ON BOHR SETS ARISING FROM PARTITIONS OF $\mathbb{Z}$

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ABSTRACT. We prove that for any finite partition $\mathbb{Z} = \cup_{i=1}^{r} A_i$, there exists $i \in \{1, \dots, r\}$ such that for all $s_1, s_2 \in \mathbb{Z}^+$, the set $s_1 A_i - s_1 A_i + s_2 A_i$ contains a Bohr set. This is a generalization of a result of Anh Le and the first author.

1. INTRODUCTION

Let G be an abelian topological group. For a finite set Λ of characters (i.e. continuous homomorphisms from G to $S^1 := \{z \in \mathbb{C} : |z| = 1\}$) and $\eta > 0$, the set

$$B(\Lambda; \eta) := \{x \in G : |\gamma(x) - 1| < \eta \text{ for all } \gamma \in \Lambda\}$$

is called a *Bohr set* or a *Bohr neighborhood of* 0. We refer to η as the *radius* and $|\Lambda|$ as the *rank* (or *dimension*) of the Bohr set.

If $A, B \subset G$, the sumset and difference set of A and B are $A \pm B := \{a \pm b : a \in A, b \in B\}$. If $c \in \mathbb{Z}$, we define $cA := \{ca : a \in A\}$.

If $A \subset \mathbb{Z}$, the *upper Banach density* of A is

$$d^*(A) := \lim_{N \to \infty} \sup_{M \in \mathbb{Z}} \frac{|A \cap [M+1, M+N]|}{N}.$$

The existence of Bohr sets in sumsets is a central topic in additive combiatorics. A classical theorem of Bogolyubov [1] says that if $A \subset \mathbb{Z}$ and $d^*(A) > 0$, then $A + A - A - A$ contains a Bohr set whose rank and dimension depend only on $d^*(A)$. Bergelson-Ruzsa [2] proved that if $A \subset \mathbb{Z}$ and $d^*(A) > 0$, then for any $s_1, s_2, s_3 \in \mathbb{Z} \setminus \{0\}$ with $s_1 + s_2 + s_3 = 0$, $s_1 A + s_2 A + s_3 A$ contains a Bohr set whose rank and dimension depend only on $d^*(A)$.

Recently these results were generalized in compact groups [6] and discrete groups [4]. Furthermore, besides density results, the authors of [6] and [4] also obtained analogous results for partitions. In particular, the authors of [6] proved the following.

Theorem 1 ([6, Theorem 1.4]). *Let $\mathbb{Z} = \bigcup_{i=1}^{r} A_i$ be a partition.*

(a) For any $s_1, s_2 \in \mathbb{Z} \setminus \{0\}$, there exists $i \in \{1, 2, \dots, r\}$ such that the set $s_1 A_i - s_1 A_i + s_2 A_i$ contains a Bohr set whose rank and radius depend only on r and s_1, s_2.

(b) There exists $i \in \{1, 2, \dots, r\}$ such that for any $s \in \mathbb{Z} \setminus \{0\}$, the set $A_i - A_i + sA_i$ contains a Bohr set.

Theorem 1(b) is a step towards a well-known open problem of Katznelson [5] and Ruzsa [7, Chapter 5], which asks if there must be an i such that the set $A_i - A_i$ contains a Bohr set. See [3] for a comprehensive discussion of this problem.

In this note, we prove the following "upgrade" of part (b) of Theorem 1: we show that part (b) can be extended to include all sumsets $s_1 A_i - s_1 A_i + s_2 A_i$ considered in part (a).

Theorem 2. *Let $\mathbb{Z} = \bigcup_{i=1}^r A_i$ be an arbitrary partition. Then there exists $i \in \{1, \dots, r\}$ such that for all $s_1, s_2 \in \mathbb{Z}^+$, the set $s_1 A_i - s_1 A_i + s_2 A_i$ contains a Bohr set.*

Note that, similarly to part (b) of Theorem 1, we do not obtain quantitative bounds on the rank and radius of the Bohr set in $s_1 A_i - s_1 A_i + s_2 A_i$.

2. Proof of Theorem 2

By symmetry of Bohr sets, we may assume that $s_1, s_2 > 0$. For a set $A \subset \mathbb{Z}$, we write $A^{(N)} := A \cap [-N, N]$. Our proof uses the same methods as [6] and relies on the following lemma, whose proof is implicit [6, p.30].

Lemma 3. *Let $A \subset \mathbb{Z}$, and let $s_1, s_2 \in \mathbb{Z}^+$. Suppose there exists $c > 0$ and an infinite set $\mathcal{N} \subset \mathbb{Z}^+$ such that*

$$\sum_{|x|,|y| \le N} 1_{A^{(N)}}(s_1 y) 1_{A^{(N)}}(x) 1_{A^{(N)}}(x + s_2 y) \ge cN^2$$

for every $N \in \mathcal{N}$. Then $s_1 A - s_1 A + s_2 A$ contains a Bohr set.

We also need the following counting result from [6].

Proposition 4 ([6, Proposition 5.1])**.** *For any $k, \ell, r > 0$, there is a constant $c(k, \ell, r) > 0$ such that the following holds: For any partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$ and sufficiently large N, there exists $1 \le i \le r$ such that*

$$\sum_{|x|,|y| \le N} 1_{A_i^{(N)}}(\ell y) \prod_{j=0}^{k} 1_{A_i^{(N)}}(x + jy) \ge c(k, \ell, r) N^2.$$

Using Proposition 4, we will now deduce the following, which is a generalization of [6, Proposition 5.2].

Proposition 5. *Let $\mathcal{P}$ denote an arbitrary partition $\mathbb{Z} = \bigcup_{i=1}^r A_i$. Then there exists some $1 \le i \le r$ with the following property: For every $s_1, s_2 \in \mathbb{Z}^+$, there is a constant $c(s_1, s_2, \mathcal{P}) > 0$ such that*

$$\sum_{|x|,|y| \le N} 1_{A_i^{(N)}}(s_1 y) 1_{A_i^{(N)}}(x) 1_{A_i^{(N)}}(x + s_2 y) \ge c(s_1, s_2, \mathcal{P}) N^2 \tag{1}$$

for infinitely many $N \in \mathbb{Z}^+$.

Proof. Enumerate the set $\{(s_1, s_2) : s_1, s_2 \in \mathbb{Z}^+\}$ as $\{(s_1^{(i)}, s_2^{(i)})\}_{i=1}^{\infty}$. For any k, let

$$S_k = \prod_{j=1}^{k} s_1^{(j)}, \qquad T_k^{(i)} = s_2^{(i)} \prod_{\substack{j=1, \\ j \ne i}}^{k} s_1^{(j)} \qquad \text{and} \qquad T_k = \max_{1 \le i \le k} T_k^{(i)}.$$

Applying Proposition 4 with S_k in place of ℓ and T_k in place of k, we see that for every sufficiently large N, there exists $i \in \{1, \dots, r\}$ such that

$$\sum_{|u|,|v| \le N} 1_{A_i^{(N)}}(S_k v) \prod_{j=0}^{T_k} 1_{A_i^{(N)}}(u + jv) \ge c(k, r) N^2. \tag{2}$$

Here the constant $c(k,r) > 0$ is independent of N.

For each k, there exists $i = f(k)$ such that for infinitely many N, (2) holds. Hence, there is $i_0 \in \{1, \dots, r\}$ and an infinite set K such that for all $k \in K$, we have $f(k) = i_0$. In other words, for all $k \in K$ and for infinitely many N (depending on k), we have

$$\sum_{|u|,|v|\le N} 1_{A_{i_0}^{(N)}}(S_k v) \prod_{j=0}^{T_k} 1_{A_{i_0}^{(N)}}(u + jv) \ge c(k,r)N^2. \tag{3}$$

Now let $s_1, s_2 \in \mathbb{Z}^+$ be arbitrary and suppose $(s_1, s_2) = (s_1^{(\ell)}, s_2^{(\ell)})$. Pick $k \in K$ such that $k \ge \ell$. Then for infinitely many N,

$$\begin{aligned} &\sum_{|u|,|v|\le N} 1_{A_{i_0}^{(N)}}(S_k v) 1_{A_{i_0}^{(N)}}(u) 1_{A_{i_0}^{(N)}}(u + T_k^{(\ell)} v) \\ &\ge \sum_{|u|,|v|\le N} 1_{A_{i_0}^{(N)}}(S_k v) \prod_{j=0}^{T_k} 1_{A_{i_0}^{(N)}}(u + jv) \ge c(k,r)N^2. \end{aligned} \tag{4}$$

Upon writing $x = u, y = \left(\prod_{\substack{j=1,\\ j\ne\ell}}^{k} s_1^{(j)}\right) v$ and $N' = \left(\prod_{\substack{j=1,\\ j\ne\ell}}^{k} s_1^{(j)}\right) N$, we have that for infinitely many $N' \in \mathbb{Z}^+$,

$$\sum_{|x|,|y|\le N'} 1_{A_{i_0}^{(N')}}(s_1 y) 1_{A_{i_0}^{(N')}}(x) 1_{A_{i_0}^{(N')}}(x + s_2 y) \ge c(k,r) \left(\prod_{\substack{j=1,\\ j\ne\ell}}^{k} s_1^{(j)} \right)^{-2} N'^2, \tag{5}$$

as desired. □

Theorem 2 follows immediately from Lemma 3 and Proposition 5.

Finally, we remark that the proof of Theorem 2 can be easily generalized to number fields and function fields in place of $\mathbb{Z}$. We refer the reader to Sections 5.1 and 5.2 of [6] for the necessary modifications.

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