.

Write $z_j = \sum_{i \in A_j} \phi(x_i)$ for $1 \le j \le n'$, and $z_{n'+1} = x_{n+1} + \sum_{i \in A_0} f_i(x_i)$. Then all the z_j have nonempty and disjoint supports, and all elements of the form

$$\psi(z_{n'+1}) + \sum_{j=1}^{n'} \zeta_j(z_j), \qquad \zeta_j \in \Phi_{m'+1},$$

have the same color.

By the inductive hypothesis, there exists a sequence $y_1, \dots, y_\ell \in S_{m'}(z_1, \dots, z_n')$ having nonempty and disjoint supports such that for each $i = 1, \dots, \ell$, the elements

$$\psi(y_i) + \sum_{1 \le j \le i-1} \xi_j(y_j) \qquad \text{where } \xi_j \in \{0, \psi, \phi_1, \dots, \phi_k\}$$

have the same color. We now set $y_{\ell+1} = z_{n'+1}$. Clearly the elements

$$\phi(y_{\ell+1}) + \sum_{1 \le j \le \ell} \xi_j(y_j) \qquad \text{where } \xi_j \in \{0, \psi, \phi_1, \dots, \phi_k\}$$

are of the form

$$\phi(z_{n'+1}) + \sum_{j=1}^{n'} \zeta_j(z_j), \qquad \zeta_j \in \Phi_{m'+1},$$

and so they have the same color. Thus Proposition 3.3 is proved. □

Proof of Theorem 3.1. Applying Proposition 3.3 with $\ell = r + 1$, we can find a sequence $y_1, \dots, y_{r+1}$ satisfying the conclusion of that proposition. Let $c(i)$ be the color of

$$\psi(y_i) + \sum_{1 \le j \le i-1} \xi_j(y_j) \qquad \text{where } \xi_j \in \{0, \psi, \phi_1, \dots, \phi_k\}.$$

Then there exist $1 \le u < v \le r + 1$ such that $c(u) = c(v)$. Hence the elements

$$\psi(y_u), \psi(y_v), \psi(y_v) + \phi_1(y_u), \dots, \psi(y_v) + \phi_k(y_u),$$

have the same color, and we are done (with $x = \psi(y_v), y = y_u$). □

3.2. **Proofs of Theorem 1.11 and Theorem 1.5.** Using Theorem 3.1 we can now prove Theorem 1.11, which we recall for convenience:

Theorem. *Suppose $\phi, \psi_1, \dots, \psi_k : G \to G$ are continuous homomorphisms satisfying:*

(1) $\psi(G), \phi_1(G), \dots, \phi_k(G)$ have finite indices in G, and

(2) $\psi, \phi_1, \dots, \phi_k$ are commuting.

Suppose $G = \bigcup_{i=1}^r A_i$ is a partition of G into measurable sets. Then

$$\sum_{i=1}^r \iint_{G^2} 1_{A_i}(\psi(t)) 1_{A_i}(x) 1_{A_i}(x + \phi_1(t)) \cdots 1_{A_i}(x + \phi_k(t)) \, d\mu(x) d\mu(t) \ge c_3$$

for some positive constant c_3 depending only on r, k and the aforementioned indices.

Proof. Consider the set $S_m(x_1, \dots, x_n)$ given by Theorem 3.1, where we now let $x_1, \dots, x_n$ vary over G. Note that for any $\psi \in \Phi_m$, we have $[G : \psi(G)] < \infty$ by Lemma 2.6. Let R be the set of all pairs (z, t) where $z, t \in S_m(x_1, \dots, x_n)$ have nonempty and disjoint supports.

Suppose $G = \bigcup_{i=1}^r A_i$. For $i \in [r]$, we define

$$T_i := \int_{G^2} 1_{A_i}(\psi(y)) 1_{A_i}(x) 1_{A_i}(x + \phi_1(y)) \cdots 1_{A_i}(x + \phi_k(y))\, d\mu(x) d\mu(y).$$

Let $(z, t) \in R$ be arbitrary, and suppose

$$z = \sum_{u \in U} \zeta_u(x_u) \qquad \text{and} \qquad t = \sum_{v \in V} \xi_v(x_v)$$

where $U, V \subset [n]$ are nonempty and disjoint and $\zeta_u, \xi_v \in \Phi_m \setminus \{0\}$. We have

$$\begin{aligned}
& \int_{G^m} 1_{A_i}(\psi(t)) 1_{A_i}(z) 1_{A_i}(z + \phi_1(t)) \cdots 1_{A_i}(z + \phi_k(t))\, d\mu(x_1) \cdots d\mu(x_m) \\
= \; & \int_{G^m} 1_{A_i}\left(\sum_{v \in V} \psi(\xi_v(x_v))\right) 1_{A_i}\left(\sum_{u \in U} \zeta_u(x_u)\right) 1_{A_i}\left(\sum_{u \in U} \zeta_u(x_u) + \phi_1\left(\sum_{v \in V} \xi_v(x_v)\right)\right) \cdots \\
& \qquad 1_{A_i}\left(\sum_{u \in U} \zeta_u(x_u) + \phi_k\left(\sum_{v \in V} \xi_v(x_v)\right)\right) d\mu(x_1) \cdots d\mu(x_m) \\
\ll \; & \int_{G^m} 1_{A_i}\left(\sum_{v \in V} \psi(x_v)\right) 1_{A_i}\left(\sum_{u \in U} x_u\right) 1_{A_i}\left(\sum_{u \in U} x_u + \phi_1\left(\sum_{v \in V} x_v\right)\right) \cdots \\
& \qquad 1_{A_i}\left(\sum_{u \in U} x_u + \phi_k\left(\sum_{v \in V} x_v\right)\right) d\mu(x_1) \cdots d\mu(x_m) \\
= \; & \int_{G^2} 1_{A_i}(\psi(y)) 1_{A_i}(x + \phi_1(y)) \cdots 1_{A_i}(x + \phi_k(y))\, d\mu(x)\, d\mu(y) \\
= \; & T_i,
\end{aligned}$$

by $|U| + |V|$ applications of Lemma 2.8.

Now Theorem 3.1 implies that

$$\begin{aligned}
1 \;\; & \leq \int_{G^m} \sum_{i=1}^r \sum_{(z,t) \in R} 1_{A_i}(\psi(t)) 1_{A_i}(z) 1_{A_i}(z + \phi_1(t)) \cdots 1_{A_i}(z + \phi_k(t))\, d\mu(x_1) \cdots d\mu(x_m) \\
& \leq \sum_{i=1}^r \int_{G^m} \sum_{(z,t) \in R} 1_{A_i}(\psi(t)) 1_{A_i}(z) 1_{A_i}(z + \phi_1(t)) \cdots 1_{A_i}(z + \phi_k(t))\, d\mu(x_1) \cdots d\mu(x_m) \\
& \ll \sum_{i=1}^r T_i,
\end{aligned}$$

thus finishing the proof. $\square$

To prove Theorem 1.5, we will need the following proposition. With an eye to potential applications, we state and prove a slightly stronger version than what is needed.

Proposition 3.4. *Let $\phi, \psi : G \to G$ be commuting continuous homomorphisms with images having finite indices. Suppose $f_1, \dots, f_r : G \to [0, 1]$ are measurable functions such* *that* $\sum_{i=1}^r f_i \geq 1$ *pointwise. For* $w \in G$*, define*

$$R_i(w) = \iint f_i(\psi(y)) f_i(x+w) f_i(x+\phi(y)) \, d\mu(x) d\mu(y).$$

Then there are $c, k, \eta > 0$ *depending only on* r *and the indices above such that for some* $i \in [r]$*, the set* $\{w \in G : R_i(w) > c\}$ *contains a Bohr-*(k, η) *set.*

Proof. For $i \in [r]$, let $A_i = \{x \in G : f_i(x) \geq 1/r\}$. Since $\sum_{i=1}^r f_i \geq 1$ pointwise, $G = \bigcup_{i=1}^r A_i$. In light of Theorem 1.11, there exists a constant c depending only on r and the indices and an $i \in [r]$ such that

$$\iint_{G^2} 1_{A_i}(x) 1_{A_i}(x+\phi(y)) 1_{A_i}(\psi(y)) \, d\mu(x) d\mu(y) > c.$$

It then follows that

$$R_i(0) \geq \frac{c}{r^3}.$$

On the other hand, by Lemma 2.12, for every $w \in G$,

$$|R_i(w) - R_i(0)| \ll \|\hat{f} - \widehat{f_w}\|_\infty,$$

where the implicit constant depends only on the indices of $\phi(G), \psi(G), (\phi - \psi)(G)$ in G. Hence, there exists a constant c' such that $R_i(w) \geq \frac{c}{2r^3}$ if

$$\|\widehat{f_i} - \widehat{f_{i,w}}\|_\infty < c'.$$

By Lemma 2.1, the set of such w contains a Bohr-(k, η) set, where k and η depend only on c'. □

Theorem 1.5 is now a special case of the next theorem with $\psi_1 = \phi_2$ and $\psi_2 = \phi_1$.

Theorem 3.5. *Let* $G = \bigcup_{i=1}^r A_i$ *be a partition into measurable sets. Let* $\phi_1, \phi_2, \psi_1, \psi_2 : G \to G$ *be continuous homomorphisms satisfying the followings:*

(1) $\phi_2 \circ \psi_2 = \phi_1 \circ \psi_1$*,*
(2) $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1$*,*
(3) $\phi_1(G), \psi_1(G), \psi_2(G)$ *have finite indices in* G*.*

Then for some $1 \leq i \leq r$*, the set* $\phi_1(A_i) - \phi_1(A_i) + \phi_2(A_i)$ *contains a Bohr-*(k, η) *set, where* k *and* η *depend only on* r *and the indices of* $\phi_1(G), \psi_1(G), \psi_2(G)$ *in* G*.*

Proof. Suppose $G = \bigcup_{i=1}^r A_i$. We apply Proposition 3.4 with $f_i = 1_{A_i}$ and (ψ_1, ψ_2) in place of (ψ, ϕ). Then for some i, the set $\{w \in G : R_i(w) > c\}$ contains a Bohr-(k, η) set B. This means that for $w \in B$, there exist $x, y \in G$ such that

$$x + w, \psi_2(y), x + \psi_1(y) \in A_i.$$

Since

$$\phi_1(x+w) + \phi_2(\psi_2(y)) - \phi_1(x + \psi_1(y)) = \phi_1(w),$$

we conclude that $\phi_1(B) \subset \phi_1(A_i) + \phi_2(A_i) - \phi_1(A_i)$. Our theorem now follows from Lemma 2.10. □

Remark 5. If ϕ_1 is an automorphism and $[G:\phi_2]<\infty$ then the hypothesis of Theorem 3.5 is also satisfied. Indeed, we let $\psi_1=\phi_1^{-1}\circ\phi_2$ and $\psi_2=\mathrm{Id}$. Then the first two conditions of Theorem 3.5 are satisfied. As for the third condition, we have $\psi_1(G)=\phi_1^{-1}\circ\phi_2(G)$, which has finite index in G by Lemma 2.6. Similarly, we see that if ϕ_2 is an automorphism and $[G:\phi_1]<\infty$, then the conditions of Theorem 3.5 is also satisfied.

4. Bohr sets and sets of positive measure

4.1. **A regularity lemma.** The goal of this section is to prove Proposition 4.2. As mentioned in the introduction, this argument has its genesis in Bourgain [8]. Bourgain's ideas were elaborated by Tao [31], who proved Roth's theorem in compact abelian groups that are 2-divisible; and by Bergelson-Host-McCutcheon-Parreau [2, Theorem 4.1], who proved Roth's theorem for dilations on the torus $\mathbb{R}/\mathbb{Z}$. We streamline and generalize Bergelson-Host-McCutcheon-Parreau's argument to deal with homomorphisms on arbitrary compact abelian groups. This generalization requires non-trivial modifications; especially, we will make use of Lemma 2.1 and Lemma 2.4.

Lemma 4.1 (cf. [2, Lemma 4.2]). *Let $\phi,\psi:G\to G$ be continuous homomorphisms such that $\phi(G),\psi(G)$ and $(\phi-\psi)(G)$ have finite indices in G. For $f\in L^\infty(G)$, define*

$$J(f)=\iint_{G^2} f(x)f(x+\phi(y))f(x+\psi(y))\,d\mu(x)d\mu(y).$$

Then for any measurable functions $f,g:G\to[0,1]$,

$$|J(f)-J(g)|\ll\|\widehat{f}-\widehat{g}\|_\infty,$$

where the implicit constant depends only on the aforementioned indices.

Proof. We have

$$\begin{aligned} J(f)-J(g) &= \iint_{G^2}(f-g)(x)\cdot f(x+\phi(y))\cdot f(x+\psi(y))\,d\mu(x)d\mu(y)\\ &\quad+\iint_{G^2} g(x)\cdot(f-g)(x+\phi(y))\cdot f(x+\psi(y))\,d\mu(x)d\mu(y)\\ &\quad+\iint_{G^2} g(x)\cdot g(x+\phi(y))\cdot(f-g)(x+\psi(y))\,d\mu(x)d\mu(y).\end{aligned}$$

The lemma now follows from Lemma 2.11 and the assumptions on ϕ and ψ. $\square$

Proposition 4.2 (Regularity Lemma). *Let $f:G\to[0,1]$ be a measurable function with $\int_G f\,d\mu=\delta>0$. Let $\phi,\psi:G\to G$ be continuous homomorphisms such that $\phi(G),\psi(G)$ and $(\phi-\psi)(G)$ have finite indices in G. Then for every $\epsilon>0$, there exist a constant C that depends only on δ,ϵ and the indices above, a kernel $K:G\to\mathbb{R}_{\geq 0}$, and a decomposition $f=f_{st}+f_{er}+f_{un}$ such that*

(1) $\|K\|_\infty<C$,

(2) $\|f_{st}\|_\infty\leq 1$, $\|f_{er}\|_\infty\leq 2$ *and* $\|f_{un}\|_\infty\leq 2$,

(3) $J'(f_{st}):=\iint_{G^2} f_{st}(x)f_{st}(x+\phi(t))f_{st}(x+\psi(t))K(t)\,d\mu(x)d\mu(t)>\delta^3-\epsilon$,

(4) $\|f_{er}\|_2<\epsilon$,

(5) $\|\widehat{f}_{un}\|_\infty \|\widehat{K}\|_1 < \epsilon$.

Proof. For $t \in G$, let

$$d(t) := \max\left(\|\widehat{f} - \widehat{f}_t\|_\infty, \|\widehat{f} - \widehat{f_{\phi(t)}}\|_\infty, \|\widehat{f} - \widehat{f_{\psi(t)}}\|_\infty\right).$$

Fixing $\epsilon > 0$, we define sequences $\eta_n \in (0, 1/2]$, $\kappa_n \in (0, \infty)$ and finite sets $\Lambda_n \subseteq \Gamma$ recursively as follows:

First set $\eta_0 = 1/2$. For $n \geq 0$, Lemma 2.1 implies that there exists a set $\Lambda_n \in \Gamma$ with $|\Lambda_n| \leq 12/\eta_n^2$ such that $D(\eta_n) := \{t \in G : d(t) \leq \eta_n\}$ contains a Bohr set $B(\Lambda_n; \eta_n)$. For $\eta \in (0, 1/2]$, define $\nu(\eta) = (C_0\eta)^{12/\eta^2}$ where C_0 is the constant found in Lemma 2.4; in particular, $\nu(\eta_n) = (C_0\eta_n)^{12/\eta_n^2} \leq (C_0\eta_n)^{|\Lambda_n|}$. Put

$$\kappa_n = \nu(\eta_n)^{-1/2} \text{ and } \eta_{n+1} = \min\left\{\eta_n, \frac{\epsilon^2}{4\kappa_n^2}, \epsilon\nu\left(\frac{\epsilon}{2\kappa_n}\right)\right\}.$$

In view of Lemma 2.4, for $n \geq 0$, there is a kernel $K_n : G \to [0, \infty)$ such that

$$\widehat{K}_n \geq 0, \|\widehat{K_n}\|_1 = \|K_n\|_\infty \leq 1/\nu(\eta_n)$$

and K_n is supported on $B(\Lambda_n; \eta_n) \subseteq D(\eta_n)$. We define

$$f_n = f * K_n.$$

Claim 1:

$$\|\widehat{f} - \widehat{f_n}\|_\infty = \sup_{\gamma\in\Gamma}\left|\widehat{f}(\gamma)(1 - \widehat{K_n}(\gamma))\right| \leq \eta_n.$$

Indeed, by construction, K_n is supported on $D(\eta_n)$; and every $t \in D(\eta_n)$ satisfies $\left|\widehat{f}(\gamma)(1 - \gamma(t))\right| \leq \eta_n$ for all $\gamma \in \Gamma$. Therefore, for all $\gamma \in \Gamma$,

$$\begin{aligned}
\left|\widehat{f}(\gamma)\right|\left|1 - \widehat{K_n}(\gamma)\right| &\leq \left|\widehat{f}(\gamma)\right|\int_G K_n(x)\left|1 - \overline{\gamma(x)}\right| d\mu(x) \\
&= \left|\widehat{f}(\gamma)\right|\int_{D(\eta_n)} K_n(x)\left|1 - \overline{\gamma(x)}\right| d\mu(x) \\
&= \int_{D(\eta_n)} K_n(x)\left|\widehat{f}(\gamma)\right|\left|1 - \overline{\gamma(x)}\right| d\mu(x) \\
&\leq \eta_n \int_{D(\eta_n)} K_n(x)\, d\mu(x) \leq \eta_n.
\end{aligned}$$

Claim 2:

$$\|f_{n+1} - f_n\|_2^2 \leq \|f_{n+1}\|_2^2 - \|f_n\|_2^2 + 2\eta_{n+1}\kappa_n^2.$$

Indeed, we have

$$\begin{aligned}
\|f_{n+1}-f_n\|_2^2 &= \|\widehat{f_{n+1}}-\widehat{f_n}\|_2^2 \\
&= \|\widehat{f_{n+1}}\|_2^2-\|\widehat{f_n}\|_2^2-\sum_{\gamma\in\Gamma}\left(-\widehat{f_{n+1}}(\gamma)\overline{\widehat{f_n}(\gamma)}+\overline{\widehat{f_{n+1}}(\gamma)}\widehat{f_n}(\gamma)\right) \\
&= \|\widehat{f_{n+1}}\|_2^2-\|\widehat{f_n}\|_2^2+2\sum_{\gamma\in\Gamma}\left|\widehat{f}(\gamma)\right|_2^2\widehat{K_n}(\gamma)\left(\widehat{K_n}(\gamma)-\widehat{K_{n+1}}(\gamma)\right) \\
&\le \|\widehat{f_{n+1}}\|_2^2-\|\widehat{f_n}\|_2^2+2\sum_{\gamma\in\Gamma}\left|\widehat{f}(\gamma)\right|^2\widehat{K_n}(\gamma)\left(1-\widehat{K_{n+1}}(\gamma)\right) \\
&\le \|\widehat{f_{n+1}}\|_2^2-\|\widehat{f_n}\|_2^2+2\sup_{\gamma\in\Gamma}|\widehat{f}(\gamma)|\left(1-\widehat{K_{n+1}}(\gamma)\right)\cdot\|\widehat{K_n}\|_1 \\
&\le \|\widehat{f_{n+1}}\|_2^2-\|\widehat{f_n}\|_2^2+2\eta_{n+1}\kappa_n^2
\end{aligned}$$

and the claim is proved.

Since $\eta_{n+1}\le\epsilon^2/(4\kappa_n^2)$, we have

$$\|f_{n+1}-f_n\|_2^2\le\|f_{n+1}\|_2^2-\|f_n\|_2^2+\epsilon^2/2.$$

Let M be the smallest integer such that $M\ge 2/\epsilon^2$. Then

$$\sum_{n=0}^{M-1}\|f_{n+1}-f_n\|_2^2\le\|f_M\|_2^2-\|f_0\|_2^2+M\epsilon^2/2\le 1+M\epsilon^2/2\le M\epsilon^2.$$

Therefore there exists $0\le n\le M-1$ such that

$$\|f_{n+1}-f_n\|_2\le\epsilon.$$

From now on, we fix this n. Next consider the expression

$$I_n(t)=\int_G f_n(x)f_n(x+\phi(t))f_n(x+\psi(t))\,d\mu(x)\quad\text{for } t\in G.$$

We have

$$|I_n(0)-I_n(t)|\le\|f_n-(f_n)_{\phi(t)}\|_1+\|f_n-(f_n)_{\psi(t)}\|_1.$$

Note that

$$\begin{aligned}
\|f_n-(f_n)_{\phi(t)}\|_1^2 &\le \|f_n-(f_n)_{\phi(t)}\|_2^2=\|(f-f_{\phi(t)})*K_n\|_2^2 \\
&= \sum_{\gamma\in\Gamma}\left|\widehat{K_n}(\gamma)\right|^2\left|\widehat{f}(\gamma)\right|^2|1-\gamma(\phi(t))|^2 \\
&\le \|\widehat{K_n}\|_1 d(t)^2\le\kappa_n^2 d(t)^2.
\end{aligned}$$

The same estimate holds for $\|f_n-(f_n)_{\psi(t)}\|_1^2$. Hence $|I_n(0)-I_n(t)|\le 2\kappa_n d(t)$ for any $t\in G$.

Since $I_n(0)=\|f_n\|_1^3=\|f\|_1^3\ge\delta^3$, it follows that

$$I_n(t)\ge\delta^3-2\kappa_n d(t)\text{ for all } t\in G.$$

Note that $d(t)\le\epsilon/(2\kappa_n)$ for t in the set $D(\epsilon/(2\kappa_n))$ and so $I_n(t)\ge\delta^3-\epsilon$ in this set.

Let $\eta = \epsilon/(2\kappa_n)$. In view of Lemma 2.4, there exists a kernel K supported on $D(\eta)$ such that $\|K\|_\infty \le 1/\nu(\eta)$. We then have

$$J'(f_n) := \int_G I_n(t)K(t)\, d\mu(t) \ge (\delta^3 - \epsilon)\int_{B(\epsilon/(2\kappa_n))} K(t)\, d\mu(t) \ge \delta^3 - \epsilon.$$

Letting $f_{st} = f_n$, $f_{er} = f_{n+1} - f_n$ and $f_{un} = f - f_{n+1}$, we obtain

(1) $\|K\|_\infty \le 1/\nu(\epsilon/(2\kappa_n)) \le 1/\nu(\epsilon/(2\kappa_M))$,

(2) $\|f_{er}\|_2 = \|f_{n+1} - f_n\|_2 < \epsilon$,

(3) $\|\hat{f}_{un}\|_\infty \|K\|_\infty = \|\hat{f} - \hat{f}_{n+1}\|_\infty \|K\|_\infty < \eta_{n+1}/\nu(\eta) < \epsilon$ because $\eta_{n+1} \le \epsilon\nu(\epsilon/(2\kappa_n)) = \epsilon\nu(\eta)$.

(4) $J'(f_{st}) = J'(f_n) \ge \delta^3 - \epsilon$.

Our proof finishes. $\square$

4.2. **Proof of density results.** The goal of this section is to prove Theorem 1.6 and Theorem 1.9. First we recall Theorem 1.9 for the reader's convenience.

Theorem (Khintchine-Roth theorem for compact abelian groups)**.** *Let $f : G \to [0,1]$ be a mensurable function with $\int_G f\, d\mu > \delta$. Let $\phi, \psi : G \to G$ be continuous homomorphisms such that $[G : \phi(G)], [G : \psi(G)]$ and $[G : (\phi - \psi)G]$ are finite. Then for every $\epsilon > 0$, there exists a constant c_1 that depends only on δ, ϵ and the indices above such that the set*

$$B = \left\{ t \in G : \int_G f(x)f(x+\phi(t))f(x+\psi(t))\, d\mu(x)d\mu(t) > \delta^3 - \epsilon \right\}$$

has measure greater than c_1. As a consequence, there exists a constant c_2 that depends only on δ and indices of $\phi(G), \psi(G)$ such that

$$J(f) := \iint_{G^2} f(x)f(x+\phi(t))f(x+\psi(t))d\mu(x)d\mu(t) > c_2.$$

Proof. Fix $\epsilon > 0$ and let constant C, kernel K and the decomposition $f = f_{st} + f_{er} + f_{un}$ be as found in Proposition 4.2. Define

$$J'(f) := \iint_{G^2} f(x)f(x+\phi(t))f(x+\psi(t))K(t)\ d\mu(x)d\mu(t)$$

and

$$J'(f_{st}) := \iint_{G^2} f_{st}(x)f_{st}(x+\phi(t))f_{st}(x+\psi(t))K(t)\ d\mu(x)d\mu(t).$$

Applying the decomposition $f = f_{st} + f_{er} + f_{un}$ and expanding $J'(f)$, we see that the difference $J'(f) - J'(f_{st})$ will have 26 terms. The terms that contain f_{er} can be bounded by 4ϵ since for $f_1, f_2, f_3 \in L^\infty(G)$,

$$\iint_{G^2} f_1(x)f_2(x+\phi(t))f_3(x+\psi(t))K(t)\, d\mu(x)d\mu(t) \le \max_i \|f_i\|_\infty^2 \max_i \|f_i\|_1 \|K\|_1.$$

On the other hand, in view of Lemma 2.11, the terms containing f_{un} are bounded by $O(\|\hat{f}_{un}\|_\infty \|\widehat{K}\|_1)$ which is $O(\epsilon)$ thanks to the properties of the decomposition. Therefore,

$$J'(f) > J'(f_{st}) - O(\epsilon) > \delta^3 - c_0\epsilon \tag{15}$$

where the constant c_0 depends only on the indices of $\phi(G), \psi(G)$ and $(\phi - \psi)(G)$ in G.

Define
$$I_f(t) = \int_G f(x)f(x+\phi(t))f(x+\psi(t))\,d\mu(x)$$
and
$$B = \{t \in G : I_f(t) > \delta^3 - 2c_0\epsilon\}.$$
We then have
$$J'(f) = \int_G I_f(t)K(t)\,d\mu(t) = \int_B I_f(t)K(t)\,d\mu(t) + \int_{G\setminus B} I_f(t)K(t)\,d\mu(t) \leq$$
$$\int_B K(t)\,d\mu(t) + (\delta^3 - 2c_0\epsilon)\int_{G\setminus B} K(t)\,d\mu(t) \leq \|K\|_\infty \mu(B) + (\delta^3 - 2c_0\epsilon). \quad (16)$$
Combining (15) and (16), we deduce that
$$\mu(B) > c_0\epsilon/\|K\|_\infty.$$
Letting $c_1 = c_0\epsilon/\|K\|_\infty$, we obtain the first part of the theorem.

Now we have
$$J(f) = \int_G I_f(t)\,d\mu(t) > (\delta^3 - 2c_0\epsilon)c_1.$$
Letting $c_2 = c_1(\delta^3 - 2c_0\epsilon)$, we obtain the second part of the theorem. □

In order to prove Theorem 1.6, we need the following proposition. For our future applications, we will state and prove a slightly more general version than what is necessary.

Proposition 4.3. *Suppose $\phi, \psi : G \to G$ are continuous homomorphisms such that $\phi(G), \psi(G), (\phi - \psi)(G)$ have finite indices in G. Let $f : G \to [0,1]$ such that $\int_G f\,d\mu = \delta > 0$. For $w \in G$, define*
$$R(w) = \iint_{G^2} f(x+w)f(x+\phi(y))f(x+\psi(y))\,d\mu(x)d\mu(y)$$
Then there are $c, k, \eta > 0$ depending only on δ and the indices above such that the set $\{w \in G : R(w) > c\}$ contains a Bohr-(k,η) set.

Proof. By Lemma 2.11, we have
$$|R(w) - R(0)| \ll \|\hat{f} - \widehat{f_w}\|_\infty$$
where implicit constant depends only on the indices of $\phi(G), \psi(G), (\phi-\psi)(G)$ in G. By Theorem 1.9, we know that $R(0) > c$ for some constant $c > 0$ depending on these indices and δ. It follows that there exists a constant c' such that $R(w) > c/2$ if
$$\|\hat{f} - \widehat{f_w}\|_\infty < c'. \quad (17)$$
Lemma 2.1 implies that the set of such w contains a Bohr-(k,η) set, where k and η depend only on c'. □

We can now formulate and prove our main theorem for sets of positive measure.

Theorem 4.4. *Let $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2 : G \to G$ be continuous homomorphisms satisfying the following*

(1) $\phi_1 + \phi_2 + \phi_3 = 0$,

(2) $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$,

(3) $\phi_3(G), \psi_1(G), \psi_2(G), (\psi_1 + \psi_2)(G)$ *have finite indices in* G.

The for any measurable set $A \subset G$, $\mu(A) = \delta > 0$, *the set* $\phi_1(A) + \phi_2(A) + \phi_3(A)$ *contains a Bohr-*(k, η) *set, where* k *and* η *depend only on* δ *and the indices above.*

Proof. Applying Proposition 4.3 for $f = 1_A$ and ψ_1, ψ_2 in place of ϕ and ψ, we see that there exists a Bohr-(k, η) set B such that for all $w \in B$, there are $x, y \in G$ such that

$$x + w, x + \psi_1(y) \text{ and } x - \psi_2(y) \in A.$$

Note that

$$\phi_3(x + w) + \phi_1(x + \psi_1(y)) + \phi_2(x - \psi_2(y)) = \phi_3(w)$$

and so that $\phi_1(A) + \phi_2(A) + \phi_3(A) \supseteq \phi_3(B)$. Our theorem then follows from Lemma 2.10. □

Theorem 1.6 is now a special case of Theorem 4.4 when $\psi_1 = \phi_2$ and $\psi_2 = \phi_1$.

Remark 6. If ϕ_1 is an automorphism and $[G : \phi_2(G)], [G : \phi_3(G)] < \infty$ then the hypothesis of Theorem 4.4 is also satisfied. Indeed, we let $\psi_1 = \phi_1^{-1} \circ \phi_2$ and $\psi_2 = \mathrm{Id}$. Then the first two conditions of Theorem 4.4 are satisfied. As for the third condition, we have $\psi_1(G) = \phi_1^{-1} \circ \phi_2(G)$ and $(\psi_1 + \psi_2)(G) = \phi_1^{-1} \circ (\phi_2 + \phi_1)(G)$. Both of these have finite indices in G by Lemma 2.6.

5. Bohr sets in sumsets in number fields and function fields

In this section we prove Theorems 1.4, 1.7 and 1.8 using a strategy similar to Bergelson and Ruzsa's proof of Theorem 1.2. To prove Theorem 1.2, one could embed $A \cap [N]$ naturally in $\mathbb{Z}_N$, and invoke the counting result (for example, Theorem 1.9) in $\mathbb{Z}_N$. However, one has to deal with the "wraparound effect": A solution to $s_1 x + s_2 y + s_3 z = 0$ in $\mathbb{Z}_N$ does not necessarily correspond to a solution in $\mathbb{Z}$. To overcome this issue, Bergelson and Ruzsa embedded $A \cap [N]$ in $\mathbb{Z}_{N'}$ for some $N' \gg N$. Then $A \cap [N]$ remains dense in $\mathbb{Z}_{N'}$ and a solution in $\mathbb{Z}_{N'}$ found in $A \cap [N]$ is now a solution in $\mathbb{Z}$.

For partitions, the corresponding counting result would be Theorem 1.11. However, if this theorem were applied directly, we would have a partition of the whole group $\mathbb{Z}_{N'}$ which again causes the wrap-around effect. To avoid this problem, we need to modify Theorem 1.11 so that it allows for partitions of a subset $[-N, N] \subset \mathbb{Z}_{N'}$ instead of the whole group.

Proposition 5.1. *For any* $k, \ell, r > 0$, *there is a constant* $c(k, \ell, r) > 0$ *such that the following holds: For sufficiently large* N, *if* $[-N, N] = \bigcup_{i=1}^r A_i$, *then for some* $1 \le i \le r$, *we have*

$$\sum_{|x|, |y| \le N} 1_{A_i}(\ell y) 1_{A_i}(x) 1_{A_i}(x + y) \cdots 1_{A_i}(x + ky) \ge c(k, \ell, r) N^2.$$

Remark 7. Proposition 5.1 also follows from Frankl-Graham-Rödl [12, Theorem 1], but our proof shows that it is directly in line with Theorem 1.11. Furthermore, our proof easily generalizes to other rings such as $\mathbb{Z}[i]$ and $\mathbb{F}_q[t]$.

Proof. We applying Theorem 3.1 with $\psi(y) = \ell y$ and $\phi_j(y) = jy$ for $1 \leq j \leq k$. Then there exist m and n depending only on r and k such that for any r-coloring of $S_m(x_1, \ldots, x_n)$, there are x and y of nonempty and disjoint support such that the configuration

$$\{\ell y, x, x+y, \ldots, x+ky\}$$

is monochromatic.

Note that elements of $S_m(x_1, \ldots, x_n)$ are all linear forms in $x_1, \ldots, x_n$ with bounded integer coefficients. We now let $x_1, \ldots, x_n$ vary over $[-cN, cN]$ where c is a small constant. Then $S_m(x_1, \ldots, x_n) \subset [-N, N]$. Under the partition $[-N, N] = \bigcup_{i=1}^r A_i$, each set $S_m(x_1, \ldots, x_n)$ contains a monochromatic configuration $\{\ell y, x, x+y, \ldots, x+ky\}$. There are $\gg N^n$ monochromatic configurations arising in this way. However, a configuration may come from many different sets $S_m(x_1, \ldots, x_n)$. We will show that the number of tuples $(x_1, \ldots, x_n)$ giving rise to the same configuration $\{\ell y, x, x+y, \ldots, x+ky\}$ is $\ll N^{n-2}$.

Indeed, let I, J be disjoint nonempty subsets of $[n]$ such that x and y are linear combinations with bounded coefficients of $(x_i)_{i \in I}$ and $(x_j)_{j \in J}$, respectively. For fixed I and J, the number of choices for $(x_i)_{i \in I}$ is $\ll N^{|I|-1}$, since any choice of $(|I|-1)$ of the x_i's gives at most one value for the remaining x_i. For the same reason, the number of choices for $(x_j)_{j \in J}$ is $\ll N^{|J|-1}$. Since there are finitely many pairs (I, J), we see that the number of $(x_1, \ldots, x_n)$ that give rise to $\{\ell y, x, x+y, \ldots, x+ky\}$ is $\ll N^{n-2}$. Hence the number of monochromatic configurations in $[N]$ is $\gg N^2$, and we are done. ☐

Our next statement is essentially a diagonalization argument.

Proposition 5.2. *Let $\mathcal{P}$ denote an arbitrary partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$. Then there exists some $1 \leq i \leq r$ with the following property: For every $\ell \geq 0$, there is a constant $c(\ell, \mathcal{P})$ such that*

$$\sum_{|x|,|y| \leq N} 1_{A_i}(y) 1_{A_i}(x) 1_{A_i}(x+\ell y) \geq c(\ell, \mathcal{P}) N^2$$

for infinitely many $N \in \mathbb{N}$.

Proof. Invoking Proposition 5.1, for each $k \in \mathbb{N}$, there is $i = f(k)$ such that for infinitely many N, we have

$$\sum_{|x|,|y| \leq N} 1_{A_i}(y) 1_{A_i}(x) 1_{A_i}(x+y) \cdots 1_{A_i}(x+ky) \geq c(k, 1, r) N^2.$$

Hence there exist an $i \in \{1, \ldots, r\}$ and an infinite set K such that $f(k) = i$ for all $k \in K$.

Let ℓ be arbitrary and pick $k \in K, k \geq \ell$. We have, for infinitely many N,

$$\sum_{|x|,|y| \leq N} 1_{A_i}(y) 1_{A_i}(x) 1_{A_i}(x+\ell y)$$
$$\geq \sum_{|x|,|y| \leq N} 1_{A_i}(y) 1_{A_i}(x) 1_{A_i}(x+y) \cdots 1_{A_i}(x+ky) \geq c(k, 1, r) N^2,$$

thus proving the desired claim. ☐

Remark 8. In the proof above, we do not have any control on $c(k,1,r)$ since we do not have control on k. As a result, the constant $c(\ell,\mathcal{P})$ above depends on the partition. It is interesting to see if this dependence is indeed necessary.

We can now prove Theorem 1.4.

Proof of Theorem 1.4(a). Let $\mathbb{Z}=\bigcup_{i=1}^r A_i$ be an arbitrary partition and $s_1,s_2\in\mathbb{Z}\setminus\{0\}$. Without loss of generality, we assume $s_1,s_2>0$. For a set $A\subset\mathbb{Z}$ and $N>0$, we write $A^{(N)}$ to denote $A\cap[-N,N]$.

By Proposition 5.1, there exist $i\in[r]$ and an infinite set $\mathcal{N}$ such that

$$\sum_{|x|,|y|\le N} 1_{A_i^{(N)}}(s_1y)1_{A_i^{(N)}}(x)1_{A_i^{(N)}}(x+s_2y)\ge cN^2 \tag{18}$$

for any $N\in\mathcal{N}$, where $c>0$ is a constant independent of N.

Let N' be the smallest odd integer greater than $(2s_1+s_2+1)N$. We identify $\mathbb{Z}_{N'}$ with $[-\frac{N'-1}{2},\frac{N'-1}{2}]$. Then (18) implies that

$$\sum_{x,y\in\mathbb{Z}_{N'}} 1_{A_i^{(N)}}(s_1y)1_{A_i^{(N)}}(x)1_{A_i^{(N)}}(x+s_2y)\ge c'N'^2 \tag{19}$$

for some constant $c'>0$ independent of N. Define

$$R(w)=\sum_{x,y\in\mathbb{Z}_{N'}} 1_{A_i^{(N)}}(s_1y)1_{A_i^{(N)}}(x+w)1_{A_i^{(N)}}(x+s_2y).$$

Then by the same argument as the proof of Proposition 3.4, the set $\{w\in\mathbb{Z}_{N'}:R(w)>0\}$ contains a Bohr-(k,η) set in $\mathbb{Z}_{N'}$, where k and η are independent of N. Note that $R(w)>0$, implies there are $a,a',a''\in A_i^{(N)}$ and $x,y\in\mathbb{Z}_{N'}$ such that

$$s_1y\equiv a,\quad x+w\equiv a',\quad\text{and}\quad x+s_2y\equiv a''\pmod{N'}.$$

Therefore,

$$s_1w=s_1(x+w)-s_1(x+s_2y)+s_2(s_1y)\equiv s_1a'-s_1a''+s_2a\pmod{N'}.$$

If $|w|\le N$ then this congruence is an equality in $\mathbb{Z}$ thanks to the way we choose N' and the fact that $|a|,|a'|,|a''|\le N$. We have thus proved that, for each $N\in\mathcal{N}$, there exist $x_1,\dots,x_k\in[-\frac{N'-1}{2},\frac{N'-1}{2}]$ such that

$$(s_1A_i-s_1A_i+s_2A_i)/s_1\supset[-N,N]\cap\left\{w\in\mathbb{Z}:\left|e\left(\frac{x_jw}{N'}\right)-1\right|<\eta\quad\forall j=1,\dots,k\right\}.$$

Here we are using the notation A/c defined in (2).

As $N\in\mathcal{N},N\to\infty$ and by passing to a subsequence if necessary, the sequence $(\frac{x_1}{N'},\dots,\frac{x_k}{N'})$ converges to a point $(\alpha_1,\dots,\alpha_k)$ in $(\mathbb{R}/\mathbb{Z})^k$. Hence,

$$(s_1A_i-s_1A_i+s_2A_i)/s_1\supset\left\{w\in\mathbb{Z}:|e(\alpha_jw)-1|<\frac{\eta}{2}\quad\forall j=1,\dots,k\right\}.$$

This implies that

$$s_1A_i-s_1A_i+s_2A_i\supset\left\{n\in\mathbb{Z}:\left|e\left(\frac{\alpha_jn}{s_1}\right)-1\right|<\frac{\eta}{2}\quad\forall j=1,\dots,k\right\}\cap s_1\mathbb{Z}.$$

Since $s_1\mathbb{Z}$ is a Bohr set and the intersection of two Bohr sets is a Bohr set, our proof finishes. $\square$

Proof of Theorem 1.4(b). We proceed similarly to part (a), using Proposition 5.2 instead of Proposition 5.1. Let $\mathcal{P}$ be an arbitrary partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$. Let i be given by Proposition 5.2. Let $s \in \mathbb{N}$ be arbitrary. Then there is an infinite set $\mathcal{N}_s \subset \mathbb{N}$ such that for any $N \in \mathcal{N}_s$, we have

$$\sum_{|x|,|y|\leq N} 1_{A_i^{(N)}}(y)1_{A_i^{(N)}}(x)1_{A_i^{(N)}}(x+sy) \geq c(s,\mathcal{P})N^2. \tag{20}$$

for some constant $c(s,\mathcal{P}) > 0$ independent of N. Note that

$$w = (w+x) - (w+sy) + sy.$$

The rest is identical to part (a). $\square$

5.1. **Sumsets in $\mathbb{Z}[i]$.** Even though the corresponding tori in the cases of $\mathbb{Z}[i]$ and $\mathbb{F}_q[t]$ are slightly different from $\mathbb{Z}$, the general approaches are very similar. Therefore, we will be brief and highlight only the differences.

The following proposition is needed for the proof of Theorem 1.7(b,c). We omit its proof since it is identical to the ones of Propositions 5.1 and 5.2.

Proposition 5.3.

(a) Let $b, a_1, \ldots, a_k \in \mathbb{Z}[i]$ and $r > 0$. There is a constant $c = c(b, a_1, \ldots, a_k, r) > 0$ such that the following holds: For N sufficiently large, if $[-N,N]^2 = \bigcup_{j=1}^r A_j$, then for some $1 \leq j \leq r$, we have

$$\sum_{x,y\in[-N,N]^2} 1_{A_j}(by)1_{A_j}(x)1_{A_j}(x+a_1y)\cdots 1_{A_i}(x+a_ky) \geq cN^4.$$

(b) Let $\mathcal{P}$ denote an arbitrary partition $\mathbb{Z}[i] = \bigcup_{j=1}^r A_j$. Then there exists some $1 \leq j \leq r$ with the following property: For each $\ell \in \mathbb{Z}[i]$, there is a constant $c(\ell,\mathcal{P})$ such that

$$\sum_{x,y\in[N]^2} 1_{A_j}(y)1_{A_j}(x)1_{A_j}(x+\ell y) \geq c(\ell,\mathcal{P})N^4.$$

for infinitely many $N \in \mathbb{N}$.

Proof of Theorem 1.7 (a). Suppose $A \subset \mathbb{Z}[i]$ has $\overline{d}(A) = \delta > 0$. Then for infinitely many N, we have $|A^{(N)}| \geq \delta N^2$, where $A^{(N)} = A \cap [-N,N]^2$.

Let $N' = 2(|s_1|+|s_2|+|s_3|)N + 1$. We identify $[-\frac{N'-1}{2}, \frac{N'-1}{2}]^2$ with $\mathbb{Z}_{N'} \times \mathbb{Z}_{N'}$. By Theorem 1.6, the set $s_1A + s_2A + s_3A$ contains a Bohr set in $\mathbb{Z}_{N'} \times \mathbb{Z}_{N'}$, which is of the form

$$\left\{(w,v) \in \mathbb{Z}_{N'} \times \mathbb{Z}_{N'} : \left| e\left(\frac{wx_j + vy_j}{N'}\right) - 1\right| < \eta \quad \forall j = 1,\ldots,k\right\}$$

for some $x_1, \ldots, x_k, y_1, \ldots, y_k \in [-\frac{N'-1}{2}, \frac{N'-1}{2}]$, where k and η depend only on δ and s_1, s_2, s_3.

If (w,v) is in the Bohr set above and $|w|, |v| \leq N$, then there exist $a, a', a'' \in A^{(N)}$ such that

$$(w,v) = s_1a + s_2a' + s_3a'',$$

where the equality is in $\mathbb{Z}[i]$ and not just in $\mathbb{Z}_{N'} \times \mathbb{Z}_{N'}$. Hence,

$$s_1A+s_2A+s_3A \supset [-N,N]^2 \cap \left\{(w,v) \in \mathbb{Z}[i] : \left| e\left(\frac{wx_j+vy_j}{N'}\right) - 1\right| < \eta \quad \forall j=1,\ldots,k\right\}.$$

Letting N go to infinity along some subsequence, we have that

$$s_1A+s_2A+s_3A \supset \left\{(w,v) \in \mathbb{Z}[i] : \left| e\left(w\alpha_j+v\beta_j\right) - 1\right| < \frac{\eta}{2} \quad \forall j=1,\ldots,k\right\},$$

where $(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$ is a limit point of $(\frac{x_1}{N'},\ldots,\frac{x_k}{N'},\frac{y_1}{N'},\ldots,\frac{y_k}{N'})$, and we are done. □

Proof of Theorem 1.7(b). Using Proposition 5.3(a) and arguing similarly to the proof of Theorem 1.4(a), we see that for some $1 \le j \le r$, for infinitely many N, we have

$$(s_1A_j-s_1A_j+s_2A_j)/s_1 \supset [-N,N]^2 \cap \left\{(w,v) \in \mathbb{Z}[i] : \left| e\left(\frac{wx_j+vy_j}{N'}\right) - 1\right| < \eta \quad \forall j=1,\ldots,k\right\}$$

Letting N go to infinity, we have

$$(s_1A_j-s_1A_j+s_2A_j)/s_1 \supset \left\{(w,v) \in \mathbb{Z}[i] : \left| e\left(w\alpha_j+v\beta_j\right) - 1\right| < \frac{\eta}{2} \quad \forall j=1,\ldots,k\right\},$$

where $(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$ is a limit point of $(\frac{x_1}{N'},\ldots,\frac{x_k}{N'},\frac{y_1}{N'},\ldots,\frac{y_k}{N'})$. Note that

$$w\alpha_j + v\beta_j = \Re((w+iv)(\alpha_j - i\beta_j))$$

and hence,

$$s_1A-s_1A+s_2A \supset \left\{z \in \mathbb{Z}[i] : \left| e\left(\Re\left(z\frac{\alpha_j - i\beta_j}{s_1}\right)\right) - 1\right| < \frac{\eta}{2} \quad \forall j=1,\ldots,k\right\} \cap s_1\mathbb{Z}[i],$$

which is a Bohr set by Lemma 2.2. □

The proof of Theorem 1.7(c) is similar to part (b), using Proposition 5.3(b) instead of Proposition 5.3(a).

5.2. **Sumsets in $\mathbb{F}_q[t]$.** Let p be a prime and q be a power of p. First, let us introduce some standard facts about $\mathbb{F}_q[t]$. Let $\mathbb{K} = \mathbb{F}_q(t)$ be the field of fractions of $\mathbb{F}_q[t]$. For $f/g \in \mathbb{K}$ we define $|f/g| = q^{\deg(f)-\deg(g)}$ and $|0| = 0$. The completion of $\mathbb{K}$ with respect to $|\cdot|$ is $\mathbb{K}_\infty = \mathbb{F}_q((\frac{1}{t})) = \left\{\sum_{i=-\infty}^{n} a_i t^i : a_i \in \mathbb{F}_q, n \in \mathbb{Z}\right\}$. Let $\mathbb{T}_q = \left\{\sum_{i=-\infty}^{-1} a_i t^i : a_i \in \mathbb{F}_q\right\}$. Then $\mathbb{F}_q[t], \mathbb{K}, \mathbb{K}_\infty, \mathbb{T}_q$ are the analogs of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$, respectively.

For $x \in \mathbb{F}_q$, we write $e_q(x) = e\left(\frac{\mathrm{Tr}(x)}{p}\right)$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map.[4] It can be checked that $x \mapsto e_q(ax)$ (where $a \in \mathbb{F}_q$) are all the additive characters of $\mathbb{F}_q$.

If $\alpha = \sum_{i=-\infty}^{n} a_i t^i \in \mathbb{K}_\infty$, we write $(\alpha)_{-1} = a_{-1}$ and $E(\alpha) = e_q(a_{-1})$. It can be checked that $f \mapsto E(f\alpha)$, where $\alpha \in \mathbb{T}_q$, are all the continuous characters of $\mathbb{F}_q[t]$. This also shows that $\mathbb{T}_q$ is the dual of $\mathbb{F}_q[t]$.

Any Bohr set B in $\mathbb{F}_q[t]$ is of the form

$$B = \{f \in \mathbb{F}_q[t] : |E(f\alpha_i) - 1| < \eta \text{ for } i = 1,\ldots,k\},$$

where $\alpha_1,\ldots,\alpha_k \in \mathbb{T}_q$. If $\eta < |e(1/p) - 1|$ then

$$B = \{f \in \mathbb{F}_q[t] : \mathrm{Tr}((f\alpha_i)_{-1}) = 0 \text{ for } i = 1,\ldots,k\}.$$

[4]That is, $\mathrm{Tr}(x)$ is the trace of the $\mathbb{F}_p$-linear map $y \mapsto xy$ from $\mathbb{F}_q$ to $\mathbb{F}_q$, when $\mathbb{F}_q$ is viewed as a $\mathbb{F}_p$-vector space. In particular, $\mathrm{Tr}(x) \in \mathbb{F}_p$.

This is an $\mathbb{F}_p$-subspace and not necessarily an $\mathbb{F}_q$-subspace. However, it contains the $\mathbb{F}_q$-subspace

$$\{f \in \mathbb{F}_q[t] : (f\alpha_i)_{-1} = 0 \text{ for } i = 1, \ldots, k\}.$$

We write $G_N = \{f \in \mathbb{F}_q[t] : \deg(f) < N\}$. For a set $A \subset \mathbb{F}_q[t]$, we write $A^{(N)}$ for $A \cap G_N$. Moreover, for each N, we fix a polynomial $P_N \in \mathbb{F}_q[t]$ of degree N. Then $G_N \cong \mathbb{F}_q[t]/(P_N)$. While G_N is already a group, we work with $\mathbb{F}_q[t]/(P_N)$ since the multiplication $f \mapsto sf$ is a homomorphism on the latter.

Using the same arguments as in Propositions 5.1 and 5.2, we can prove the following:

Proposition 5.4.

(a) Let $b, a_1, \ldots, a_k \in \mathbb{F}_q[t]$ and $r > 0$. There is a number $c = c(q, b, a_1, \ldots, a_k, r) > 0$ such that the following holds. For N sufficiently large, if $G_N = \bigcup_{j=1}^r A_i$, then for some $1 \leq i \leq r$, we have

$$\sum_{x,y \in G_N} 1_{A_i}(by)1_{A_i}(x)1_{A_i}(x + a_1 y) \cdots 1_{A_i}(x + a_k y) \geq cq^{2N}.$$

(b) Let $\mathcal{P}$ denote an arbitrary partition $\mathbb{F}_q[t] = \bigcup_{i=1}^r A_i$. Then there exists some $1 \leq i \leq r$ with the following property: For each $\ell \in \mathbb{F}_q[t]$, there is a constant $c(\ell, \mathcal{P})$ such that

$$\sum_{x,y \in \mathbb{F}_q[t]} 1_{A_i}(y)1_{A_i}(x)1_{A_i}(x + \ell y) \geq c(\ell, \mathcal{P})q^{2N}$$

for infinitely many $N \in \mathbb{N}$.

Proof of Theorem 1.8. We will sketch the proof of Theorem 1.8(b). Parts (a) and (c) can be proved along the same lines.

Let $\mathbb{F}_q[t] = \bigcup_{i=1}^r A_i$ be an arbitrary partition and $s_1, s_2 \in \mathbb{F}_q[t] \setminus \{0\}$. By Proposition 5.4(a), we know that there exist $1 \leq i \leq r$ and an infinite set $\mathcal{N}$ such that

$$\sum_{x,y \in G_N} 1_{A_i^{(N)}}(s_1 y)1_{A_i^{(N)}}(x)1_{A_i^{(N)}}(x + s_2 y) \gg q^{2N} \tag{21}$$

for each $N \in \mathcal{N}$.

Let $N' = \max(\deg s_1, \deg s_2) + N$. We identify $G_{N'}$ with $\mathbb{F}_q[t]/(P_{N'})$. Arguing similarly to the proof of Theorem 1.4(a) and using (21), we find that

$$(s_1 A_i - s_1 A_i + s_2 A_i)/s_1 \supset G_N \cap \left\{ w \in \mathbb{F}_q[t] : (w\frac{x_i}{P_{N'}})_{-1} = 0 \quad \forall j = 1 \ldots, k \right\},$$

for some $x_1, \ldots, x_k \in G_{N'}$.

Letting $N \to \infty$ and using compactness of $\mathbb{T}_q$, we have

$$(s_1 A_i - s_1 A_i + s_2 A_i)/s_1 \supset \{w \in \mathbb{F}_q[t] : (w\alpha_i)_{-1} = 0 \quad \forall j = 1, \ldots, k\}$$

for some $\alpha_1, \ldots, \alpha_k \in \mathbb{T}_q$. Therefore,

$$s_1 A_i - s_1 A_i + s_2 A_i \supset \left\{ f \in \mathbb{F}_q[t] : (f\frac{\alpha_i}{s_1})_{-1} = 0 \quad \forall j = 1, \ldots, k \right\} \cap s_1 \mathbb{F}_q[t],$$

which is clearly an $\mathbb{F}_q$-subspace of bounded codimension. $\square$

6. Open questions

Theorem 1.4(b) says that in any partition $\mathbb{Z} = \bigcup_{i=1}^{r} A_i$, there exists an $i \in \{1, \ldots, r\}$ such that $A_i - A_i + sA_i$ contains a Bohr set for every $s \in \mathbb{Z} \setminus \{0\}$. Inspired by Katznelson and Ruzsa's question, Theorem 1.4(b) naturally gives rise to the following question.

Question 6.1. *Suppose $A \subseteq \mathbb{Z}$ does not contain a Bohr set and $B \subseteq \mathbb{Z}$ such that $B + sA$ contains a Bohr set for every $s \in \mathbb{Z} \setminus \{0\}$. Must it be true that B contains a Bohr set?*

An positive answer to Question 6.1 would lead to a resolution of Katznelson-Ruzsa's question. However, it is likely that the answer to Question 6.1 is negative.

As mentioned in the introduction, we do not know whether the commuting conditions in Theorem 1.5 and Theorem 1.6 can be removed entirely or not. It is interesting to answer the following.

Question 6.2. *Can the commuting conditions in Theorem 1.5 and Theorem 1.6 be removed?*

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